Quantum Theory of Gravity. II. The Manifestly Covariant Theory*

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Contrary to the situation which holds for the canonical theory described in the first paper of this series, there exists at present no tractable pure operator language on which to base a manifestly covariant quantum theory of gravity. One must construct the theory by analogy with conventional S-matrix theory, using the c-number language of Feynman amplitudes when nothing else is available. The present paper undertakes this construction. It begins at an elementary level with a treatment of the propagation of small disturbances on a classical background. The classical background plays a fundamental role throughout, both as a technical instrument for probing the vacuum (i.e., analyzing virtual processes) and as an arbitrary fiducial point for the quantum fluctuations. The problem of the quantized light cone is discussed in a preliminary way, and the formal structure of the invariance group is displayed. A condensed notation is adopted which permits the Yang-Mills field to be studied simultaneously with the gravitational field. Generally covariant Green's functions are introduced through the imposition of covariant supplementary conditions on small disturbances. The transition from the classical to the quantum theory is made via the Poisson bracket of Peierls. Commutation relations for the asymptotic fields are obtained and used to define the incoming and outgoing states. Because of the non-Abelian character of the coordinate transformation group, the separation of propagated disturbances into physical and nonphysical components requires much greater care than in electrodynamics. With the aid of a canonical form for the commutator function, two distinct Feynman propagators relative to an arbitrary background are defined. One of these is manifestly covariant, but propagates nonphysical as well as physical quanta; the other propagates physical quanta only, but lacks manifest covariance. The latter is used to define external-line wave functions and non-radiatively-corrected amplitudes for scattering, pair production, and pair annihilation by the background field. The group invariance of these amplitudes is proved. A fully covariant generalization of the complete S matrix is next proposed, and Feynman's *tree theorem* on the group invariance of non-radiatively-corrected n-particle amplitudes is derived. The big problem of radiative corrections is then confronted. The resolution of this problem is carried out in steps. The single-loop contribution to the vacuum-to-vacuum amplitude is first computed with the aid of the formal theory of continuous determinants. This contribution is then functionally differentiated to obtain the lowest-order radiative corrections to the n-quantum amplitudes. These amplitudes split automatically into *Feynman baskets*, i.e., sums over tree amplitudes (bare scattering amplitudes) in which all external lines are on the mass shell. This guarantees their group invariance. The invariance can be made partially manifest by converting from the noncovariant Feynman propagator to the covariant one, and this leads to the formal appearance of *fictitious quanta* which compensate the nonphysical modes carried by the covariant propagator. Although avoidable in principle, these quanta necessarily appear whenever manifestly covariant expressions are employed, e.g., in renormalization theory. The fictitious quanta, however, appear only in closed loops and are coupled to real quanta through vertices which vanish when the invariance group is Abelian. The vertices are nonsymmetric and always occur with a uniform orientation around any fictitious quantum loop. The problem of splitting radiative corrections into Feynman baskets becomes more difficult in higher orders, when overlapping loops occur. This problem is approached with the aid of the Feynman functional integral. It is shown that the "measure" or "volume element" for the functional integration plays a fundamental role in the decomposition into Feynman baskets and in guaranteeing the invariance of radiative corrections under arbitrary changes in the choice of basic field variables. The "measure" has two effects. Firstly, it removes from all closed loops the *noncausal chains* of cyclically connected advanced (or retarded) Green's functions, thereby breaking them open and ensuring that at least one segment of every loop is on the mass shell. Secondly it adds certain nonlocal corrections to the operator field equations, which vanish in the classical limit $\hbar \rightarrow 0$. The question arises why these removals and corrections are always neglected in conventional field theory without apparent harm. It is argued that the usual procedures of renormalization theory automatically take care of them. In practice the criteria of locality and unitarity are replaced by analyticity statements and Cutkosky rules. It is virtually certain that the "measure" may be similarly ignored (set equal to unity) in gravity theory, and that attention may therefore be confined to *primary diagrams*, i.e., diagrams which contain Feynman propagators only, with no noncausal chains removed. A general algorithm is given for obtaining the primary diagrams of arbitrarily high order, including all fictitious quantum loops, and the group invariance of the amplitudes thereby defined is proved. Essential to all these derivations is the use of a background field satisfying the classical "free" field equations. It is never necessary to employ external sources, and hence the well-known difficulties arising with sources in a non-Abelian context are avoided.

1. INTRODUCTION

IN the first paper of this series[1] an attempt was made to show what happens when canonical Hamiltonian quantization methods are applied to the gravitational field. Attention was focused on some of the bizarre features of the resulting formalism which arise in the case of finite worlds, and which are of possible cosmological and even metaphysical significance. Such

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[1] B. S. DeWitt, Phys. Rev. 160, 1113 (1967). This paper will be referred to as I.

prosaic questions as the scattering, production, absorption, and decay of individual quanta were left untouched. The main reason for this was that the canonical theory does not lend itself easily to the study of these questions when physical conditions are such that the effects of vacuum processes must be taken into account. A manifestly covariant formalism is needed instead. It is the task of the present paper to provide such a formalism.

We must begin by making clear precisely what is meant by "manifest covariance." In conventional S-matrix theory (whether based on a conventional field theory or not) "manifest covariance" means "manifest Lorentz covariance." In the context of a theory of gravity the question arises whether it should mean more than this, since the classical theory from which one starts has "manifest *general* covariance." Here one must be careful. There is an important difference between general covariance and ordinary Lorentz covariance, and neither one implies the other. Lorentz covariance is the expression of a *geometrical* symmetry possessed by a system. In gravity theory it has relevance at most to the asymptotic state of the field. As has been emphasized by Fock,[2] the word "relativity" in the name "general relativity" has connotations of symmetry which are misleading. Far from being more relativistic than special relativity, general relativity is in fact less relativistic. For as soon as space-time acquires bumps (i.e., curvature) it becomes absolute in the sense that one may be able to specify position or velocity with respect to these bumps, provided they are sufficiently pronounced and distinguishable from one another. Only when the bumps coalesce into regions of uniform curvature does space-time regain its relativistic properties. It never becomes *more* relativistic than flat space-time, which is characterized by the 10-parameter Poincaré group.

The technical method of distinguishing between the Poincaré group and the general coordinate transformation group is to confine the operations of the latter group to a finite (but arbitrary) region of space-time. The asymptotic coordinates are then left undisturbed by general coordinate transformations, and only the operations of the Poincaré group (if that is indeed the asymptotic symmetry group of the problem) are allowed to change them. The general coordinate transformation group thus becomes a *gauge* group which, although historically an offspring of the Poincaré group and the equivalence principle, plays technically the rather obscure role of providing the analytic means by which the Einstein equations can be obtained from a variational principle and their essential locality displayed.[3]

This, however, is not the whole story, for the general coordinate transformation group still has, even as a gauge group, profound physical implications. Some of these we have already encountered in I, and some we shall encounter in the present paper. Others will appear in the final paper of this series, which is to be devoted to applications of the covariant theory. If it were not for these implications there would be little interest in pushing our investigations further, for there is no likelihood that such "prosaic" processes as graviton-graviton scattering or curvature induced vacuum polarization will ever be experimentally observed.[4] The real reason for studying the quantum theory of gravity is that by uniting quantum theory and general relativity one may discover, at no cost in the way of new axioms of physics, some previously unknown consequences of general coordinate invariance, which suggest new interesting things that can be done with quantum field theory as a whole.

Our problem will be to develop a formalism which makes manifest the extent to which general covariance permeates the theory. This will be accomplished by introducing, instead of a flat background, an adjustable c-number background metric. Use of such a metric has the following fundamental technical advantages: (1) It facilitates the introduction of particle propagators which are generally covariant rather than merely Lorentz-covariant. (2) It reduces the study of radiative corrections to the study of the vacuum. (3) It makes possible the generally covariant isolation of divergences, which is essential to any renormalization program. (4) It renders theorems analogous to the Ward identity almost trivial. (5) It makes possible, in principle, the extension of the theory of radiative corrections to worlds for which space-time is not asymptotically flat and which may even be closed and finite. These advantages are typical of what we shall mean by the phrase "manifest covariance." Use of the phrase, however, is not to be understood as implying that the simple trick of introducing a variable background metric makes *everything* obvious. The generally covariant propagators will not be unique but will be choosable in various ways, analogous to the gauge choices in quantum electrodynamics, and we shall have to undertake a separate investigation, just as in quantum electrodynamics, to verify that the choice is irrelevant. This investigation turns out to be much more complicated than in the case of quantum electrodynamics.

Of the five advantages listed above as stemming from the use of a variable background metric only the first two will appear in the present paper. The third

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[2] V. Fock, *The Theory of Space-Time and Gravitation* (Pergammon Press, New York, 1959).

[3] The content of the Einstein equations can be expressed in an intrinsic coordinate-independent form only at the cost of introducing nonlocal structures. (See, for example, Ref. 32). It can be argued [see S. Weinberg, Phys. Rev. **138**, B988 (1965)] that the general coordinate transformation group is simply a consequence of the zero rest mass of the gravitational field and its long-range character.

[4] Although one might hope for some very indirect cosmological evidence for such processes.

and fourth will be demonstrated in the following paper of this series, while the fifth remains a program for the future. It is not out of place here, however, to speculate briefly on this ultimate program. As long as the conventional S matrix is our chief concern it is appropriate to choose a background metric which is asymptotically flat. We shall see that Lorentz invariance of the S matrix then follows almost trivially from the formalism, in the limit in which the background metric becomes everywhere Minkowskian. Now it is obvious that scattering processes are also possible in an infinite world which is not asymptotically flat. In such a world it should be possible to construct a generalized S matrix in which the conventional plane-wave momentum eigenfunctions are replaced by wave functions appropriate to the altered asymptotic geometry. The asymptotic geometry itself would be fixed by choosing the background metric appropriately.

In a closed world no rigorous S matrix exists. The continuum of scattering states is replaced by a regime of discrete quantization, and, as we have seen in I, the wave function of the universe may even be unique. It may be conjectured that the formalism most appropriate to this case is obtained by choosing the background metric to be *not* a c number but rather an operator depending on a small number (e.g., *one*) of quantum variables similar to the operator R representing the radius of the Friedmann universe studied in I. These variables would be quantized by the canonical method, while the full q-number metric would continue to be treated by manifestly covariant methods. (Conditions of constraint would, of course, have to be imposed on the latter metric to take into account the fact that some of its degrees of freedom have been transferred to the background metric.) The resulting simultaneous use of both the canonical and covariant theories might help to reveal the relationship between them.

As has been remarked in I, no rigorous mathematical link has thus far been established between the canonical and covariant theories. In the case of infinite worlds it is believed that the two theories are merely two versions of the same theory, expressed in different languages, but no one knows for sure. The analysis of radiative corrections has turned out to be of such intricacy that the covariant theory has had to be developed completely within its own framework and independently of the canonical theory. Although the structure of the covariant theory is suggested by the formalism of field operators, and hence maintains a few points of contact with conventional field theory, the language of operators is dropped at a certain key stage and c-number criteria are thenceforth exclusively employed to maintain internal consistency. It turns out that the language of operators is a peculiarly unwieldy one in which to discuss questions of consistency when the invariance group of the theory is non-Abelian. The language of graphs and the S matrix is much more direct.

The latter language, embracing as it does many different particle theories at once, is also much less dependent on the detailed Lagrangian structure of the field theory on which it is based. It assumes that virtual processes may be described by an infinite set of basic diagrams, the combinatorial properties of which are the same for all field theories. In working out the details of how this language is to be extended to the non-Abelian case, we have attempted to develop it within as broad a framework as possible. Every theorem in this paper will therefore apply not only to the gravitational field but also to the Yang-Mills field[5] which, like the gravitational field, possesses a non-Abelian invariance group.[6]

Section 2 begins with the introduction of a notation which is sufficiently general to embrace all boson field theories and at the same time condensed enough to reduce the highly complex analysis of subsequent sections to manageable proportions. A table is included to facilitate comparison of the condensed notation with the detailed forms which the various symbols take in the case of the Yang-Mills and gravitational fields. The notation is particularly useful in dealing with the second functional derivative of the action, which plays the role of the differential operator governing the propagation of infinitesimal disturbances on an arbitrary background field. It is also useful in dealing with the higher functional derivatives, which are the bare vertex functions of the theory. The problem of the quantized light cone is discussed in a preliminary way in Sec. 3, and its relationship to the "nonrenormalizability" of the theory is noted. Attention is called to the various roles of the background metric, one of which is to define the concepts of "past" and "future." Green's theorem for an arbitrary differential operator is then derived.

Section 4 introduces a notation for the basic structures governing the action of the invariance group on the field variables. The relationship between manifest covariance and linearity of the group transformation laws is emphasized. In Sec. 5 it is pointed out that the infinitesimal disturbances themselves are determined only modulo an Abelian transformation group. This group, which is the tangent group of the full group, affects only the field variables but not physical observables. The latter are necessarily group-invariant. Infinitesimal disturbances satisfying retarded or ad-

[5] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

[6] The term "invariance group," as used in this paper, will always refer to the infinite dimensional "gauge" group of the theory, and not to the finite dimensional (≤ 10) asymptotic isometry group, which is undetermined *a priori*. It is not hard to show that the Yang-Mills field and its "gauge" group can be given a metrical interpretation which suggests a physical kinship between the Yang-Mills and gravitational fields which is closer than the formal mathematical similarities between them alone indicate. [See B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach Science Publishers, Inc., New York, 1965), problem 77, p. 139.]

vanced boundary conditions can be computed with the aid of corresponding Green's functions provided supplementary conditions are imposed. For convenience these supplementary conditions are chosen in a manifestly covariant way, but their essential arbitrariness is emphasized.

Use of the covariant Green's functions in connection with Cauchy data for infinitesimal disturbances is discussed in Sec. 6, and the fundamental reciprocity relations of propagator theory are established. Transition from the classical to the quantum theory is made via the Poisson bracket of Peierls (see Ref. 20), which is determined solely by the behavior of infinitesimal disturbances. The reciprocity relations are used to show that Peierls' Poisson bracket satisfies all the usual identities. Section 7 introduces the important concept of the asymptotic fields, which obey the field equations of the linearized theory. From the asymptotic fields one can construct asymptotic invariants, which may be used to characterize completely the physical state of the field. The asymptotic invariants are conditional invariants, i.e., invariants *modulo* the field equations. It is emphasized that their commutators (i.e., Poisson brackets) are nonetheless well defined. A direct proof is given that the asymptotic invariants satisfy the commutation relations of the linearized theory, a result which is nontrivial when a group is present. This result is used in Sec. 8 to construct the creation and annihilation operators for real (i.e., physical) quanta in the remote past and future. The detailed structures of the asymptotic Yang-Mills and gravitational fields must be investigated separately, but a condensed notation (for the asymptotic wave functions) is again introduced, which embraces both fields at once and emphasizes their similarities. A table is included to facilitate the comparison. The quanta of both fields are transverse and differ only in spin. States are labeled by helicity, which is readily shown to be Lorentz-invariant.

Continuing the uniform treatment of the two fields, Sec. 9 shows that the asymptotic commutator functions of both can be expressed in a standard canonical form. A special notation is introduced for the projection of the canonical form into the physical subspace. With the aid of this projection two distinct Feynman propagators are defined relative to an arbitrary background field. Both serve to describe the propagation of field quanta in nonasymptotic regions as well as at infinity. One is manifestly covariant but propagates nonphysical as well as physical quanta; the other propagates physical quanta only but lacks manifest covariance. The latter is used in Sec. 10 to define the external line wave functions which enter into the ultimate definition of the S matrix. These functions serve to generalize the asymptotic wave functions to the case in which an arbitrary background field is present. They satisfy a number of important relations following from a fundamental lemma which is proved in this section. The lemma is used again in Sec. 11 to prove that the non-radiatively-corrected amplitudes for scattering, pair production and pair annihilation by the background field are group-invariant. "Group invariance" here implies invariance under group transformations of the background field, under gauge changes of the propagators, and under radiation gauge changes in the asymptotic wave functions. The amplitudes are also shown to satisfy a set of relations which are the relativistic generalizations of the well known optical theorem for nonrelativistic scattering.

Construction of the full S matrix of the theory is begun in Sec. 12. The field operators are separated into two parts, a classical background satisfying the classical field equations, and a quantum remainder. Vacuum states associated with the remote past and future are defined *relative* to the background field. Vacuum matrix elements of chronological products are constructed by varying the vacuum-to-vacuum amplitude with respect to the background field. It turns out that all physical amplitudes can be obtained in this way despite the fact that the variations in the background field are subject to the constraint that the classical field equations never be violated. The well-known difficulties arising with the use of external sources in a non-Abelian context are thus avoided. When no invariance group is present the vacuum matrix elements of chronological products are expressible in terms of functions having the combinatorial structure of tree diagrams. Use of these functions constitutes an essential part of the program for constructing the S matrix as given in this paper. Since these functions are initially defined only in the absence of an invariance group, however, we are at this point forced to abandon the strict operator formalism. Section 13 displays the structure of the S matrix and its unitarity conditions when no invariance group is present. Section 14 then begins the long and intricate task of generalizing this structure to the case in which a group is present. Aside from an invariance lemma which is used to suggest the desired generalization, the important proof of this section is the *tree theorem*. The tree theorem says that the lowest-order (i.e., non-radiatively corrected) contributions to any scattering process can always be calculated by elementary methods, using any choice of gauge for the propagators of the internal lines and any choice of gauge for the external-line wave functions. The result will be independent of the gauge choices provided all the tree diagrams contributing to the given process are summed together.

There remains only the question of the vacuum-to-vacuum amplitude itself. Since all radiative corrections can be obtained by functionally differentiating this amplitude with respect to the background field, a proof of its group invariance would complete the proof of the invariance of the entire S matrix. The real problem, however, is to *construct* the amplitude, and the invariance criterion must therefore be used as a guide rather than as an *a posteriori* consistency check. Section 15 pauses briefly to review the question of Lorentz invariance, to point out that the theory should also be invariant under changes in the specific variables with which one works, and to comment upon the utility of using c-number language exclusively. Section 16 then plunges into the main problem. The single-loop contribution to the vacuum-to-vacuum amplitude is computed with the aid of the formal theory of continuous determinants, and various alternative forms for it are given. There is no ambiguity about this contribution, and its group invariance is readily demonstrated. This contribution is functionally differentiated in Sec. 17 to yield the lowest-order contribution to single quantum production by the background field. The latter splits into two parts, one involving the covariant propagator for normal quanta and the other involving the covariant propagator for a set of *fictitious quanta* which compensate the nonphysical quanta that the first propagator also carries. The fictitious quanta are coupled to real quanta through asymmetric vertices which vanish when the invariance group is Abelian. With the aid of the fundamental lemma of Sec. 10 and a collection of new identities it is shown that the fictitious quanta can be formally avoided by replacing the covariant propagator by the noncovariant one which carries physical quanta only. The covariant propagators, however, are needed for the practical implementation of any renormalization program.

The lowest-order radiative corrections to the n-quantum amplitudes are analyzed in Sec. 18. These amplitudes split automatically into *Feynman baskets,* i.e., sums over tree amplitudes (lowest-order scattering amplitudes) in which all external lines are on the mass shell. The tree theorem then guarantees their group invariance. This invariance can be made partially manifest by converting from the noncovariant propagator to the covariant one, and the fictitious quanta again make their appearance.

The problem of splitting the radiative corrections into Feynman baskets becomes more difficult in higher orders, when overlapping loops occur. This problem is approached in Sec. 19 with the aid of the Feynman functional integral. When no invariance group is present it is shown that the "measure" or "volume element" for the functional integration plays a fundamental role in the decomposition into Feynman baskets and in guaranteeing the invariance of the vacuum-to-vacuum amplitude under arbitrary changes in the choice of basic field variables. The "measure" has two effects. Firstly, it removes from all closed loops the *noncausal chains* of cyclically connected advanced (or retarded) Green's functions, thereby breaking them open and insuring that at least one segment of every loop is on the mass shell. Secondly, it adds certain nonlocal corrections to the operator field equations, which vanish in the classical limit $\hbar \rightarrow 0$. The question arises why these removals and corrections are always neglected in conventional field theory without apparent harm. It is argued that the usual procedures of renormalization theory automatically take care of them and that in practice the criteria of locality and unitarity are replaced by analyticity statements and Cutkosky rules (see Ref. 52). A detailed investigation of these corrections when a group is present is undertaken in Sec. 20. The two-loop Feynman-basket decomposition of the preceding section is appropriately generalized and the result is reexpressed in terms of covariant propagators, including the fictitious quanta. It turns out that the total two-loop amplitude is obtainable from a set of covariant *primary diagrams* (containing Feynman propagators only, and hence off-mass-shell contributions in all lines) by a process of removing noncausal chains and adding nonlocal corrections, which is completely analogous to that of the no-group case. Moreover, the primary diagrams, taken together, are group-invariant as they stand, *independently of the tree theorem.* This suggests that even when a group is present the noncausal chains and nonlocal corrections may be neglected as in conventional field theory. The problem therefore becomes one of finding a general algorithm for obtaining the primary diagrams of arbitrarily high order, including all fictitious quantum loops. The remainder of Sec. 20 is devoted to the construction of such an algorithm. The generator for the algorithm is a Feynman functional integral for the vacuum-to-vacuum amplitude, which includes fields representing the fictitious quanta. The group invariance of this integral is explicitly demonstrated, and the fictitious quanta are shown formally to obey Fermi statistics despite their integral spin. No physical criteria are violated, however, since the fictitious quanta never occur outside of closed loops. Finally, the rules for inserting external lines into the primary vacuum diagrams are given, and the asymmetric vertices contained in the fictitious quantum loops are shown to have a uniform orientation around each loop.

2. NOTATION. INFINITESIMAL DISTURBANCES. BARE VERTEX FUNCTIONS

A quantum field theory begins with the selection of an action functional S. If the theory is *local* this functional is expressible in the form

$$S \equiv \int \mathfrak{L} dx, \quad dx \equiv dx^0 dx^1 dx^2 dx^3, \tag{2.1}$$

where $\mathfrak{L}$—the Lagrangian (density)—is a function of the dynamical variables and a finite number of their space-time derivatives at a single point. Various criteria such as covariance, self-consistency of the field equations, the existence of the vacuum as a state of lowest energy, and positive definiteness of the quantum-mechanical Hilbert space in practice drastically limit the possible choices for $\mathfrak{L}$. However, many different choices exist for the Lagrangian of a given field. Thus it is always possible to add a trivial divergence to the Lagrangian without changing the field equations at all. Moreover, the field variables may be replaced by arbitrary functions of themselves; this replaces the field equations by linear combinations of themselves. Finally, even the number of field variables is not unique; for example, alternative Lagrangians may be found leading to field equations which express some of the variables in terms of derivatives of others. What is important is that the choice of Lagrangian is basically irrelevant to the development of the theory of a given field and should be determined only by convenience. The quantum theory of a given field must be constructed in such a way that it is invariant under changes in the mode of description of the field.

It will prove convenient in what follows to adopt a highly condensed notation. The field variables (assumed here to be real) will be denoted by φ^i,[7] and commas followed by indices from the middle of the Greek alphabet will be used to denote differentiation with respect to the space-time coordinates. The first part of the Greek alphabet will be reserved for *group* indices, to be introduced presently. Primes will be used to distinguish different points of space-time; they will also appear on associated indices, or on field symbols themselves, when it is desired to avoid cumbersome explicit appearances of the x's. In most cases, however, the primes will be simply omitted. This corresponds to making the indices i, j, etc. do double duty as discrete labels for field components and as continuous labels over the points of space-time. That is, an index such as i will really stand for the quintuple (i, x^0, x^1, x^2, x^3) and the summation convention for repeated indices will be extended to include integrations over the x's. The significance of the indices thus becomes almost purely combinatorial. When this notation is employed it is necessary to remember that expressions such as M_{ij} are really elements of continuous matrices and that the symbol $\delta^i{}_j$ involves a 4-dimensional δ function.

For most purposes the form of the field equations is more important than the value of the action functional. Therefore, the domain of integration in (2.1) is unimportant; when otherwise unspecified it is to be understood as being large enough to embrace all points at which it may be desired to perform functional differentiations. Functional differentiation with respect to the field variables will be denoted by a comma followed by one or more Latin indices. Thus the field equations will be expressed in the symbolic form

$$S_{,i}=0. \qquad (2.2)$$

[7] In this paper no restriction is imposed on the range of Latin indices. Other conventions, to the extent they overlap, are the same as in I.

Suppose the form of the action functional suffers the following change:

$$S\rightarrow S+\epsilon A\,, \qquad (2.3)$$

where ϵ is an infinitesimal constant. Such a change may be thought of as being brought about by weak coupling to some external agent. The coupling produces an infinitesimal disturbance $\delta\varphi^i$ in the field, which satisfies the linear inhomogeneous equation

$$S_{,ij}\delta\varphi^j=-\epsilon A_{,i}. \qquad (2.4)$$

That is, $\varphi^i+\delta\varphi^i$ satisfies the field equations of the system $S+\epsilon A$ if φ^i satisfies those of the system S. The undisturbed field φ^i may be regarded as a *background field* upon which the disturbance $\delta\varphi^i$ propagates. The concept of the background field proves to be a useful one in the covariant theory, and will occur repeatedly in what follows.

For local theories the quantity $S_{,ij}$ has the form of a linear combination of δ functions and derivatives of δ functions, with functions of the field variables and their derivatives as coefficients. In Eq. (2.4) $S_{,ij}$ therefore plays the role of a linear differential operator with variable coefficients. The reader will find it useful to consult Table I, which lists the explicit forms which this and various other abstract symbols of the general formalism take in the cases of the Yang-Mills field and the gravitational field, respectively.

In the case of linear theories $S_{,ij}$ corresponds to a linear differential operator with constant coefficients, and the higher functional derivatives $S_{,ijk}$, etc., vanish. In nonlinear theories the higher functional derivatives are known as *bare vertex functions*. They describe the basic interactions between *finite* disturbances, the propagation of which, as will be seen later, provides a direct classical model for the quantum S matrix.

It is frequently convenient to introduce a further condensation of notation, namely to make the replacement

$$S_{,i_1\cdots i_n}\rightarrow S_n \qquad (2.5)$$

and to drop the indices altogether. Equations (2.2) and (2.4) are then replaced by

$$S_1=0 \qquad (2.6)$$

and

$$S_2\delta\varphi=-\epsilon A_1, \qquad (2.7)$$

respectively. If the basic field variables are properly chosen the number of nonvanishing bare vertex functions is finite in the case of both the Yang-Mills and gravitational fields. Thus, for the Yang-Mills field we have $S_n=0$ for $n>4$ when the field variables are chosen as in Table I, while for the gravitational field we have $S_n=0$ for $n>9$ if the quantities $\varphi^{\mu\nu}\equiv g^{5/18}g^{\mu\nu}-\eta^{\mu\nu}$ are

TABLE I. Expressions for the Yang-Mills and gravitational fields corresponding to quantities appearing in the abstract formalism.

Abstract symbol or equation	Corresponding expression for the Yang-Mills field	Corresponding expression for the gravitational field
φ^i	$A^\alpha{}_\mu$; $\mu=0,1,2,3$; $\alpha=1\cdots n$	$\varphi_{\mu\nu}\equiv g_{\mu\nu}-\eta_{\mu\nu}$; $\mu,\nu=0,1,2,3$; $\varphi_{\mu\nu}\equiv\varphi_{\nu\mu}$
S	$S\equiv-\frac{1}{4}\int F_{\alpha\mu\nu}F^{\alpha\mu\nu}dx,$ $F^\alpha{}_{\mu\nu}\equiv A^\alpha{}_{\nu,\mu}-A^\alpha{}_{\mu,\nu}+c^\alpha{}_{\beta\gamma}A^\beta{}_\mu A^\gamma{}_\nu.$ The indices μ, ν are raised and lowered by means of the Minkowski metric $\eta_{\mu\nu}\equiv\mathrm{diag}(-1,1,1,1)$ and its inverse $\eta^{\mu\nu}$. The indices α, β, γ are raised and lowered by means of the Cartan metric, $\gamma_{\alpha\beta}\equiv-c^{-2}c^\gamma{}_{\alpha\delta}c^\delta{}_{\beta\gamma}$ and its inverse $\gamma^{\alpha\beta}$. The c's are the structure constants of a compact n-dimensional semi-simple Lie group, and the constant c^2 is chosen so that $\det(\gamma_{\alpha\beta})=1$.	$S\equiv\int g^{1/2}\,{}^{(4)}R\,dx,$ $g\equiv-\det(g_{\mu\nu})$, ${}^{(4)}R\equiv R_\mu{}^\mu$, $R_{\mu\nu}\equiv R_{\sigma\mu\nu}{}^\sigma$, $R_{\mu\nu\sigma}{}^\tau\equiv\Gamma_{\nu\sigma}{}^\tau{}_{,\mu}-\Gamma_{\mu\sigma}{}^\tau{}_{,\nu}+\Gamma_{\nu\sigma}{}^\rho\Gamma_{\mu\rho}{}^\tau-\Gamma_{\mu\sigma}{}^\rho\Gamma_{\nu\rho}{}^\tau$, $\Gamma_{\mu\nu}{}^\sigma\equiv\frac{1}{2}g^{\mu\tau}(g_{\mu\tau,\nu}+g_{\nu\tau,\mu}-g_{\mu\nu,\tau})$. The indices μ, ν, ρ, σ, τ are raised and lowered by means of the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. In the remaining entries of this table the symbol ${}^{(4)}R$ is replaced simply by R.
$0=S_{,i}$	$0=\delta S/\delta A^\alpha{}_\mu\equiv-F_\alpha{}^{\mu\nu}{}_{;\nu}$.	$0=\delta S/\delta\varphi_{\mu\nu}\equiv-g^{1/2}(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R)$.
$\delta^i{}_j$	$\delta^\alpha{}_{\mu\beta'}{}^{\nu'}\equiv\delta^\alpha{}_\beta\delta_\mu{}^\nu\delta(x,x')$	$\delta_{\mu\nu}{}^{\sigma'\tau'}\equiv\frac{1}{2}(\delta_\mu{}^\sigma\delta_\nu{}^\tau+\delta_\nu{}^\sigma\delta_\mu{}^\tau)\delta(x,x')$.
$S_{,ij}$	$\delta^2S/\delta A^\alpha{}_\mu\delta A^{\beta'}{}_{\nu'}\equiv\delta_\alpha{}^\mu{}_{\beta'}{}^{\nu'}{}_{;\sigma}{}^\sigma-\delta_\alpha{}^\sigma{}_{\beta'}{}^{\nu'}{}_{;}{}^\mu{}_\sigma+c^\epsilon{}_{\alpha\gamma}F^{\gamma\mu}{}_\sigma\delta_\epsilon{}^\sigma{}_{\beta'}{}^{\nu'}$	$\delta^2S/\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\equiv\frac{1}{4}g^{1/2}(g^{\mu\rho}g^{\nu\lambda}+g^{\mu\lambda}g^{\nu\rho}-g^{\mu\nu}g^{\rho\lambda})g^{\iota\kappa}$ $\times(\delta_{\rho\lambda}{}^{\sigma'\tau'}{}_{;\iota\kappa}+\delta_{\iota\kappa}{}^{\sigma'\tau'}{}_{;\rho\lambda}-\delta_{\rho\iota}{}^{\sigma'\tau'}{}_{;\lambda\kappa}-\delta_{\lambda\kappa}{}^{\sigma'\tau'}{}_{;\rho\iota})$ $-\frac{1}{2}g^{1/2}(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R)\delta_\rho{}^{\rho\sigma'\tau'}$ $+g^{1/2}(R^\mu{}_\rho\delta^{\rho\nu\sigma'\tau'}+R^\nu{}_\rho\delta^{\rho\mu\sigma'\tau'}-\frac{1}{2}R\delta^{\mu\nu\sigma'\tau'}$ $-\frac{1}{2}g^{\mu\nu}R^{\rho\lambda}\delta_{\rho\lambda}{}^{\sigma'\tau'})$
$\delta\xi^\alpha$	The infinitesimal group parameters are functions $\delta\xi^\alpha(x)$ which assign to each point x a corresponding infinitesimal transformation of the generating Lie group. Under inner automorphisms they transform according to the adjoint representation of the full group.	The infinitesimal group parameters are the functions $\delta\xi^\mu(x)$ appearing in the infinitesimal coordinate transformation $\bar{x}^\mu=x^\mu+\delta\xi^\mu$. Under inner automorphisms they transform as contravariant vectors. Note that group and coordinate indices coincide in the case of the general coordinate transformation group.
$R^i{}_\alpha$	$R^\alpha{}_{\mu\beta'}\equiv-\delta^\alpha{}_{\beta';\mu}$, $\delta^\alpha{}_{\beta'}\equiv\delta^\alpha{}_\beta\delta(x,x')$	$R_{\mu\nu\sigma'}\equiv-\delta_{\mu\sigma';\nu}-\delta_{\nu\sigma';\mu}$, $\delta_{\mu\sigma'}\equiv g_{\mu\sigma}\delta(x,x')$
$\delta\varphi^i=R^i{}_\alpha\delta\xi^\alpha$	$\delta A^\alpha{}_\mu=-\delta\xi^\alpha{}_{;\mu}=-\delta\xi^\alpha{}_{,\mu}-c^\alpha{}_{\gamma\beta}A^\gamma{}_\mu\delta\xi^\beta$ Semicolons denote invariant differentiation. A field quantity φ which has the group transformation law $\delta\varphi=G_\alpha\varphi\delta\xi^\alpha$, where the G_α are the generators of a matrix representation of the generating Lie group, is defined to have the invariant derivative $\varphi_{;\mu}\equiv\varphi_{,\mu}+G_\alpha A^\alpha{}_\mu\varphi$. Invariant differentiation leaves transformation properties intact. It has the commutation law $\varphi_{;\mu\nu}-\varphi_{;\nu\mu}=-G_\alpha F^\alpha{}_{\mu\nu}\varphi$.	$\delta\varphi_{\mu\nu}=-\delta\xi_{\mu;\nu}-\delta\xi_{\nu;\mu}=-g_{\mu\nu,\sigma}\delta\xi^\sigma-g_{\sigma\nu}\delta\xi^\sigma{}_{,\mu}-g_{\mu\sigma}\delta\xi^\sigma{}_{,\nu}$ Semicolons denote covariant differentiation. A field quantity φ which has the group transformation law $\delta\varphi=-\varphi_{,\mu}\delta\xi^\mu+G^\nu{}_\mu\varphi\delta\xi^\mu{}_{,\nu}=-\varphi_{;\mu}\delta\xi^\mu+G^\nu{}_\mu\varphi\delta\xi^\mu{}_{;\nu}$ where the $G^\nu{}_\mu$ are the generators of a matrix representation of the linear group, is defined to have the covariant derivative $\varphi_{;\mu}\equiv\varphi_{,\mu}+G^\nu{}_\sigma\Gamma_{\mu\nu}{}^\sigma\varphi$. Covariant differentiation adds one covariant index. It has the commutation law $\varphi_{;\mu\nu}-\varphi_{;\nu\mu}=-G^\sigma{}_\tau R_{\mu\nu\sigma}{}^\tau\varphi$.
$S_{,i}R^i{}_\alpha\equiv0$	$F^{\alpha\mu\nu}{}_{;\mu\nu}\equiv0$ This identity is a consequence of the antisymmetry of $F^\alpha{}_{\mu\nu}$ and of the structure constants $c^\gamma{}_{\alpha\beta}$. $F^\alpha{}_{\mu\nu}$ transforms according to the adjoint representation of the group and also satisfies the cyclic identity $F^\alpha{}_{\mu\nu;\sigma}+F^\alpha{}_{\nu\sigma;\mu}+F^\alpha{}_{\sigma\mu;\nu}\equiv0$.	$-2[g^{1/2}(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R)]_{;\nu}\equiv0$. This identity results from contracting the *Bianchi identity* $R_{\mu\nu\sigma}{}^\tau{}_{;\rho}+R_{\nu\rho\sigma}{}^\tau{}_{;\mu}+R_{\rho\mu\sigma}{}^\tau{}_{;\nu}\equiv0$ which can be verified by straightforward computation using the fact that the *Riemann tensor* $R_{\mu\nu\sigma}{}^\tau$ transforms as a mixed tensor of the fourth rank.
$R^i{}_{\alpha,j}$	$\delta R^\alpha{}_{\mu\beta'}/\delta A^{\gamma''}{}_{\nu''}\equiv c^\alpha{}_{\beta\gamma}\delta_\mu{}^\nu\delta(x,x')\delta(x,x'')\equiv c^\alpha{}_{\delta\epsilon}\delta^\delta{}_{\beta'}\delta^\epsilon{}_{\mu\gamma''}{}^{\nu''}$	$\delta R_{\mu\nu\sigma'}/\delta\varphi_{\rho''\tau''}\equiv-\delta_{\mu\nu}{}^{\rho''\tau''}{}_{;\lambda}\delta^\lambda{}_{\sigma'}-\delta_{\lambda\nu}{}^{\rho''\tau''}\delta^\lambda{}_{\sigma';\mu}-\delta_{\mu\lambda}{}^{\rho''\tau''}\delta^\lambda{}_{\sigma';\nu}$
$c^\alpha{}_{\beta\gamma}$	$c^\alpha{}_{\beta'\gamma''}\equiv c^\alpha{}_{\beta\gamma}\delta(x,x')\delta(x,x'')$	$c^\mu{}_{\nu'\sigma''}\equiv\delta^\mu{}_\nu\delta_{,\sigma}(x,x')\delta(x,x'')-\delta^\mu{}_\sigma\delta_{,\nu}(x,x'')\delta(x,x')$ $=\delta^\mu{}_{\nu';\tau}\delta^\tau{}_{\sigma''}-\delta^\mu{}_{\sigma'';\tau}\delta^\tau{}_{\nu'}$
γ_{ij}	$\gamma_\alpha{}^\mu{}_{\beta'}{}^{\nu'}\equiv-\delta_\alpha{}^\mu{}_{\beta'}{}^{\nu'}$	$\gamma^{\mu\nu\sigma'\tau'}\equiv-\frac{1}{2}g^{1/2}(\delta^{\mu\nu\sigma'\tau'}-\frac{1}{2}g^{\mu\nu}\delta_\rho{}^{\rho\sigma'\tau'})$
$\hat{F}_{\alpha\beta}$	$\hat{F}_{\alpha\beta'}\equiv\delta_{\alpha\beta';\sigma}{}^\sigma$	$\hat{F}_{\mu\nu'}\equiv g^{1/2}(\delta_{\mu\nu';\sigma}{}^\sigma+R_\mu{}^\sigma\delta_{\sigma\nu'})$
$\tilde{\gamma}_{\alpha\beta}$	$\tilde{\gamma}_{\alpha\beta'}\equiv-\delta_{\alpha\beta'}$	$\tilde{\gamma}_{\mu\nu'}\equiv-g^{1/2}\delta_{\mu\nu'}$
$\tilde{\gamma}^{-1\alpha\beta}$	$\tilde{\gamma}^{-1\alpha\beta'}\equiv-\delta^{\alpha\beta'}$	$\tilde{\gamma}^{-1\mu\nu'}\equiv-\delta^{\mu\nu'}g'^{-1/2}$
F_{ij}	$F_\alpha{}^\mu{}_{\beta'}{}^{\nu'}\equiv\delta_\alpha{}^\mu{}_{\beta'}{}^{\nu'}{}_{;\sigma}{}^\sigma+2c^\epsilon{}_{\alpha\gamma}F^{\gamma\mu}{}_\sigma\delta_\epsilon{}^\sigma{}_{\beta'}{}^{\nu'}$	$F^{\mu\nu\sigma'\tau'}\equiv\frac{1}{4}g^{1/2}(g^{\mu\rho}g^{\nu\lambda}+g^{\mu\lambda}g^{\nu\rho}-g^{\mu\nu}g^{\rho\lambda})$ $\times(\delta_{\rho\lambda}{}^{\sigma'\tau'}{}_{;\kappa}{}^\kappa-2R_{\rho\iota\lambda\kappa}\delta^{\iota\kappa\sigma'\tau'}-R_{\rho\lambda}\delta_\kappa{}^{\kappa\sigma'\tau'}$ $+2R_\rho{}^\kappa\delta_{\kappa\lambda}{}^{\sigma'\tau'}-g_{\rho\lambda}R_{\iota\kappa}\delta^{\iota\kappa\sigma'\tau'}-R\delta_{\rho\lambda}{}^{\sigma'\tau'}$ $+\frac{1}{2}g_{\rho\lambda}R\delta_\kappa{}^{\kappa\sigma'\tau'})$ The last five terms inside the parentheses may be omitted when the field equations are satisfied.

chosen as the basic field variables.[8] With the conventional choice of Table I the number of nonvanishing gravitational vertex functions is infinite.

For a local theory a typical term in S_n involves the product of $n-1$ δ functions or derivatives of δ functions. In momentum space with a constant (e.g., flat) background field these reduce to a single δ function, which expresses the conservation of momentum of the n field quanta taking part in the elementary process described by the vertex in question. The calculation of specific processes is usually most conveniently performed in momentum space; the development of the general theory, however—in particular, the demonstration of the covariance of renormalization procedures—is best done in coordinate space.

Because of the commutativity of functional differentiation the bare vertex functions $S_{,ijk\cdots}$ are completely symmetric in their indices, and $S_{,ij}$ corresponds to a self-adjoint linear operator. When employing the notation (2.5) we may regard the symbol S_2 as actually representing this operator. Note: The abstract notation must be used with a measure of caution because the associative law of matrix multiplication does not always hold. If Φ^i and Ψ^i are two functions which do not vanish rapidly at infinity, the value of the expression $\Phi^i S_{,ij} \Psi^j$ may depend on which implicit integration is performed first. This ambiguity may be removed by using arrows to distinguish the two possibilities: $\Phi^{\sim}\overrightarrow{S}_2\Psi$ and $\Phi^{\sim}\overleftarrow{S}_2\Psi$.[9]

The present discussion will be limited to boson fields. For the extension of the formalism to the case of fermion fields, which involves anticommutative differentiation and antisymmetric vertex functions, the reader may consult the reference given in Ref. 6. This reference contains detailed proofs of some of the important theorems to be stated in what follows. We shall therefore restrict ourselves here to sketch-proofs or simple statements of these theorems but will take the occasion to improve their presentation.

3. THE QUANTIZED LIGHT CONE AND THE DEFINITION OF TIME. GREEN'S THEOREM

In a standard hyperbolic wave theory the operator S_2 defines a class of characteristic hypersurfaces which separate spacelike from timelike directions. To this operator therefore falls the task of providing the definition of time. Since S_2 generally depends on the background field it is evident that the background field may play a role in this definition. For most field theories, the characteristic surfaces are in fact unaffected by the fields themselves; only in the case of general relativity does the background field exert an influence.

In the quantum theory S_2 is not only a differential operator but also a quantum-mechanical operator. In gravity theory the position of the light cone thus becomes a q number. Critics of the program to quantize gravity frequency ask "What can this mean?" A good answer to this question does not yet exist. However, there are some indications where the answer may lie. We have seen in I that the canonical formalism can be developed to a considerable degree without the question arising. This is particularly true when the discussion is carried out in the "metric representation," in which the metric appears as a c number. It is also true in Leutwyler's analysis[10] of transition amplitudes as Feynman sums over *classical* histories. Where do these analyses break down, or rather, where must they be supplemented by more sophisticated reasoning? They must be modified at precisely the point at which it becomes necessary to account for radiative corrections and field renormalization.

In the covariant theory we shall not make use of a q number S_2. Our approach will be that of perturbation theory, with all its limitations. We thereby gain, however, the possibility of working exclusively with c numbers. The background field will play two roles simultaneously. Firstly it will serve as a classical reference point about which the quantum fluctuations may be assumed to take place. Secondly it will serve as a useful technical instrument. By varying the background field we can reproduce the effect which individual field quanta have on a variety of fundamental processes, *including the laws of propagation* (i.e., on the light cone). By allowing these effects to superpose nonlinearly we achieve the full S-matrix expansion, including all radiative corrections.[11] The only limitation is that we never consider more than a finite number of quanta at once. The total perturbation series is never summed, and thus we never determine the answer to Pauli's speculation[12] that quantization of gravity may yield an intrinsic cutoff by "smearing out" the light cone, which would at the same time be the definitive answer to the question of the meaning of a q number S_2.

A clue to the eventual answer may perhaps be found in the fact that quantum gravidynamics is not, by standard criteria, a renormalizable theory. It is not difficult to see that the strongest divergences (which, from a perturbation point of view, are responsible for the nonrenormalizability) are precisely those which arise from the fluctuations of the light cone. It may be hoped that these divergences will some day prove to be summable to a finite correction embodying Pauli's

[8] An alternative choice of basic variables is $g^{-5/22}g_{\mu\nu}-\eta_{\mu\nu}$, which yields $S_n=0$ for $n>11$. Neither choice, however, can in general avoid an infinity of nonvanishing bare vertex functions if other fields, necessarily coupled to the gravitational field, are present. Peres has proposed to treat the latter case by a method which makes use of an additional constraint. [See A. Peres, Nuovo Cimento **28**, 865 (1963).] It should be remarked, however, that the presence of an infinity of bare vertices does not pose an essential difficulty for the theory, as we shall see.

[9] The symbol $\sim$ denotes transposition.

[10] H. Leutwyler, Phys. Rev. **134**, B1155 (1964).

[11] Hence the present formalism takes into account all nonlinear effects, classical as well as quantum, and is not merely the theory of a linearized field on an arbitrary background.

[12] W. Pauli, Helv. Phys. Acta Suppl. **4**, 69 (1956).

cutoff. Present evidence on this matter, as well as the significance of the Planck length, will be discussed briefly in the final paper of this series.

We assume, then, that S_2 defines two classes of timelike directions, one of which will be arbitrarily called *the past* and the other *the future*. We assume furthermore that the classical background possesses no geometrical singularities and that both time and space are infinite in extent. The work of Brill[13] gives us confidence that nontrivial backgrounds (i.e., other than flat space-time) having these properties exist. We note that such backgrounds are *absolutely* classical, not only from a mathematical but also from a physical point of view. A space of infinite volume has the capacity for an infinite amount of action[14] and hence can serve as the classical bedrock for setting boundary conditions. At the same time we note that the real world, even if it is finite, is so huge that it is *effectively* classical. For example, by renouncing only slightly the infinite precision usually ascribed to the energy and momentum labels of S-matrix elements, and by using the terms *remote past* and *remote future* in a relative sense, we can have an effective S-matrix theory which is extremely precise, based on a background which becomes asymptotically flat at the boundaries of a finite but large region.

The remote past and remote future will be denoted by $-\infty$ and ∞, respectively. If the space-time point which is associated with an index i lies to the future of a spacelike hypersurface Σ we shall write $i > \Sigma$. If it lies to the past of Σ we write $\Sigma > i$. If, relative to two points associated with indices i and j, respectively, a space-like hypersurface can be found such that $i > \Sigma$ and $\Sigma > j$ then we write $i > j$. It is possible to have both $i > j$ and $j > i$ simultaneously. In this case the associated points are separated by a spacelike interval. Evidently $\infty > i > -\infty$ and $\infty > \Sigma > -\infty$ for all i and Σ.

Consider now the following expression:

$$\int (\Phi^i S_{,ij'}\Psi^{j'} - \Phi^{j'} S_{,j'i}\Psi^i)dx',$$

in which, owing to the δ-function character of $S_{,ij'}$, only those points x' in the immediate neighborhood of x make any contribution to the integral. By symmetry the integral of this expression over all x vanishes, provided the functions Φ^i and Ψ^i vanish sufficiently rapidly at infinity so that the integral exists. Since S_2 is a differential operator this implies that the expression itself must be expressible as a divergence, of the form

$$\int (\Phi^i S_{,ij'}\Psi^{j'} - \Phi^{j'} S_{,j'i}\Psi^i)dx' \equiv \int dx' \int dx'' (\Phi^{i'} s^{\mu}{}_{i'j''}\Psi^{j''})_{,\mu}, \quad (3.1)$$

where $s^{\mu}{}_{i'j''}$ is a certain homogeneous quadratic combination of delta functions and their derivatives. It is not hard to show that the self-adjointness of S_2 implies

$$s^{\mu}{}_{i'j''} \equiv -s^{\mu}{}_{j''i'}. \quad (3.2)$$

More generally we have

$$\int (\Phi^i F_{ij'}\Psi^{j'} - \Phi^{j'} F_{j'i}\Psi^i)dx' \equiv \int dx' \int dx'' (\Phi^{i'} f^{\mu}{}_{i'j''}\Psi^{j''})_{,\mu} \quad (3.3)$$

for any differential operator $F_{ij'}$ regardless of whether it is self-adjoint or not, and for any pair of functions Φ^i, Ψ^i regardless of their behavior at infinity. The differential operator $f^{\mu}{}_{i'j''}$, however, has the symmetry (3.2) only in the self-adjoint case.

If Σ is the boundary of a finite domain Ω then (3.3) implies

$$\int_{\Omega} dx \int dx' (\Phi^i F_{ij'}\Psi^{j'} - \Phi^{j'} F_{j'i}\Psi^i) = \int_{\Sigma} d\Sigma_{\mu} \int dx' \int dx'' \Phi^{i'} f^{\mu}{}_{i'j''}\Psi^{j''} d\Sigma_{\mu}, \quad (3.4)$$

where $d\Sigma_{\mu}$ is the directed surface element of Σ. If Φ^i and Ψ^i vanish sufficiently rapidly at spatial infinity we obtain, on letting Ω expand without limit,

$$\Phi^{\sim}\overrightarrow{F}\Psi - \Phi^{\sim}\overleftarrow{F}\Psi = \left(\int_{\infty} - \int_{-\infty}\right) \Phi^{\sim}\overleftrightarrow{f}^{\mu}\Psi d\Sigma_{\mu}. \quad (3.5)$$

Here the condensed notation has been employed, and the double arrow $\leftrightarrow$ has been placed above f^{μ} to emphasize that as a differential operator it has components which act to the right as well as components which act to the left. In a similar manner we write

$$\Phi^{\sim}\overrightarrow{S}_2\Psi - \Phi^{\sim}\overleftarrow{S}_2\Psi = \left(\int_{\infty} - \int_{-\infty}\right) \Phi^{\sim}\overleftrightarrow{s}^{\mu}\Psi d\Sigma_{\mu}. \quad (3.6)$$

4. THE INVARIANCE GROUP, PHYSICAL OBSERVABLES, AND MANIFEST COVARIANCE

The invariance groups of both the Yang-Mills and gravitational fields are infinite dimensional and non-Abelian. In the abstract notation the change produced in the field by an infinitesimal group transformation will be expressed in the form

$$\delta\varphi^i = R^i{}_{\alpha}\delta\xi^{\alpha} \quad \text{or, more simply,} \quad \delta\varphi = R\delta\xi. \quad (4.1)$$

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[13] D. Brill, Ann. Phys. (N. Y.) 7, 466 (1959); State University of Iowa Report No. SUI 61-4, 1961 (unpublished).

[14] It is for this reason that it is impossible to quantize a Friedmann universe of negative curvature for which no periodic identification of points is assumed and which is therefore "open" (i.e., infinite).

Here the $\delta\xi^\alpha$ are the infinitesimal *group parameters* and the $R^i{}_\alpha$ are certain linear combinations of the δ function and its derivatives, with coefficients depending on the φ's (see Table I). As functions of the x's the $\delta\xi^\alpha$ are assumed to be differentiable and to vanish outside a finite domain of space-time but to be otherwise arbitrary. The group property is expressed in the form of a functional differential condition on the $R^i{}_\alpha$:

$$R^i{}_{\alpha,j}R^j{}_\beta - R^i{}_{\beta,j}R^j{}_\alpha \equiv R^i{}_\gamma c^\gamma{}_{\alpha\beta}\,, \tag{4.2}$$

where the $c^\gamma{}_{\alpha\beta}$ are the *structure constants* of the group, which in turn satisfy

$$c^\delta{}_{\alpha\epsilon}c^\delta{}_{\beta\gamma} + c^\delta{}_{\beta\epsilon}c^\epsilon{}_{\gamma\alpha} + c^\delta{}_{\gamma\epsilon}c^\epsilon{}_{\alpha\beta} \equiv 0\,. \tag{4.3}$$

A functional A of the φ's is regarded as a *physical observable* if it is a group invariant. The condition for this is

$$A_{,i}R^i{}_\alpha \equiv 0\,. \tag{4.4}$$

The action functional in particular is a group invariant:

$$S_{,i}R^i{}_\alpha \equiv 0\,. \tag{4.5}$$

By functionally differentiating the latter identity we learn that under a group transformation the field equations are replaced by linear combinations of themselves:

$$\delta S_{,i} \equiv S_{,ij}\delta\varphi^j \equiv S_{,ij}R^j{}_\alpha\delta\xi^\alpha \equiv -S_{,j}R^j{}_{\alpha,i}\delta\xi^\alpha\,, \tag{4.6}$$

and hence that solutions go into solutions. We also learn that S_2 is a singular operator, at least when the field equations are satisfied, for it then possesses the $R^i{}_\alpha$ as zero-eigenvalue eigenvectors of compact support;

$$S_2 R = 0\,. \tag{4.7}$$

With the conventional choice of field variables given in Table I the dependence of the $R^i{}_\alpha$ on the φ's is linear inhomogeneous, so that $R^i{}_{\alpha,jk}$ vanishes. This has important consequences for the manifest covariance of the formalism. For example, by repeatedly differentiating (4.5) we find for the transformation law of the nth vertex function

$$\delta S_{,i_1\cdots i_n} = -(S_{,ji_2\cdots i_n}R^j{}_{\alpha,i_1} + \cdots + S_{,i_1\cdots i_{n-1}j}R^j{}_{\alpha,i_n})\delta\xi^\alpha\,. \tag{4.8}$$

Similarly, from (4.2) we find

$$\delta R^i{}_\alpha = (R^i{}_{\beta,j}R^j{}_\alpha - R^i{}_\gamma c^\gamma{}_{\beta\alpha})\delta\xi^\beta\,. \tag{4.9}$$

These simple linear laws permit the transformation character of many quantities appearing in the formalism to be inferred at once from the positions of the field and group indices attached to them. In general, when introducing new quantities, we shall be careful to insure that they obey the following transformation laws, of which (4.8) and (4.9) are special cases: A quantity bearing several indices will transform according to the direct product of a corresponding number of (continuous) matrix representations of the group. A field index in the upper position will correspond to the representation generated by the matrices $R^i{}_{\alpha,j}$, while a field index in the lower position will correspond to the contragredient representation. Similarly a group index in the upper position will correspond to the adjoint representation of the group, while one in the lower position will correspond to the contragredient representation. The adjoint representation of the Yang-Mills group is the infinite direct sum of the adjoint representation of the generating Lie group taken repeatedly over the points of space-time; the adjoint representation of the coordinate transformation group of general relativity is that of a contravariant vector field.

Both the $R^i{}_{\alpha,j}$ and the structure constants $c^\gamma{}_{\alpha\beta}$ are homogeneous quadratic combinations of the δ function and its derivatives, independent of the φ's. In the theory of radiative corrections we encounter the expressions $R^i{}_{\alpha,i}$ and $c^\beta{}_{\alpha\beta}$ which are mathematically meaningless, involving the δ function and its derivatives at $x'=x$. We shall find it necessary to assign vanishing values to these expressions in order to maintain internal consistency of the theory;

$$R^i{}_{\alpha,i} = 0\,, \quad c^\beta{}_{\alpha\beta} = 0\,. \tag{4.10}$$

The reasonableness of these assignments may be made apparent by noting the transformation laws of the quantities in question. Both transform contragrediently to the adjoint representation of the group. In the case of general relativity they are therefore covariant vector densities of unit weight[15] and may be presumed to vanish by virtue of the fact that space-time has no metric-independent preferred directions. In the case of the Yang-Mills group they may be presumed to vanish by virtue of the fact that the corresponding quantities vanish for the generating Lie group which, for physical reasons is necessarily compact.

5. BOUNDARY CONDITIONS, SUPPLEMENTARY CONDITIONS, AND GREEN'S FUNCTIONS

Suppose the infinitesimal change (2.3) in the action functional corresponds to alterations in the structure of the physical system which are limited to a finite region of space-time. The functional A will then be constructed out of field variables evaluated at points within this region, and its functional derivative $A_{,i}$ will vanish at points outside. Under these circumstances we may distinguish two particular solutions of Eq. (2.4) which are of special importance, the *retarded* and *advanced*, denoted by $\delta_A{}^-\varphi^i$ and $\delta_A{}^+\varphi^i$, respectively, characterized by the boundary conditions

$$\lim_{i\to\pm\infty} \delta_A{}^\pm\varphi^i = 0\,. \tag{5.1}$$

Because of the singularity of the operator S_2 the

[15] The δ functions in Table I are to be regarded as densities of zero weight at their first argument and unit weight at their second.

conditions (5.1) do not determine the $\delta_A{}^{\pm}\varphi^i$ completely, but only *modulo* transformations of the form

$$\delta_A{}^{\pm}\varphi'^i=\delta_A{}^{\pm}\varphi^i+R^i{}_\alpha\delta\xi^\alpha. \quad (5.2)$$

Since such transformations can be additively superimposed, they constitute an *Abelian* "gauge" group for infinitesimal disturbances. Unlike the situation which holds for the familiar gauge group of electrodynamics, the *scale* of these transformations varies from point to point owing to the dependence of $R^i{}_\alpha$ on the background field. This fact is responsible for *all* of the formal complications which arise in the quantum theory of non-Abelian gauge fields.

Although Eqs. (5.1) do not suffice to determine the $\delta_A{}^{\pm}\varphi^i$ completely they provide unique *physical* boundary conditions. Because of the invariance condition $B_{,i}R^i{}_\alpha=0$ the disturbance produced in any physical observable B is unaffected by the transformation (5.2):

$$\delta_A{}^{\pm}B'=B_{,i}\delta_A{}^{\pm}\varphi'^i=B_{,i}\delta_A{}^{\pm}\varphi^i=\delta_A{}^{\pm}B. \quad (5.3)$$

Nevertheless, in practice it is a convenience to restrict the $\delta_A{}^{\pm}\varphi^i$ by adding further conditions known as *supplementary conditions.*

As the standard form for supplementary conditions we shall choose

$$R_{i\alpha}\delta_A{}^{\pm}\varphi^i=0, \quad (5.4)$$

$$R_{i\alpha}\equiv\gamma_{ij}R^j{}_\alpha, \quad (5.5)$$

where γ_{ij} is a matrix which may be used to lower field indices[16] and which is *arbitrary* except for a single essential requirement, namely that it be such that the operator corresponding to the matrix

$$\hat{F}_{\alpha\beta}\equiv R_{i\alpha}R^i{}_\beta \quad (5.6)$$

shall be nonsingular and have unique advanced and retarded Green's functions $\hat{G}^{\pm\alpha\beta}$ satisfying

$$\hat{F}_{\alpha\gamma}\hat{G}^{\pm\gamma\beta}=-\delta_\alpha{}^\beta, \quad (5.7)$$

$$\lim_{\alpha\to\pm\infty}\hat{G}^{\pm\alpha\beta}=\lim_{\beta\to\mp\infty}\hat{G}^{\pm\alpha\beta}=0. \quad (5.8)$$

If the supplementary conditions (5.4) are not initially satisfied they may be made to hold by carrying out a transformation of the form (5.2), with

$$\delta\xi^\alpha=\hat{G}^{\pm\alpha\beta}R_{i\beta}\delta_A{}^{\pm}\varphi^i, \quad (5.9)$$

and the $\delta_A{}^{\pm}\varphi^i$ thus restricted will generally be unique.

Although the arbitrariness of γ_{ij} in the general formalism must be stressed, it is nevertheless a great convenience in practice to impose the following three additional conditions: (1) that γ_{ij} shall be symmetric in its indices; (2) that it shall have the group transformation law suggested by the position of its indices; and (3) that it shall be such as to make $\hat{F}_{\alpha\beta}$ correspond to a *local* (i.e., differential) operator. Condition (1) insures that $\hat{F}$ will be self-adjoint. Condition (2) maintains the manifest covariance of the formalism by insuring that $\hat{F}_{\alpha\beta}$ will transform according to the law suggested by the position of *its* indices. Condition (3) enables (5.8) to be replaced by the stricter relations

$$\hat{G}^{+\alpha\beta}=0 \quad \text{for} \quad \alpha>\beta, \quad \hat{G}^{-\alpha\beta}=0 \quad \text{for} \quad \beta>\alpha. \quad (5.10)$$

In addition to the matrix γ_{ij} we shall also introduce a matrix $\tilde{\gamma}_{\alpha\beta}$ for the purpose of lowering group indices. Like γ_{ij} it may be chosen in a completely arbitrary way except for a single essential requirement. The requirement in this case is that $\tilde{\gamma}_{\alpha\beta}$ shall be nonsingular and possess a unique inverse $\tilde{\gamma}^{-1\alpha\beta}$, which may be used to raise group indices.[17] It is then not difficult to show that the matrix

$$F_{ij}\equiv S_{,ij}+R_{i\alpha}\tilde{\gamma}^{-1\alpha\beta}R_{j\beta} \quad (5.11)$$

is nonsingular, provided (as is true in the cases of interest) the $R^i{}_\alpha$ constitute a complete set of zero-eigenvalue eigenvectors of $S_{,ij}$ having compact support.

Although the arbitrariness of $\tilde{\gamma}_{\alpha\beta}$, like that of γ_{ij}, must be stressed in the general formalism, it is again a practical convenience (and for the same reasons) to impose three additional conditions, similar to those imposed on γ_{ij}: (1) that $\tilde{\gamma}_{\alpha\beta}$ shall be symmetric in its indices; (2) that it shall have the group-transformation law suggested by the position of its indices; (3) that it shall be such as to make F_{ij} correspond to a local (differential) operator.

In the case of the Yang-Mills and gravitational fields it turns out that if all of the above conditions are satisfied then only one additional requirement, namely that the Green's functions of $\hat{F}$ and F shall have the weakest possible singularities on the light cone, leads to choices for γ_{ij} and $\tilde{\gamma}_{\alpha\beta}$ which are unique up to a constant factor. These are the choices shown in Table I. They are the generalizations, to the case of arbitrary background fields, of the well-known Lorentz and DeDonder conditions of the corresponding linearized theories. Any other choices lead to more singular Green's functions.

We note that the supplementary conditions are here imposed on the infinitesimal disturbances rather than on the fields themselves. The differences between this approach and that of more familiar formulations of gauge theory will become apparent as the discussion progresses.

When the supplementary conditions (5.4) are satisfied, Eq. (2.4) may be replaced by

$$F_{ij}\delta\varphi^j=-\epsilon A_{,i}, \quad (5.12)$$

which has the unique advanced and retarded solutions

$$\delta_A{}^{\pm}\varphi^i=\epsilon G^{\pm ij}A_{,j}, \quad (5.13)$$

[16] If γ_{ij} has a unique inverse this inverse may be used to raise field indices, but this is not essential.

[17] The Cartan metric $\gamma_{\alpha\beta}\equiv-c^\gamma{}_{\alpha\delta}c^\delta{}_{\beta\gamma}$ cannot be defined for an infinite dimensional group, and hence cannot be employed here.

the $G^{\pm ij}$ being the Green's functions of F, satisfying

$$F_{ik}G^{\pm kj}=-\delta_i{}^j \tag{5.14}$$

and [in virtue of condition (3) above]

$$G^{+ij}=0 \quad \text{for} \quad i>j, \quad G^{-ij}=0 \quad \text{for} \quad j>i. \tag{5.15}$$

6. CAUCHY DATA AND RECIPROCITY RELATIONS. THE POISSON BRACKET

Instead of studying disturbances which are produced by physical alterations in the system it is frequently of interest to consider disturbances which originate at infinity and which satisfy the homogeneous equation

$$S_2\delta\varphi=0. \tag{6.1}$$

(We here employ the supercondensed notation.) If the supplementary condition

$$R^{\sim}\gamma\delta\varphi=0 \tag{6.2}$$

is imposed [cf. (5.4)] then these disturbances also satisfy

$$F\delta\varphi=0, \quad F\equiv S_2+\gamma R\tilde{\gamma}^{-1}R^{\sim}\gamma, \tag{6.3}$$

and the value of $\delta\varphi$ is determined throughout space-time if it and its derivatives are known over any space-like hypersurface Σ. With the aid of Eq. (3.5) it is not difficult to derive the following integral realization of these facts:

$$\delta\varphi=\int_\Sigma \tilde{G}\overleftrightarrow{f}^\mu\delta\varphi d\Sigma_\mu, \tag{6.4}$$

where

$$\tilde{G}\equiv G^+-G^- \tag{6.5}$$

One has only to make use of the kinematics of the $G^\pm$ and to assume that they are *left* inverses of $-F$[18]:

$$G^\pm\overleftarrow{F}=-1, \quad \tilde{G}\overleftarrow{F}=0. \tag{6.6}$$

From (6.3) and the arbitrariness of the *Cauchy data* $\overleftrightarrow{f}^\mu\delta\varphi$ it then follows that they are right inverses as well:

$$\overrightarrow{F}G^\pm=-1, \quad F\tilde{G}=0. \tag{6.7}$$

Equations (6.4) to (6.7) hold regardless of the symmetry of F. If its self-adjointness is taken into account, the following additional laws are obtained:

$$G^{\pm\sim}=G^\mp, \tag{6.8}$$

$$\tilde{G}^\sim=-\tilde{G}. \tag{6.9}$$

Combining these laws with Eqs. (5.3) and (5.13), one obtains the important reciprocity relations

$$\delta_A{}^\pm B=\epsilon B_1{}^\sim G^\pm A_1=\epsilon A_1{}^\sim G^\mp B_1=\delta_B{}^\mp A, \tag{6.10}$$

which may be loosely expressed in the words, *the retarded effect of A on B equals the advanced effect of B on A, and vice versa.* Although the use of (6.8), which holds when γ and $\tilde{\gamma}$ are symmetric, is the easiest way to prove these relations, it is to be emphasized that since they involve physical observables (invariants) only, these relations are independent of such conditions. In particular it can be shown explicitly that $\delta_A{}^\pm B$ and $\delta_B{}^\pm A$ remain invariant under arbitrary changes in the γ's, including changes which destroy the symmetry and group-transformation properties of the γ's.

Another important relation which can be obtained is the following:

$$R\hat{G}^\pm\tilde{\gamma}=G^\pm\gamma R, \tag{6.11}$$

which is proved by making use of (4.7), (5.6), (5.11), and the kinematic structure of the Green's functions. Since (4.7) generally holds only when the background field satisfies the field equations, it is important to remember that Eq. (6.11) holds only in this case. The transpose of Eq. (6.11) may be used in a straightforward way in combination with (4.4) to show that the solutions (5.13) of the equation for infinitesimal disturbances are consistent with the supplementary conditions which were used to get them in the first place. Equation (6.11) also finds repeated use in the theory of radiative corrections.

The above results provide the starting point for a covariant theory of the Poisson bracket. In the canonical theory equal-time Poisson brackets are defined for arbitrary functions of the $g_{\mu\nu}$ and their conjugate momenta, and the physical Hilbert space of the quantum theory is determined by constraints imposed on the state vectors. In the manifestly covariant theory Poisson brackets are defined only for observables, and hence it is possible in principle to work within the physical Hilbert space from the very beginning.[19] Moreover, the covariant theory makes no distinction between equal-time Poisson brackets and others.

The definition, which is due to Peierls,[20] is

$$(A,B)\equiv D_AB-D_BA, \tag{6.12}$$

where

$$D_AB\equiv\lim_{\epsilon\to0}\epsilon^{-1}\delta_A{}^-B. \tag{6.13}$$

With the aid of (6.5) and (6.10) this may be converted to

$$(A,B)=A_1{}^\sim\tilde{G}B_1. \tag{6.14}$$

Peierls' definition makes immediately manifest the fundamental role played by the Poisson bracket in the theory of mutual disturbances in measurement processes, and provides the most natural bridge to the quantum commutator and the uncertainty principle. In its quantum form,

$$[\mathbf{A},\mathbf{B}]=i(D_\mathbf{A}\mathbf{B}-D_\mathbf{B}\mathbf{A})=i\mathbf{A}_1{}^\sim\tilde{\mathbf{G}}\mathbf{B}_1, \tag{6.15}$$

[18] Kronecker δ's and δ functions are replaced by the unit symbol 1 in the supercondensed notation.

[19] Since a complete operator formalism does not yet exist for the covariant theory this idea will not be fully developed here.

[20] R. E. Peierls, Proc. Roy. Soc. (London) A214, 143 (1952).

it allows one to derive in a straightforward manner the variational formula

$$\delta\langle \mathbf{A}'|\mathbf{B}'\rangle = i\langle \mathbf{A}'|\bar{\delta}\mathbf{S}|\mathbf{B}'\rangle, \tag{6.16}$$

which, in Schwinger's hands, has been used to derive all of quantum electrodynamics. Here, and in the future, we use boldface to distinguish quantum operators from the corresponding functionals of the classical background field. In Eq. (6.16), $|\mathbf{A}'\rangle$ and $|\mathbf{B}'\rangle$ are eigenvectors of **A** and **B**, respectively; the field variables out of which **A** is constructed are assumed to be taken at points all of which lie to the future of the points at which the variables making up **B** are taken; and $\bar{\delta}\mathbf{S}$, which represents a change in the functional form of the action, is assumed to be constructed from field variables taken at intermediate points.

We shall make no use of Eq. (6.16) in this paper, firstly because in the absence of a complete operator theory we cannot be sure how to order the factors occurring in **A**, **B**, etc., and secondly because it is necessary in a generally covariant theory, to handle the problem of the relative temporal location of the operators **A**, **B**, and $\bar{\delta}\mathbf{S}$ in a completely intrinsic way. Instead of attempting to alter the form of the action functional we shall develop alternative techniques based on variations of the background field.

It is worthy of note that the Poisson bracket is determined solely by the behavior of *infinitesimal* disturbances. Since the commutators of the quantum theory completely determine the physical Hilbert space, this suggests that the quantum theory is obtained merely by appending a theory of infinitesimal disturbances to the classical theory. Such a view is defective in that it ignores (a) the factor ordering problems arising in the definition of the quantum operators (which like their classical counterparts are involved in nonlinear field equations) and (b) the existence, in the quantum theory, of nonclassical phase effects which manifest themselves in virtual processes and radiative corrections. Nevertheless, if the word "infinitesimal" is modified to "finite but small" we shall see that this view accords quite well with the perturbation theoretic approach to quantum field theory. Moreover, because of the uniqueness of the formalism which emerges, it will appear that the exact theory is already completely determined by the behavior of infinitesimal disturbances.

Peierls' Poisson bracket satisfies all of the usual identities. The only one which is not immediately evident is the Poisson-Jacobi identity. For any three observables A, B, C, we have

$$\begin{aligned}(A,(B,C))&+(B,(C,A))+(C,(A,B))\\ &=A_{,i}\tilde{G}^{il}(B_{,j}\tilde{G}^{jk}C_{,k})_{,l}+B_{,j}\tilde{G}^{jl}(C_{,k}\tilde{G}^{ki}A_{,i})_{,l}\\ &\qquad +C_{,k}\tilde{G}^{kl}(A_{,i}\tilde{G}^{ij}B_{,j})_{,l}\\ &=A_{,il}B_{,j}C_{,k}(\tilde{G}^{ij}\tilde{G}^{kl}+\tilde{G}^{jl}\tilde{G}^{ki})+A_{,i}B_{,jl}C_{,k}\\ &\quad\times(\tilde{G}^{jk}\tilde{G}^{il}+\tilde{G}^{kl}\tilde{G}^{ij})+A_{,i}B_{,j}C_{,kl}(\tilde{G}^{ki}\tilde{G}^{jl}+\tilde{G}^{il}\tilde{G}^{jk})\\ &\quad+A_{,i}B_{,j}C_{,k}(\tilde{G}^{il}\tilde{G}^{jk}{}_{,l}+\tilde{G}^{jl}\tilde{G}^{ki}{}_{,l}+\tilde{G}^{kl}\tilde{G}^{ij}{}_{,l}).\end{aligned} \tag{6.17}$$

The first three terms of the expanded form vanish on account of (6.9) and the commutativity of functional differentiation. In order to show that the fourth term likewise vanishes an expression for the functional derivative of $\tilde{G}$ must be obtained.

The desired expression is a special case of a general relation obtained by varying Eq. (5.14). Under an arbitrary infinitesimal variation δF in the operator F the $G^{\pm}$ suffer variations satisfying

$$F\delta G^{\pm} = -\delta F G^{\pm}, \tag{6.18}$$

which, taking into account kinematics, has the solution[21]

$$\delta G^{\pm} = G^{\pm}\delta F G^{\pm}. \tag{6.19}$$

Therefore,

$$\begin{aligned}G^{\pm ij}{}_{,c}&=G^{\pm ia}F_{ab,c}G^{\pm bj}\\ &=G^{\pm ia}(S_{,abc}+R_{a\alpha,c}R_b{}^{\alpha}+R_{a\alpha}R_b{}^{\alpha}{}_{,c})G^{\pm bj}\\ &=G^{\pm ia}S_{,abc}G^{\pm bj}+G^{\pm ia}R_{a\alpha,c}\hat{G}^{\pm\alpha\beta}R^j{}_{\beta}\\ &\qquad +R^i{}_{\beta}\hat{G}^{\pm\beta}{}_{\alpha}R_b{}^{\alpha}{}_{,c}G^{\pm bj},\end{aligned} \tag{6.20}$$

in which (5.11) and (6.11) have been used.

Breaking $\tilde{G}$ up into its advanced and retarded parts and inserting (6.20) into (6.17), we see that in virtue of the group invariance of A, B, and C, only the terms involving the third functional derivative of the action survive. These terms, however, cancel among themselves, as may be seen by writing them out in the form

$$\begin{aligned}A_{,i}B_{,j}C_{,j}[&(G^{+ia}-G^{-ia})(G^{+jb}G^{-kc}-G^{-jb}G^{+kc})\\ &+(G^{+jb}-G^{-jb})(G^{+kc}G^{-ia}-G^{-kc}G^{+ia})\\ &+(G^{+kc}-G^{-kc})(G^{+ia}G^{-jb}-G^{-ia}G^{+jb})]S_{,abc},\end{aligned} \tag{6.21}$$

in which use has been made of (6.8).

We finally remark that Peierls' Poisson bracket, being defined for pairs of invariants, is itself a group invariant. More precisely, it remains unchanged not only when a group transformation is performed on the background field but also when a transformation of the form (5.2) is performed on the infinitesimal disturbances, corresponding to an arbitrary change in the γ's and hence in the supplementary conditions. The demonstration is straightforward and will be left to the reader.

7. CONDITIONAL INVARIANTS AND ASYMPTOTIC FIELDS

The functional A appearing in (2.3) must be a group invariant. Otherwise the equation (2.4) for infinitesimal disturbances will not be consistent with the singularity condition (4.7). The invariance condition (4.4), however, need not hold as an identity but may hold in consequence of the field equations. That is, (4.4) may be replaced by an identity of the form

$$A_{,i}R^i{}_{\alpha} \equiv S_{,i}a^i{}_{\alpha}. \tag{7.1}$$

[21] Kinematics assure the associatiativity of the matrix product in (6.19).

When the a's are nonvanishing A will be called a *conditional invariant.*

Poisson brackets are as unique and well-defined for conditional invariants as they are for exact invariants. Therefore any invariant, whether conditional or exact, is an observable. The chief tool for proving these statements is the lemma

$$S_2\tilde{G} = -\gamma R(\hat{G}^+ - \hat{G}^-)R^\sim, \tag{7.2}$$

which is a corollary of (6.6), (6.7), and (6.11). With its aid it is straightforward to show that the Poisson bracket of two conditional invariants is itself a conditional invariant and that transformations of the form $A \to A + S_{,i}a^i$, $B \to B + S_{,i}b^i$ leave the Poisson bracket unaffected. Evidently observables are defined only *modulo* the field equations.

An important class of conditional invariants are those which can be constructed out of the *asymptotic fields.* The asymptotic fields are defined by

$$\begin{aligned}\varphi^{\pm i} &\equiv \varphi^i - G_0^{\pm ij}(S_{,j} - S_{,jk}{}^0\varphi^k) \\ &\equiv \varphi^i - G_0^{\pm ij}(\tfrac{1}{2}S_{,jkl}{}^0\varphi^k\varphi^l + \cdots),\end{aligned} \tag{7.3}$$

the notation here being based on the formal expansion of the action

$$S \equiv \frac{1}{2!}S_{,ij}{}^0\varphi^i\varphi^j + \frac{1}{3!}S_{,ijk}{}^0\varphi^i\varphi^j\varphi^k + \cdots. \tag{7.4}$$

The index 0, in either the upper or lower position, indicates that the quantity to which it is affixed is to be evaluated at the zero point $\varphi^i = 0$, which, with the conventions of Table I, corresponds to flat empty space-time. In Eq. (7.4) terms linear in the φ's are absent since $\varphi^i = 0$ is a solution of the field equations, and constant terms are irrelevant.

If the amount of "energy" contained in the field is finite, e.g., if the field has the form of one or more essentially finite wave packets[22] (which inevitably spread in both past and future), then the fields φ^+ and φ^- will coincide with φ in the remote future and past, respectively. The quadratic dependence of the leading term in the expansion of $S_1 - S_2{}^0\varphi$ ensures that the difference between φ and $\varphi^\pm$ will behave like the potential due to a distribution of charge which becomes more and more diffuse in the remote past and future. Because of the spreading of the field the effect of nonlinearities diminishes with time, and we anticipate that the asymptotic fields will satisfy the linear equation $S_2{}^0\varphi^\pm = 0$. The formal proof is immediate. Making use of (4.7), (5.11), (6.11), and (7.3), we have

$$\begin{aligned}S_2{}^0\varphi^\pm &\equiv S_2{}^0\varphi - (F_0 - \gamma_0 R_0\tilde{\gamma}_0{}^{-1}R_0{}^\sim\gamma_0)G_0{}^\pm(S_1 - S_2{}^0\varphi) \\ &\equiv (1 + \gamma_0 R_0\hat{G}_0{}^\pm R_0{}^\sim)S_1 = 0.\end{aligned} \tag{7.5}$$

It is to be noted that this equation holds regardless of the choice of the γ's. In fact it can be shown that the only effect of a change in the γ's is to produce a gauge transformation of the asymptotic fields, having the form

$$\delta\varphi^\pm = R_0\delta\zeta^\pm. \tag{7.6}$$

A group transformation (4.1) of the field φ has a similar effect. Thus

$$\begin{aligned}\delta\varphi^\pm &\equiv \delta\varphi - G_0{}^\pm[\delta S_1 - (F_0 - \gamma_0 R_0\tilde{\gamma}_0{}^{-1}R_0{}^\sim\gamma_0)\delta\varphi] \\ &= -R_0\hat{G}_0{}^\pm R_0{}^\sim\gamma_0\delta\varphi,\end{aligned} \tag{7.7}$$

which takes the form (7.6) with[23]

$$\delta\zeta^\pm = -\hat{G}_0{}^\pm R_0{}^\sim\gamma_0 R\delta\xi. \tag{7.8}$$

For this reason the asymptotic fields can be used to construct group invariants by the dozen. One has only to introduce a set of field-independent quantities I_{Ai}, I_{Bi}... satisfying

$$I_A R_0 = 0, \quad I_B R_0 = 0 \cdots, \tag{7.9}$$

and then define

$$A^\pm \equiv I_A\varphi^\pm, \quad B^\pm \equiv I_B\varphi^\pm \cdots. \tag{7.10}$$

Since (7.7) holds only when the field equations are satisfied the latter quantities, as well as all functionals of these quantities, are conditional invariants.

In practice it is very easy to find differential coefficients I_A, $I_B \cdots$ with the desired properties, and sets of quantum invariants $A^\pm$, $B^\pm \cdots$ forming complete commuting sets in the physical Hilbert space are readily constructed. In this way the quantum states may be uniquely defined by the asymptotic behavior of the field.

Poisson brackets for the invariants $A^\pm$, $B^\pm \cdots$ are determined in a straightforward manner with the aid of the easily verified identity

$$\tilde{G}_0 \equiv (1 - G_0{}^\pm U)\tilde{G}(1 - UG_0{}^\mp), \tag{7.11}$$

where

$$U \equiv F - F_0. \tag{7.12}$$

Thus, substituting (7.3) into (7.10) and using (6.8) and (6.14), we find

$$\begin{aligned}(A^\pm, B^\pm) = I_A[1 - G_0{}^\pm(S_2 - S_2{}^0)] \\ \times\tilde{G}[1 - (S_2 - S_2{}^0)G_0{}^\mp]I_B{}^\sim.\end{aligned} \tag{7.13}$$

[22] In the quantum theory one speaks of matrix elements between analogous "wave-packet" states, and then the same arguments apply. In this case, however, a wave function renormalization constant $Z^{1/2}$ should be attached to $\varphi^\pm$ in Eq. (7.3). For simplicity we shall omit such constants both here and in our later discussion of the S matrix. The reader should supply the missing Z's when needed.

[23] The set of transformations (7.6) forms an Abelian group for the asymptotic fields. It is to be emphasized that the relation (7.8) between the parameters $\delta\zeta^\pm$ and $\delta\xi$ raises no issue of attempting to map an Abelian group on a non-Abelian group, for the $\delta\zeta^\pm$, unlike $\delta\xi$, depend on φ through the presence of the factor R.

If $S_2-S_2{}^0$ were the same as $F-F_0$ then (7.11) would be immediately applicable. The difference between the two quantities involves R's and R_0's. With the aid of (6.11) the R_0's can be brought to bear on the I's, yielding terms which vanish on account of (7.9). The R's on the other hand must be moved in the opposite direction. Using (4.7), (6.11), and $-S_2{}^0G_0{}^{\pm}I_B{}^{\sim}=I_B{}^{\sim}$ it is not difficult to see that this leads to a set of terms which mutually cancel. Hence, finally

$$(A^{\pm},B^{\pm})=I_A\tilde{G}_0I_B{}^{\sim}. \qquad (7.14)$$

8. THE LINEARIZED THEORY. ASYMPTOTIC WAVE FUNCTIONS. HELICITY AND LORENTZ INVARIANCE

The asymptotic fields satisfy the field equations and Poisson bracket relations of the so-called *linearized theory* derived from an action functional of the form $\frac{1}{2}S_{,ij}{}^0\varphi^{\pm i}\varphi^{\pm j}$. Since the linearized theory is well understood we may at this point confidently pass to the quantum theory in order to get our bearings on the ultimate goal; a covariant S-matrix theory. Our first task is to construct, from appropriate asymptotic invariants, creation and annihilation operators for incoming and outgoing field quanta.

In the case of the Yang-Mills and gravitational fields the simplest and most important invariants are, respectively, the asymptotic curl and the double curl (Riemann tensor):

$$\mathbf{F}^{\pm\alpha}{}_{\mu\nu}\equiv \mathbf{A}^{\pm\alpha}{}_{\nu,\mu}-\mathbf{A}^{\pm\alpha}{}_{\mu,\nu}, \qquad (8.1)$$

$$\mathbf{R}^{\pm}{}_{\mu\nu\sigma\tau}\equiv\tfrac{1}{2}(\varphi^{\pm}{}_{\mu\sigma,\nu\tau}+\varphi^{\pm}{}_{\nu\tau,\mu\sigma}-\varphi_{\mu\tau,\nu\sigma}-\varphi^{\pm}{}_{\nu\sigma,\mu\tau}). \qquad (8.2)$$

Both of these quantities have the linear structure (7.10) with differential coefficients satisfying (7.9). Using the well-known cyclic differential identities satisfied by these invariants (see Table I), as well as the propagation equations

$$\Box\mathbf{F}^{\pm\alpha}{}_{\mu\nu}=0, \qquad (8.3)$$

$$\Box\mathbf{R}^{\pm}{}_{\mu\nu\sigma\tau}=0, \qquad (8.4)$$

it is straightforward to derive the following Fourier decompositions:

$$\mathbf{F}^{\pm\alpha}{}_{\mu\nu}\equiv i(2\pi)^{-3/2}\sum_r\int[p_\mu(\mathbf{a}^{\pm}{}_{r+}e_{+\nu}+\mathbf{a}^{\pm}{}_{r-}e_{-\nu}) -p_\nu(\mathbf{a}^{\pm}{}_{r+}e_{+\mu}+\mathbf{a}^{\pm}{}_{r-}e_{-\nu})]\chi_r{}^\alpha e^{ip\cdot x}(2E)^{-1/2}d\mathbf{p} +\text{Hermitian conjugate}, \qquad (8.5)$$

$$\mathbf{R}^{\pm}{}_{\mu\nu\sigma\tau}\equiv-\tfrac{1}{2}(2\pi)^{-3/2}\int[p_\nu p_\tau(\mathbf{a}^{\pm}{}_+e_{+\mu}e_{+\sigma}+\mathbf{a}^{\pm}{}_-e_{-\mu}e_{-\sigma}) +p_\mu p_\sigma(\mathbf{a}^{\pm}{}_+e_{+\nu}e_{+\tau}+\mathbf{a}^{\pm}{}_-e_{-\nu}e_{-\tau}) -p_\nu p_\sigma(\mathbf{a}^{\pm}{}_+e_{+\mu}e_{+\tau}+\mathbf{a}^{\pm}{}_-e_{-\mu}e_{-\tau}) -p_\mu p_\tau(\mathbf{a}^{\pm}{}_+e_{+\nu}e_{+\sigma}+\mathbf{a}^{\pm}{}_-e_{-\nu}e_{-\sigma})]e^{ip\cdot x}E^{-1/2}d\mathbf{p} +\text{Hermitian conjugate}. \qquad (8.6)$$

Here the $\mathbf{a}$'s and e's are functions of the 3-vector $\mathbf{p}$, and the 4-vector p_μ satisfies

$$(p^\mu)=(E,\mathbf{p}),\quad p^2=0,\quad E=|\mathbf{p}|. \qquad (8.7)$$

The e's themselves are the usual complex helicity polarization vectors satisfying

$$e_{\pm\mu}{}^*=e_{\mp\mu},\quad e_\pm\cdot e_\pm=0,\quad e_\pm\cdot e_\mp=1,$$
$$p\cdot e_\pm=0,\quad n\cdot e_\pm=0, \qquad (8.8)$$

where n_μ is an arbitrary timelike unit vector;

$$n^2=-1. \qquad (8.9)$$

The 3-vectors Re $\mathbf{e}_+$, Im$\mathbf{e}_+$, and $\mathbf{p}$, in that order, are required to form a right-handed system. In the case of the Yang-Mills field the χ_r are eigenvectors of an appropriate complete set of commuting matrices within the adjoint representation of the generating Lie group, and the index r labels the corresponding internal states.[24] The χ's may be chosen real and independent of $\mathbf{p}$, satisfying

$$\chi_r{}^\alpha\chi_{s\alpha}=\delta_{rs},\quad \chi_r{}^\alpha\chi_r{}^\beta=\gamma^{\alpha\beta}, \qquad (8.10)$$

where $\gamma^{\alpha\beta}$ is the Cartan metric of the generating group.

If the quantum version of Eq. (7.14) is now used to compute commutators of the asymptotic invariants $\mathbf{F}^{\pm\alpha}{}_{\mu\nu}$ and $\mathbf{R}^{\pm}{}_{\mu\nu\sigma\tau}$ at different space-time points, and if the function $\tilde{G}_0$, which is determined by the zero-field forms of the operators F defined in Table I, is subjected to a Fourier decomposition, then it is straightforward to show that the $\mathbf{a}$'s and their Hermitian conjugates[25] satisfy the following unique commutation laws:

$$[\mathbf{a}^{\pm}{}_A,\mathbf{a}^{\pm}{}_B]=0,\quad [\mathbf{a}^{\pm}{}_A,\mathbf{a}^{\pm}{}_B{}^*]=\delta_{AB}, \qquad (8.11)$$

which identify them as annihilation and creation operators, respectively. Here the capital Latin indices are used as schematic labels for the states of the corresponding quanta. The symbol δ_{AB} is to be understood as the product of a δ function of the 3-momenta and a Kronecker delta in the helicity and internal states.

If the quanta of the Yang-Mills or gravitational field are able, through field nonlinearities and exchange of additional quanta, to bind each other into stable composite structures, then additional creation and annihilation operators for these structures will have to be introduced. Although nothing is presently known about such possibilities, we do know that the complete set of all such operators will determine the physical Hilbert space. No other operators are needed for constructing observables. In fact, if group arbitrariness is made explicit the creation and annihilation operators suffice for the field variables φ^i themselves. Comparing (8.1)

[24] The internal states are n in number, where n is the dimensionality of the generating group.

[25] Hermitian conjugation is here denoted by *. The symbol † will be reserved as an abbreviation for $\sim$* where $\sim$ denotes an additional matrix transposition in a vector space *other* than the quantum-mechanical Hilbert space.

TABLE II. Expressions for the linearized Yang-Mills and gravitational fields corresponding to quantities appearing in the abstract formalism.

Abstract symbol	Corresponding expression for the Yang-Mills field	Corresponding expression for the gravitational field
u	$u^{\alpha}{}_{\mu r\pm}(x,\mathbf{p})\equiv(2\pi)^{-3/2}\chi_r{}^{\alpha}e_{\pm\mu}\dfrac{e^{ip\cdot x}}{\surd(2E)}$	$u_{\mu\nu\pm}(x,\mathbf{p})\equiv(2\pi)^{-3/2}e_{\pm\mu}e_{\pm\nu}\dfrac{e^{ip\cdot x}}{\surd E}$
R_0	$-i\delta^{\alpha}{}_{\beta}p_{\mu}$	$-i(\eta_{\mu\sigma}p_{\nu}+\eta_{\nu\sigma}p_{\mu})$
$\bar{R}_0$	$-i\delta^{\alpha}{}_{\beta}\bar{p}_{\mu}$	$-i(\eta_{\mu\sigma}\bar{p}_{\nu}+\eta_{\nu\sigma}\bar{p}_{\mu}-\eta_{\mu\nu}\bar{p}_{\sigma})$
v	$v^{\alpha}{}_{r}(x,\mathbf{p})\equiv(2\pi)^{-3/2}\chi_r{}^{\alpha}\dfrac{e^{ip\cdot x}}{\surd(2E)}$	$(v^{\mu}{}_{a}(x,\mathbf{p}))\equiv(2\pi)^{-3/2}(e_+{}^{\mu}\quad e_-{}^{\mu}\quad p^{\mu}\quad \bar{p}^{\mu})\dfrac{e^{ip\cdot x}}{\surd(2E)}$
N	$\delta_{rs}p\cdot\bar{p}\delta(\mathbf{p},\mathbf{p}')$	$\begin{bmatrix} p\cdot\bar{p} & 0 & 0 & 0\\ 0 & p\cdot\bar{p} & 0 & 0\\ 0 & 0 & 0 & (p\cdot\bar{p})^2\\ 0 & 0 & 2(p\cdot\bar{p})^2 & 0\end{bmatrix}\delta(\mathbf{p},\mathbf{p}')$
M	$\delta_{rs}\delta(\mathbf{p},\mathbf{p}')$	$\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & p\cdot\bar{p}\\ 0 & 0 & p\cdot\bar{p} & 0\end{bmatrix}\delta(\mathbf{p},\mathbf{p}')$
$G_0{}^{(+)}$	$G_0{}^{(+)\alpha}{}_{\mu}{}^{\beta'}{}_{\nu'}\equiv\gamma^{\alpha\beta}\eta_{\mu\nu}G_0{}^{(+)}(x,x')$	$G_0{}^{(+)}{}_{\mu\nu\sigma'\tau'}\equiv(\eta_{\mu\sigma}\eta_{\nu\tau}+\eta_{\mu\tau}\eta_{\nu\sigma}-\eta_{\mu\nu}\eta_{\sigma\tau})G_0{}^{(+)}(x,x')$
$\hat{G}_0{}^{(+)}$	$\hat{G}_0{}^{(+)\alpha\beta'}\equiv\gamma^{\alpha\beta}G_0{}^{(+)}(x,x')$	$\hat{G}_0{}^{(+)\mu\nu'}\equiv\eta^{\mu\nu}G_0{}^{(+)}(x,x')$

$$G_0{}^{(+)}(x,x')\equiv\frac{1}{(2\pi)^4}\int_{C^{(+)}}\frac{e^{ip\cdot(x-x')}}{p^2}dp,\quad dp\equiv dp^0dp^1dp^2dp^3$$

The hypercontour $C^{(+)}$ runs along the real axes in the p^1, p², p^3 planes and forms a closed loop in the p^0 plane surrounding the pole at $+E$.

and (8.2) with (8.5) and (8.6) we see, in particular, that the most general form for the asymptotic fields $\varphi^{\pm i}$ is

$$\varphi^{\pm i}=u^i{}_A\mathbf{a}^{\pm}{}_A+u^i{}_A{}^*\mathbf{a}^{\pm}{}_A{}^*+R_0{}^i{}_\alpha\zeta^{\pm\alpha},$$
or
$$\varphi^{\pm}=u\mathbf{a}^{\pm}+u^*\mathbf{a}^{\pm *}+R_0\zeta^{\pm},\tag{8.12}$$

where the u's are the functions indicated in Table II and the $\zeta^{\pm}$'s are completely arbitrary Hermitian functionals of the creation and annihilation operators.

The u's appearing in Eq. (8.12) may be regarded as wave functions for the asymptotic states. Using the explicit forms given in Table II, one may verify by direct computation that they satisfy the following important orthonormality relations:

$$-i\int_\Sigma u^{\sim}\overleftrightarrow{s}_0{}^{\mu}u\,d\Sigma_\mu=0,\qquad -i\int_\Sigma u^{\dagger}\overleftrightarrow{s}_0{}^{\mu}u\,d\Sigma_\mu=1,\tag{8.13}$$

$$-i\int_\Sigma u^{\sim}\overleftrightarrow{s}_0{}^{\mu}R_0\,d\Sigma_\mu=0,\quad -i\int_\Sigma R_0{}^{\sim}\overleftrightarrow{s}_0{}^{\mu}u\,d\Sigma_\mu=0,\tag{8.14}$$

where the hypersurface Σ is completely arbitrary except that it must be asymptotically spacelike, and where 1 is the super-abbreviation for δ_{AB}. The Σ independence of these relations follows immediately from (3.4) together with

$$S_2{}^0u=0\tag{8.15}$$

and

$$S_2{}^0R_0=0.\tag{8.16}$$

The latter relation, combined with the locality of R_0, in fact permits one to infer, without computation, the vanishing of the integrals (8.14) as well as of

$$-i\int_\Sigma R_0{}^{\sim}\overleftrightarrow{s}_0{}^{\mu}R_0\,d\Sigma_\mu=0.\tag{8.17}$$

Equations (8.14) and (8.17) imply that, as representatives of the asymptotic states, the u's need be defined only up to a gauge transformation $u\rightarrow u+R_0\zeta_0$, which leaves Eqs. (8.13) unaffected. In actual practice the u's are restricted by a supplementary condition, namely the zero-field analog of (6.2):

$$R_0{}^{\sim}\gamma_0u=0.\tag{8.18}$$

When this condition holds, the u's satisfy

$$F_0u=0\tag{8.19}$$

in addition to (8.15), and Eqs. (8.13) may be replaced by

$$-i\int u^{\sim}\overleftrightarrow{f}_0{}^{\mu}u\,d\Sigma_\mu=0,\quad -i\int u^{\dagger}\overleftrightarrow{f}_0{}^{\mu}u\,d\Sigma_\mu=1.\tag{8.20}$$

The validity of the latter orthonormality relations follows from the easily verified identity

$$\overleftrightarrow{f}^{\mu}-\overleftrightarrow{s}^{\mu}\equiv\gamma X^{\mu}\tilde{\gamma}^{-1}\overrightarrow{R}^{\sim}\gamma-\gamma\overrightarrow{R}\tilde{\gamma}^{-1}X^{\mu\sim}\gamma\,, \tag{8.21}$$

where the matrix $(X^{\mu})^{i}{}_{\alpha}$ has the form

$$(X^{\nu''})^{\alpha}{}_{\mu\beta'}\equiv-\delta^{\alpha}{}_{\beta'}\delta_{\mu}{}^{\nu''} \tag{8.22}$$

for the Yang-Mills field and

$$(X^{\tau''})_{\mu\nu\sigma'}\equiv-\delta_{\mu\sigma'}\delta_{\nu}{}^{\tau''}-\delta_{\nu\sigma'}\delta_{\mu}{}^{\tau''} \tag{8.23}$$

for the gravitational field.

The supplementary condition (8.18) does not yet completely determine the u's. Equations (8.18) to (8.20) remain unaffected by gauge transformations $u\rightarrow u+R_0\zeta_0$ for which $\hat{F}_0\zeta_0=0$. To obtain the u's of Table II a further condition must be imposed, of the form

$$\bar{R}_0{}^{\sim}\gamma_0 u=0\,. \tag{8.24}$$

Many different choices for $\bar{R}_0$ can be made which lead to the same u's. It will turn out to be a convenience to choose $\bar{R}_0$ in the particular way indicated in Table II where its momentum-space forms, as well as those of R_0, are given. The 4-vector $\bar{p}_{\mu}$ appearing in the expressions for $\bar{R}_0$ is defined by

$$\bar{p}_{\mu}\equiv p_{\mu}+2n_{\mu}n^{\nu}p_{\nu}\,, \tag{8.25}$$

where n_{μ} is the timelike unit vector of Eqs. (8.8) and (8.9). It is easy to see that $\bar{p}_{\mu}$, like p_{μ}, is null. In analogy with the terminology employed for null hypersurfaces p_{μ} may be called a *characteristic* vector and $\bar{p}_{\mu}$ the *bicharacteristic* of p_{μ} relative to n_{μ}.

The presence of n_{μ} introduces a nonrelativistic element into the formalism, the effect of which must be determined by asking for the changes in the u's and **a**'s produced by changing n_{μ}. It suffices to consider infinitesimal changes δn_{μ} leaving (8.9) invariant. If Eqs. (8.8) are to remain invariant in form, one readily finds that the e's must suffer the corresponding changes

$$\delta e_{\pm}{}^{\mu}=\mp i\delta\varphi e_{\pm}{}^{\mu}-(n\cdot p)^{-1}p^{\mu}(e_{\pm}\cdot\delta n)\,, \tag{8.26}$$

where $\delta\varphi$ is an arbitrary infinitesimal angle. If, in addition, the form of the decomposition (8.12) is to remain invariant the **a**'s must be multiplied by phase factors $e^{\pm is\delta\varphi}$ where s, the *spin*, is 1 for the Yang-Mills field and 2 for the gravitational field, the + sign or − sign being chosen according as the helicity is positive or negative. The first term on the right of (8.26) produces inverse phase changes in the u's while the second term produces a gauge transformation which may be absorbed into the last term on the right of (8.12). The phase changes produce corresponding changes in the elements of the S matrix but leave transition probabilities unaffected. Observationally, therefore, the classification of states according to helicity is Lorentz-invariant.

9. THE CANONICAL FORM OF THE COMMUTATOR FUNCTION. THE FEYNMAN PROPAGATOR

Although the u's, in virtue of Eqs. (8.13), (8.14), and (8.17), may be regarded as forming a complete orthonormal wave basis for the operator $S_2{}^0$, they do not form such a basis for the operator F_0. F_0 possesses additional, nonphysical wave functions having orthonormality properties more general than (8.20). The u's define only the physical subspace of such functions.

In the case of the Yang-Mills and gravitational fields it turns out that a complete basis for F_0 is obtained simply by adjoining to the u's the functions $R_0{}^{i}{}_{\alpha}v^{\alpha}{}_{a}$ and $\bar{R}_0{}^{i}{}_{\alpha}v^{\alpha}{}_{a}$,[26] where the v's constitute any complete basis for the auxiliary operator $\hat{F}_0$:

$$\hat{F}_0 v=0\,, \tag{9.1}$$

whence also

$$F_0R_0v=0\,,\quad F_0\bar{R}_0v=0\,. \tag{9.2}$$

By straightforward computation one may verify that in addition to (8.20) we now have relations of the form

$$-i\int_{\Sigma}v^{\dagger}\bar{R}_0{}^{\sim}\overleftrightarrow{f_0}^{\mu}R_0v\,d\Sigma_{\mu}=N\,,$$

$$-i\int_{\Sigma}v^{\dagger}R_0{}^{\sim}\overleftrightarrow{f_0}^{\mu}\bar{R}_0v\,d\Sigma_{\mu}=N^{\sim}\,, \tag{9.3}$$

with all other similar "inner products" of the functions u, R_0v, $\bar{R}_0v$ and their complex conjugates vanishing. Since the matrix

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & N^{\sim}\\ 0 & N & 0\end{bmatrix} \tag{9.4}$$

is symmetric, an orthogonal basis for F_0 can be found if desired. However, since (9.4) turns out to be a nonpositive-definite matrix, the positive normalization (8.20) cannot be extended to the entire basis. It proves convenient not to insist on complete orthogonality but to leave the basis as given. In this form it will be called a *canonical basis*.

A particular choice of v's for the Yang-Mills and gravitational fields is given in Table II, along with the corresponding matrices N. Using the table it is straightforward to verify that the function $\tilde{G}_0$, which appears in the commutator of asymptotic invariants, may be given the following *canonical decomposition*:

$$\tilde{G}_0=G_0{}^{(+)}+G_0{}^{(-)}\,, \tag{9.5}$$

$$G_0{}^{(+)*}=G_0{}^{(-)}\,,\quad G_0{}^{(+)\sim}=-G_0{}^{(-)}\,, \tag{9.6}$$

$$iG_0{}^{(+)}=uu^{\dagger}+R_0vN^{-1}v^{\dagger}\bar{R}_0{}^{\sim}+\bar{R}_0vN^{-1\sim}v^{\dagger}R_0{}^{\sim}\,. \tag{9.7}$$

[26] In the case of massless fields having spins greater than 2 these functions do not suffice to complete the basis.

The function $G_0^{(+)}$ is called the *positive energy function.* In a theory with no gauge group $iG_0^{(+)}$, regarded as an Hermitian matrix, must be positive semidefinite if a state of lowest energy—the vacuum—is to exist. In the present case $iG_0^{(+)}$ need be positive semidefinite only in the physical subspace. Since the physical subspace is represented by the functions u and $u^\dagger$ we see that this requirement holds. It will be convenient to introduce a special symbol for the projection of $\tilde{G}_0$ into the physical subspace:

$$\tilde{\mathfrak{G}}_0 \equiv \mathfrak{G}_0^{(+)} + \mathfrak{G}_0^{(-)}, \tag{9.8}$$

$$\mathfrak{G}_0^{(+)*} \equiv \mathfrak{G}_0^{(-)}, \quad \mathfrak{G}_0^{(+)\sim} \equiv -\mathfrak{G}_0^{(-)}, \tag{9.9}$$

$$i\mathfrak{G}_0^{(+)} \equiv uu^\dagger. \tag{9.10}$$

The importance of the canonical form for $\tilde{G}_0$ lies in the presence of the R_0's. It is easy to see, for example, that in virtue of (7.9) the quantum version of (7.14) immediately reduces to

$$[\mathbf{A}^\pm, \mathbf{B}^\pm] = iI_A \tilde{\mathfrak{G}}_0 I_B^\sim, \tag{9.11}$$

which is obviously consistent with the decomposition (8.12). Other more important uses of the canonical form will be encountered later.

For completeness we record the following additional relations satisfied by the quantities thus far introduced:

$$\hat{F}_0 \equiv R_0^\sim \gamma_0 R_0 = \bar{R}_0^\sim \gamma_0 \bar{R}_0, \tag{9.12}$$

$$\tilde{\gamma}_0^{-1} R_0^\sim \gamma_0 \bar{R}_0 v = vM^{-1}N^\sim, \tag{9.13}$$

$$i\hat{G}_0^{(+)} = vM^{-1}v^\dagger, \tag{9.14}$$

$$-i\int_\Sigma v^\sim \overleftrightarrow{f}_0^\mu v d\Sigma_\mu = 0, \quad -i\int_\Sigma v^\dagger \overleftrightarrow{f}_0^\mu v d\Sigma_\mu = M, \tag{9.15}$$

$$\eta_{\mu\nu} \equiv e_{+\mu}e_{-\nu} + e_{+\nu}e_{-\mu} + (p\cdot\bar{p})^{-1}(p_\mu\bar{p}_\nu + p_\nu\bar{p}_\mu). \tag{9.16}$$

Here $\hat{G}_0^{(+)}$ is the positive energy part of the function $\hat{G}_0^+ - \hat{G}_0^-$, and Eq. (9.13) assures consistency of (6.11) with the decompositions (9.7) and (9.14). The explicit form of the matrix M appearing in (9.13), (9.14) and (9.15) is given in Table II for the particular v's which are adopted there. The operator $\overleftrightarrow{f}_0^\mu$ is related to $\hat{F}_0$ in the same way that $\overleftrightarrow{f}_0^\mu$ is related to F_0. The identity (9.16), which follows from Eqs. (8.8) and (8.25), is used repeatedly in the verification of the decompositions (9.7) and (9.14).

In the classical theory a dominant role is played by the Green's functions $G^\pm$. In the quantum theory this role is usurped by the Feynman propagator. For zero fields the latter is defined by

$$G_0 \equiv G_0^\pm \mp G_0^{(\pm)}, \tag{9.17}$$

the equivalence of the two forms following from (6.5) and (9.5), which lead to the boundary conditions

$$G_0^{ij} = -G_0^{(+)ij}, \quad i > j,$$
$$= G_0^{(-)ij}, \quad j > i. \tag{9.18}$$

These conditions may be generalized so as to be applicable to nonzero background fields. In the general case the Feynman propagator is defined as that Green's function which, as a function of its first argument, has only positive energy components in the remote future and only negative energy components in the remote past. These boundary conditions suffice to yield the variational law

$$\delta G = G\ \delta F\ G, \tag{9.19}$$

and the expansions

$$G = G_0(1-UG_0)^{-1} = G_0 + G_0XG_0, \tag{9.20}$$

$$X \equiv (1-UG_0)^{-1}U = U + UG_0U + \cdots. \tag{9.21}$$

The variational law (9.19) has exactly the same form as Eq. (6.19) for the advanced and retarded Green's functions. The Feynman propagator has, in addition, a symmetry not possessed by $G^\pm$, namely

$$G_0^\sim = G_0, \quad G^\sim = G, \tag{9.22}$$

which follows from (6.8), (9.6), (9.16), and the (assumed) self-adjointness of F. The Feynman propagator and its complex conjugate may therefore be characterized as the only Green's functions which, when regarded as continuous matrices, obey all the rules of finite matrix theory—a characterization which may serve to define them uniquely even when the condition of asymptotic flatness does not hold and S-matrix theory ceases to exist. In a flat Euclidean 4-space F has only one unique inverse (Green's function) which vanishes asymptotically, and the Feynman propagator is obtainable from this inverse by analytic continuation to Minkowski space-time, the "direction" of the continuation being correlated with the direction in which time is chosen to "flow". In this sense the Feynman propagator may be regarded as *the* inverse of $-F$, its complex conjugate being obtained by analytic continuation in the alternative direction.

We now record for later use a number of identities involving the various Green's functions, which are derivable by straightforward algebraic manipulation of previous equations:

$$G^\pm = G_0^\pm(1-UG_0^\pm)^{-1} = G_0^\pm + G_0^\pm X^\pm G_0^\pm, \tag{9.23}$$

$$X^\pm \equiv (1-UG_0^\pm)^{-1}U = U + UG^\pm U, \tag{9.24}$$

$$X = (1 \pm X^\pm G_0^{(\pm)})^{-1}X^\pm, \tag{9.25}$$

$$1 \pm XG_0^{(\pm)} = (1 \pm X^\pm G_0^{(\pm)})^{-1}, \tag{9.26}$$

$$1 + XG_0 = (1 \pm X^\pm G_0^{(\pm)})^{-1}(1 + X^\pm G_0^\pm) = (1-UG_0)^{-1}, \tag{9.27}$$

$$1-UG_0{}^{\pm}=(1-UG_0)(1\mp XG_0{}^{(\pm)})$$
$$=(1-UG_0)(1\pm X^{\pm}G_0{}^{(\pm)})^{-1}, \tag{9.28}$$

$$G=G^{\pm}\mp G^{(\pm)}, \tag{9.29}$$

$$G^{(\pm)}\equiv(1+G_0{}^{\pm}X^{\pm})G_0{}^{(\pm)}(1\pm X^{\pm}G_0{}^{(\pm)})^{-1}\times(1+X^{\pm}G_0{}^{\pm}) \tag{9.30a}$$

$$=(1+G_0X)G_0{}^{(\pm)}(1\mp XG_0{}^{(\pm)})^{-1}\times(1+XG_0), \tag{9.30b}$$

$$G_0-G_0{}^*=-(G_0{}^{(+)}-G^{(-)}), \tag{9.31}$$

$$X-X^*=-X(G_0{}^{(+)}-G_0{}^{(-)})X^*, \tag{9.32}$$

$$G-G^*=-(1+G_0X)(G_0{}^{(+)}-G_0{}^{(-)})\times(1+X^*G_0{}^*). \tag{9.33}$$

Equations identical in form with these are satisfied by the corresponding functions $\hat{G}_0{}^{\pm}$, $\hat{G}^{\pm}$, $\hat{G}_0$, $\hat{G}$, $\hat{G}_0{}^{(\pm)}$, $\hat{G}^{(\pm)}$, $\hat{X}^{\pm}$, $\hat{X}$, and $\hat{U}$ associated with the operator $\hat{F}$.

In the theory of the S matrix the function G plays the role of the propagator of field quanta. When an invariance group is present this function suffers from a fundamental defect, namely, it propagates nonphysical as well as physical quanta. For purposes of defining "external-line wave functions" (see Sec. 10) and checking the unitarity of the S matrix (which is defined only between real physical states) it is convenient to introduce alternative functions which propagate real quanta only:

$$\mathfrak{G}_{\pm}\equiv G^{\pm}\mp\mathfrak{G}^{(\pm)}, \tag{9.34}$$

$$\mathfrak{G}^{(\pm)}\equiv(1+G_0{}^{\pm}X^{\pm})\mathfrak{G}_0{}^{(\pm)}(1\pm X^{\pm}\mathfrak{G}_0{}^{(\pm)})^{-1}\times(1+X^{\pm}G_0{}^{\pm}). \tag{9.35}$$

The use of these functions, however, destroys manifest covariance and, when divergences are present, is limited to formal arguments. In actual calculations the functions G, $G^{\pm}$, $\hat{G}$, $\hat{G}^{\pm}$ must be employed to assure consistency of renormalization procedures. One of our tasks will be to show how to pass formally from one set of functions to the other.

The functions $\mathfrak{G}_{\pm}$, $\mathfrak{G}^{(\pm)}$, etc. satisfy a list of identities similar to those satisfied by G, $G^{(\pm)}$, *et al.*:

$$\mathfrak{G}_{0\pm}\equiv G_0{}^{\pm}\mp\mathfrak{G}_0{}^{(\pm)}, \tag{9.36}$$

$$\mathfrak{G}_{\pm}=\mathfrak{G}_{0\pm}(1-U\mathfrak{G}_{0\pm})^{-1}$$
$$=\mathfrak{G}_{0\pm}+\mathfrak{G}_{0\pm}\mathfrak{X}_{\pm}\mathfrak{G}_{0\pm}, \tag{9.37}$$

$$\mathfrak{X}_{\pm}\equiv(1-U\mathfrak{G}_{0\pm})^{-1}U=U+U\mathfrak{G}_{\pm}U, \tag{9.38a}$$

$$=(1\pm X^{\pm}\mathfrak{G}_0{}^{(\pm)})^{-1}X^{\pm}, \tag{9.38b}$$

$$1\mp\mathfrak{X}_{\pm}\mathfrak{G}_0{}^{(\pm)}=(1\pm X^{\pm}\mathfrak{G}_0{}^{(\pm)})^{-1}, \tag{9.39}$$

$$1+\mathfrak{X}_{\pm}\mathfrak{G}_{0\pm}=(1\pm X^{\pm}\mathfrak{G}_0{}^{(\pm)})^{-1}(1+X^{\pm}G_0{}^{\pm})$$
$$=(1-U\mathfrak{G}_{0\pm})^{-1}, \tag{9.40}$$

$$1-UG_0{}^{\pm}=(1-U\mathfrak{G}_{0\pm})(1\mp\mathfrak{X}_{\pm}\mathfrak{G}_0{}^{(\pm)})$$
$$=(1-U\mathfrak{G}_{0\pm})(1\pm X^{\pm}\mathfrak{G}_0{}^{(\pm)})^{-1}, \tag{9.41}$$

$$\mathfrak{G}^{(\pm)}=(1+\mathfrak{G}_{0\pm}\mathfrak{X}_{\pm})\mathfrak{G}_0{}^{(\pm)}(1\mp\mathfrak{X}_{\pm}\mathfrak{G}_0{}^{(\pm)})^{-1}\times(1+\mathfrak{X}_{\pm}\mathfrak{G}_{0\pm}), \tag{9.42}$$

$$\mathfrak{G}_{0\pm}-\mathfrak{G}_{0\pm}{}^*=-(\mathfrak{G}_0{}^{(+)}-\mathfrak{G}_0{}^{(-)}), \tag{9.43}$$

$$\mathfrak{X}_{\pm}-\mathfrak{X}_{\pm}{}^*=-\mathfrak{X}_{\pm}(\mathfrak{G}_0{}^{(+)}-\mathfrak{G}_0{}^{(-)})\mathfrak{X}_{\pm}{}^*, \tag{9.44}$$

$$\mathfrak{G}_{\pm}-\mathfrak{G}_{\pm}{}^*=-(1+\mathfrak{G}_{0\pm}\mathfrak{X}_{\pm})(\mathfrak{G}_0{}^{(+)}-\mathfrak{G}_0{}^{(-)})\times(1+\mathfrak{X}_{\pm}{}^*\mathfrak{G}_{0\pm}{}^*). \tag{9.45}$$

The only difference is that $\mathfrak{G}_{0\pm}$, $\mathfrak{G}_{\pm}$, $\mathfrak{X}_{\pm}$, unlike G_0, G, X, are nonsymmetric, which accounts for the $\pm$ signs attached to them. From (6.8) and (9.9) it follows that

$$\mathfrak{G}_{0+}{}^{\sim}=\mathfrak{G}_{0-},\quad \mathfrak{G}_{+}{}^{\sim}=\mathfrak{G}_{-},\quad \mathfrak{X}_{+}{}^{\sim}=\mathfrak{X}_{-}. \tag{9.46}$$

We must evidently ask what difference it makes if we use $\mathfrak{G}_-$ instead of $\mathfrak{G}_+$ as a replacement for G. In order to show that it in fact makes no difference we must first develop the formalism somewhat further.

10. EXTERNAL-LINE WAVE FUNCTIONS. FUNDAMENTAL LEMMA

Consider the following functions:

$$f^{\pm}\equiv\int_{\pm\infty}\tilde{G}\overleftrightarrow{f^{\mu}}u\,d\Sigma_{\mu}. \tag{10.1}$$

In virtue of Eq. (6.4) these functions satisfy

$$Ff^{\pm}=0 \tag{10.2}$$

and reduce to the asymptotic wave functions u in the remote future and past, respectively. If (as is always assumed) $\tilde{G}$ is based on a choice of γ's which corresponds to the same supplementary conditions (6.2) as those which are imposed on the u's [Eq. (8.18)] then the $f^{\pm}$'s will also satisfy the equations

$$R^{\sim}\gamma f^{\pm}=0,\quad S_2f^{\pm}=0. \tag{10.3}$$

By making use of the combination law

$$\tilde{G}=\int_{\Sigma}\tilde{G}\overleftrightarrow{f^{\mu}}\tilde{G}\,d\Sigma_{\mu}, \tag{10.4}$$

which is a special case of (6.4), and taking note of the symmetries (6.9) and $\overleftrightarrow{f^{\mu}}{}^{\sim}=-\overleftrightarrow{f^{\mu}}$ [cf. Eq. (3.2)] as well as the fact that $\overleftrightarrow{f^{\mu}}$ reduces to $\overleftrightarrow{f_0{}^{\mu}}$ in the remote past and future (because the background field is then dispersed to a state of infinite weakness), one may show that the $f^{\pm}$'s constitute two distinct complete orthonormal bases for infinitesimal disturbances on a nonvanishing background:

$$-i\int_{\Sigma}f^{\pm\sim}\overleftrightarrow{f^{\mu}}f^{\pm}d\Sigma_{\mu}=-i\int_{\Sigma}f^{\pm\sim}\overleftrightarrow{s}^{\mu}f^{\pm}d\Sigma_{\mu}=0,$$
$$-i\int_{\Sigma}f^{\pm\dagger}\overleftrightarrow{f^{\mu}}f^{\pm}d\Sigma_{\mu}=-i\int_{\Sigma}f^{\pm\dagger}\overleftrightarrow{s}^{\mu}f^{\pm}d\Sigma_{\mu}=1. \tag{10.5}$$

The $f^{\pm}$'s are basis functions for "classical" waves. In the quantum theory a different basis, satisfying boundary conditions which take pair production into account, must be employed. The method of constructing the latter basis will be most clear if we first obtain an alternative form for the $f^{\pm}$'s. Taking note of the kinematic structure of $\tilde{G}$ we may rewrite Eq. (10.1) as

$$\begin{aligned} f^{\pm} &= \left(\int_{\infty} - \int_{-\infty}\right) G^{\pm} \overleftrightarrow{f}_0{}^{\mu} u \, d\Sigma_{\mu} \\ &= G^{\pm}(\overrightarrow{F}_0 - \overleftarrow{F}_0)u = -G^{\pm}\overleftarrow{F}_0 u \\ &= (1+G_0{}^{\pm}X^{\pm})u\,, \end{aligned} \tag{10.6}$$

and, in view of the supplementary condition (8.18), also

$$f^{\pm} = -G^{\pm}\overleftarrow{S}_2{}^0 u. \tag{10.7}$$

These forms suggest that the modified functions which we seek are

$$f \equiv -G\overleftarrow{S}_2{}^0 u = -G\overleftarrow{F}_0 u = (1+G_0X)u\,, \tag{10.8}$$

in which the Green's functions $G^{\pm}$ are replaced by the Feynman propagator. However, such functions are inappropriate for the following reason: In the remote past they possess not only components from the physical basis u but also nonphysical components which have been "scattered backwards in time" and which appear because the quantity X has nonvanishing matrix elements between physical and nonphysical states.

The desired functions are obtained from (10.8) by substituting $\mathfrak{G}_{\pm}$ for G:

$$\mathfrak{f}_{\pm} \equiv -\mathfrak{G}_{\pm}\overleftarrow{S}_2{}^0 u = -\mathfrak{G}_{\pm}\overleftarrow{F}_0 u = (1+\mathfrak{G}_{0\pm}\mathfrak{X}_{\pm})u \tag{10.9a}$$

$$= (1+G_0{}^{\pm}X^{\pm})(1\mp\mathfrak{G}_0{}^{(\pm)}\mathfrak{X}_{\pm})u\,, \tag{10.9b}$$

the final form being obtained through use of (9.39) and (9.40). In virtue of the decomposition (9.10) it is apparent that these functions can be expressed as linear combinations of the functions $f^{\pm}$ and their complex conjugates. They therefore satisfy

$$F\mathfrak{f}_{\pm}=0\,, \quad R^{\sim}\gamma\mathfrak{f}_{\pm}=0\,, \quad S_2\mathfrak{f}_{\pm}=0\,. \tag{10.10}$$

The $\mathfrak{f}_{\pm}$'s are called *external-line wave functions*. It can be shown that they differ from the f's of Eq. (10.8) by an amount which cannot be expressed as a gauge transformation. The difference between $\mathfrak{f}_+$ and $\mathfrak{f}_-$, however, *can* be so expressed, and the $\pm$ signs are therefore physically irrelevant. For the proof of this we now derive a fundamental lemma.

We first introduce the functions

$$g^{\pm} \equiv (1+\hat{G}_0{}^{\pm}\hat{X}^{\pm})v\,, \tag{10.11}$$

which are related to the v's of Table II in the same way that the $f^{\pm}$'s are related to the u's. That is, they coincide with the v's in the remote future or past, and satisfy

$$\hat{F}g^{\pm}=0\,. \tag{10.12}$$

From this it follows that

$$FRg^{\pm}=0\,. \tag{10.13}$$

Since the functions $Rg^{\pm}$ coincide with R_0v in the remote future and past, respectively, we may write

$$Rg^{\pm}=(1+G_0{}^{\pm}X^{\pm})R_0v\,. \tag{10.14}$$

From (10.11) we may also write

$$R_0g^{\pm}=R_0(1+\hat{G}_0{}^{\pm}\hat{X}^{\pm})v\,. \tag{10.15}$$

Subtracting (10.15) from (10.14) and making use of the zero-field form of (6.11), we obtain

$$(R-R_0)g^{\pm}=G_0{}^{\pm}(X^{\pm}R_0-\gamma_0R_0\tilde{\gamma}_0{}^{-1}\hat{X}^{\pm})v\,. \tag{10.16}$$

The desired lemma then follows on applying the operator F_0:

$$X^{\pm}R_0v=\gamma_0R_0\tilde{\gamma}_0{}^{-1}\hat{X}^{\pm}v-F_0(R-R_0)g^{\pm}\,. \tag{10.17}$$

The quantity $(R-R_0)g^{\pm}$ appearing in the last term of (10.17) vanishes at infinity rapidly enough so that integrations by parts may be performed when it appears as part of a larger expression. This means that the operator F_0 attached to it may act in either direction. Therefore, making use of the supplementary condition (8.18), as well as $v^{\sim}R_0{}^{\sim}\gamma_0R_0=v^{\sim}\overleftarrow{F}_0=0$, we immediately obtain the corollaries

$$u^{\sim}X^{\pm}R_0v=0\,, \quad v^{\sim}R_0{}^{\sim}X^{\pm}R_0v=0\,, \tag{10.18}$$

which hold also when the u's and/or v's are replaced by their complex conjugates. Referring to Eqs. (9.10) and (9.38b) we see that these corollaries in turn imply

$$u^{\sim}\mathfrak{X}_{\pm}R_0v=0\,, \quad v^{\sim}R_0{}^{\sim}\mathfrak{X}_{\pm}R_0v=0\,, \quad \text{etc.} \tag{10.19}$$

Next, by algebraic manipulation of Eqs. (9.36), (9.37), and (9.38) we find

$$\mathfrak{X}_+-\mathfrak{X}_-=\mathfrak{X}_+(\mathfrak{G}_{0+}-\mathfrak{G}_{0-})\mathfrak{X}_-\,, \tag{10.20}$$

$$\mathfrak{G}_+-\mathfrak{G}_-=(1+\mathfrak{G}_{0+}\mathfrak{X}_+)(\mathfrak{G}_{0+}-\tilde{\mathfrak{G}}_{0-}) \times(1+\mathfrak{X}_-\mathfrak{G}_{0-})\,, \tag{10.21}$$

$$\mathfrak{G}_{0+}-\mathfrak{G}_{0-}=\tilde{G}_0-\tilde{\mathfrak{G}}_0 \tag{10.22a}$$

$$\begin{aligned} &= -iR_0vN^{-1}v^{\dagger}\bar{R}_0{}^{\sim}-i\bar{R}_0vN^{-1\sim}v^{\dagger}R_0{}^{\sim} \\ &\quad +i\bar{R}_0v^{*}N^{-1}v^{\sim}\bar{R}_0{}^{\sim}+i\bar{R}_0v^{*}N^{-1\sim}v^{\sim}R_0{}^{\sim}\,, \end{aligned} \tag{10.22b}$$

in which use has been made of the canonical decomposition (9.7). These results may finally be combined with (9.40), (10.14), and (10.19) to yield

$$\begin{aligned} \mathfrak{f}_+-\mathfrak{f}_- &= (1+\mathfrak{G}_{0+}\mathfrak{X}_+)(\mathfrak{G}_{0+}-\mathfrak{G}_{0-})\mathfrak{X}_-u \\ &= (1+G_0{}^{+}X^{+})(1+\mathfrak{G}_0{}^{(\pm)}X^{\pm})^{-1} \\ &\quad \times R_0(-ivN^{-1}v^{\dagger}+iv^{*}N^{-1}v^{\sim})\bar{R}_0{}^{\sim}\mathfrak{X}_-u \\ &= R(-ig^{+}N^{-1}v^{\dagger}+ig^{+*}N^{-1}v^{\sim})\bar{R}_0{}^{\sim}\mathfrak{X}_-u\,, \end{aligned} \tag{10.23}$$

showing that the two functions indeed differ from one another only by a gauge transformation.

11. AMPLITUDES FOR SCATTERING, PAIR PRODUCTION, AND PAIR ANNIHILATION BY THE BACKGROUND FIELD. THE OPTICAL THEOREMS WHICH THEY SATISFY. PROOF OF THEIR GROUP INVARIANCE

Another important relation may be obtained by inserting (10.22b) into (10.20) and using (10.19):

$$u^{\sim}(\mathfrak{X}_+ - \mathfrak{X}_-)u = 0\,, \quad u^{\dagger}(\mathfrak{X}_+ - \mathfrak{X}_-)u = 0\,. \tag{11.1}$$

From this it follows that the quantities

$$I \equiv u^{\dagger}\mathfrak{X}_{\pm}u\,, \tag{11.2}$$

$$V \equiv u^{\dagger}\mathfrak{X}_{\pm}u^*\,, \tag{11.3}$$

$$\Lambda \equiv u^{\sim}\mathfrak{X}_{\pm}u\,, \tag{11.4}$$

are independent of the $\pm$ signs, showing once again the irrelevance of the signs.

I, V, and Λ are, respectively, the amplitudes for *scattering*, *pair production*, *and pair annihilation* of field quanta by the background field. More precisely, they are the amplitudes for these processes when it is assumed that the quanta themselves do not interact with one another but behave as the quanta of a model field theory with action functional $\frac{1}{2}S_{,ij}\phi^i\phi^j$.

By making use of (9.46), as well as (9.44) and its transpose, one easily verifies that these amplitudes satisfy the following relations:

$$V^{\sim} = V\,, \quad \Lambda^{\sim} = \Lambda\,, \tag{11.5}$$

$$I - I^{\dagger} = i(II^{\dagger} + VV^{\dagger}) = i(I^{\dagger}I + \Lambda^{\dagger}\Lambda)\,, \tag{11.6}$$

$$\Lambda - V^{\dagger} = i(\Lambda I^{\dagger} + I^{\sim}V^{\dagger}) = i(V^{\dagger}I + I^*\Lambda)\,, \tag{11.7}$$

$$V - \Lambda^{\dagger} = i(I\Lambda^{\dagger} + VI^*) = i(I^{\dagger}V + \Lambda^{\dagger}I^{\sim})\,. \tag{11.8}$$

Equations (11.5) express the Bose statistics satisfied by the field quanta; Eqs. (11.6), (11.7), and (11.8) are relativistic generalizations of the well known *optical theorem* for nonrelativistic scattering. The latter equations play an important role in the verification of the unitarity of the S matrix, as will be demonstrated later.

The amplitudes I, V, and Λ are not only independent of the $\pm$ signs but are group-invariant as well. In the present formalism group invariance has three distinct aspects: (1) invariance under group transformations of the background field; (2) invariance under changes in the Green's functions, as well as in the asymptotic wave functions u, resulting from changes in the γ's; and (3) invariance under gauge transformations of the u's for which the gauge parameters ζ^0 satisfy $\hat{F}_0\zeta_0 = 0$.[27]

Since the parameters $\delta\xi^{\alpha}$ of Eq. (4.1) are required to vanish at infinity, the asymptotic wave functions and the zero-point field remain unaffected by group transformations of the background field. Only the Green's functions $G^{\pm}$, G, etc. change. Owing to the care which has been taken to construct these functions in a manifestly covariant manner we may write at once

$$\delta G^{\pm ij} = (R^i{}_{\alpha,k}G^{\pm kj} + R^j{}_{\alpha,k}G^{\pm ik})\delta\xi^{\alpha}\,, \tag{11.9}$$

which may be inserted into

$$\delta X^{\pm} = \overrightarrow{F}_0\delta G^{\pm}\overleftarrow{F}_0\,. \tag{11.10}$$

This in turn may be inserted into

$$\begin{aligned}\delta\mathfrak{G}^{(\pm)} = {} & G_0{}^{\pm}\delta X^{\pm}\mathfrak{G}_0{}^{(\pm)}(1+X^{\pm}\mathfrak{G}_0{}^{(\pm)})^{-1}(1+X^{\pm}G_0{}^{\pm}) \\ & \mp(1+G_0{}^{\pm}X^{\pm})\mathfrak{G}_0{}^{(\pm)}(1\pm X^{\pm}\mathfrak{G}_0{}^{(\pm)})^{-1}\delta X^{\pm}\mathfrak{G}_0{}^{(\pm)} \\ & \times(1\pm X^{\pm}\mathfrak{G}_0{}^{(\pm)})^{-1}(1+X^{\pm}G_0{}^{\pm}) + (1+G_0{}^{\pm}X^{\pm}) \\ & \times\mathfrak{G}_0{}^{(\pm)}(1\pm X^{\pm}\mathfrak{G}_0{}^{(\pm)})^{-1}\delta X^{\pm}G_0{}^{\pm}\,,\end{aligned} \tag{11.11}$$

which follows from (9.35). Owing to the boundary conditions on the $\delta\xi^{\alpha}$ the arrow on one or the other of the F_0's in (11.10) may always be reversed. As a result the second term of (11.11) vanishes, while the first and third terms together yield

$$\delta\mathfrak{G}^{(\pm)ij} = (R^i{}_{\alpha,k}\mathfrak{G}^{(\pm)kj} + R^j{}_{\alpha,k}\mathfrak{G}^{(\pm)ik})\delta\xi^{\alpha}\,, \tag{11.12}$$

which shows that $\mathfrak{G}^{(\pm)}$ and $\mathfrak{G}_{\pm}$ have the same transformation law as $G^{\pm}$ and G. Inserting this transformation law into

$$\delta\mathfrak{X}_{\pm} = \overrightarrow{F}_0\delta\mathfrak{G}_{\pm}\overleftarrow{F}_0\,, \tag{11.13}$$

and noting that one or the other of the arrows is again reversible, we immediately get the desired result:

$$\delta I = 0\,, \quad \delta V = 0\,, \quad \delta\Lambda = 0\,. \tag{11.14}$$

With the aid of (10.9a) we also get, in a similar manner,

$$\delta\mathfrak{f}_{\pm}{}^i = R^i{}_{\alpha,j}\mathfrak{f}_{\pm}{}^j\delta\xi^{\alpha}\,, \tag{11.15}$$

which will prove useful later.

The demonstration of invariance under changes in the γ's is more complicated. We first note that in order to preserve the supplementary condition (8.18) under a change in the γ's, the u's must suffer the gauge transformation[28]

$$\begin{aligned}\delta u = R_0\hat{G}_0R_0{}^{\sim}\delta\gamma_0 u = {} & G_0{}^{\pm}\gamma_0R_0\tilde{\gamma}_0{}^{-1}R_0{}^{\sim}\delta\gamma_0 u \\ & \mp R_0\hat{G}_0{}^{(\pm)}R_0{}^{\sim}\delta\gamma_0 u\,.\end{aligned} \tag{11.16}$$

From this, together with (9.9) and (9.10), it follows that

$$\begin{aligned}\delta\mathfrak{G}_0{}^{(\pm)} = {} & G_0{}^{\pm}\gamma_0R_0\tilde{\gamma}_0{}^{-1}R_0{}^{\sim}\delta\gamma_0\mathfrak{G}_0{}^{(\pm)} \\ & +\mathfrak{G}_0{}^{(\pm)}\delta\gamma_0R_0\tilde{\gamma}_0{}^{-1}R_0{}^{\sim}\gamma_0G_0{}^{\pm} \mp R_0\hat{G}_0{}^{(+)}R_0{}^{\sim}\delta\gamma_0\mathfrak{G}_0{}^{(\pm)} \\ & \mp\mathfrak{G}_0{}^{(\pm)}\delta\gamma_0R_0\hat{G}_0{}^{(-)}R_0{}^{\sim}\,.\end{aligned} \tag{11.17}$$

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[27] Changes of types (2) and (3) together yield the most general gauge transformation of the u's.

[28] The u's may also suffer an additional change $\delta u = R_0\delta\zeta_0$ where $\hat{F}_0\delta\zeta_0 = 0$. See Eq. (11.30) ff.

With the aid of (6.19), (8.18), and (9.36) this yields

$$\delta\mathfrak{G}_{0\pm}=\mathfrak{G}_{0\pm}\delta F_0\mathfrak{G}_{0\pm}-R_0\hat{G}_0{}^{(+)}R_0{}^{\sim}\delta\gamma_0\mathfrak{G}_0{}^{(\pm)}-\mathfrak{G}_0{}^{(\pm)}\delta\gamma_0R_0\hat{G}_0{}^{(-)}R_0{}^{\sim},\quad(11.18)$$

where

$$\delta F_0=\delta\gamma_0R_0\tilde{\gamma}_0{}^{-1}R_0{}^{\sim}\gamma_0+\gamma_0R_0\delta\tilde{\gamma}_0{}^{-1}R_0{}^{\sim}\gamma_0+\gamma_0R_0\tilde{\gamma}_0{}^{-1}R_0{}^{\sim}\delta\gamma_0.\quad(11.19)$$

Equation (11.18) may be used with (9.37) to obtain

$$\begin{aligned}\delta\mathfrak{G}_{\pm}=&\mathfrak{G}_{\pm}\delta F_0\mathfrak{G}_{\pm}+\mathfrak{G}_{\pm}\delta U\mathfrak{G}_{\pm}\\&\mp(1-\mathfrak{G}_{0\pm}U)^{-1}R_0\hat{G}_0{}^{(+)}R_0{}^{\sim}\delta\gamma_0\mathfrak{G}_0{}^{(\pm)}(1+U\mathfrak{G}_{\pm})\\&\mp(1-\mathfrak{G}_{0\pm}U)^{-1}\mathfrak{G}_0{}^{(\pm)}\delta\gamma_0R_0\hat{G}_0{}^{(-)}R_0{}^{\sim}(1+U\mathfrak{G}_{\pm}),\end{aligned}\quad(11.20)$$

where

$$\delta U=\delta F-\delta F_0,\quad(11.21)$$

$$\delta F=\delta\gamma R\tilde{\gamma}^{-1}R^{\sim}\gamma+\gamma R\delta\tilde{\gamma}^{-1}R^{\sim}\gamma+\gamma R\tilde{\gamma}^{-1}R^{\sim}\delta\gamma.\quad(11.22)$$

Equation (11.20) may in turn be used with (9.38a) to obtain

$$\begin{aligned}\delta\mathfrak{X}_{\pm}=&(1+\mathfrak{X}_{\pm}\mathfrak{G}_{0\pm})\delta F(1+\mathfrak{G}_{0\pm}\mathfrak{X}_{\pm})-\delta F_0-\delta F_0\mathfrak{G}_{0\pm}\mathfrak{X}_{\pm}\\&-\mathfrak{X}_{\pm}\mathfrak{G}_{0\pm}\delta F_0\mp\mathfrak{X}_{\pm}(R_0\hat{G}^{(+)}R_0{}^{\sim}\delta\gamma_0\mathfrak{G}_0{}^{(\pm)}\\&+\mathfrak{G}_0{}^{(\pm)}\delta\gamma_0R_0\hat{G}_0{}^{(-)}R_0{}^{\sim})\mathfrak{X}_{\pm}.\end{aligned}\quad(11.23)$$

We now note that in virtue of (8.18) Eq. (11.16) may be reexpressed in the form

$$\delta u=\mathfrak{G}_{0\pm}\delta F_0u\mp R_0\hat{G}_0{}^{(\pm)}R_0{}^{\sim}\delta\gamma_0u.\quad(11.24)$$

We also note that $u^{\sim}\delta F_0u=0$ and, in virtue of (10.10), $\mathfrak{f}_{\pm}{}^{\sim}\delta F\mathfrak{f}_{\pm}=0$. Therefore,

$$\begin{aligned}\delta\Lambda=&\delta u^{\sim}\mathfrak{X}_{\pm}u+u^{\sim}\delta\mathfrak{X}_{\pm}u+u^{\sim}\mathfrak{X}_{\pm}\delta u\\=&\mp u^{\sim}(\mathfrak{X}_{\pm}R_0\hat{G}_0{}^{(+)}R_0{}^{\sim}\delta\gamma_0\mathfrak{G}_0{}^{(\pm)}\mathfrak{X}_{\pm}\\&+\mathfrak{X}_{\pm}\mathfrak{G}_0{}^{(\pm)}\delta\gamma_0R_0\hat{G}_0{}^{(-)}R_0{}^{\sim}\mathfrak{X}_{\pm}+\delta\gamma_0R_0\hat{G}_0{}^{(\pm)}R_0{}^{\sim}\mathfrak{X}_{\pm}\\&+\mathfrak{X}_{\pm}R_0\hat{G}_0{}^{(\pm)}R_0{}^{\sim}\delta\gamma_0)u,\end{aligned}\quad(11.25)$$

which vanishes by (9.14) and (10.19). Similarly, $\delta V=0$ and $\delta I=0$.

As a byproduct of this demonstration we again get a transformation law for the $\mathfrak{f}_{\pm}$'s. Thus, using (10.9a) and the fact that $\delta S_2=0$ as well as $\overleftarrow{S}_2{}^0\delta u=0$, we find

$$\begin{aligned}\delta\mathfrak{f}_{\pm}=&-\delta\mathfrak{G}_{\pm}\overleftarrow{S}_2{}^0u=-\delta\mathfrak{G}_{\pm}\overleftarrow{F}_0u\\=&\mathfrak{G}_{\pm}\delta F\mathfrak{f}_{\pm}\pm(1+\mathfrak{G}_{0\pm}\mathfrak{X}_{\pm})(R_0\hat{G}_0{}^{(+)}R_0{}^{\sim}\delta\gamma_0\mathfrak{G}_0{}^{(\pm)}\\&+\mathfrak{G}_0{}^{(\pm)}\delta\gamma_0R_0\hat{G}_0{}^{(-)}R_0{}^{\sim})(1+\mathfrak{X}_{\pm}\mathfrak{G}_{0\pm})\overleftarrow{F}_0u\\=&\mathfrak{G}_{\pm}\gamma R\tilde{\gamma}^{-1}R^{\sim}\delta\gamma\,\mathfrak{f}_{\pm}\mp(1+\mathfrak{G}_{0\pm}\mathfrak{X}_{\pm})\\&\times R_0\hat{G}_0{}^{(+)}R_0{}^{\sim}\delta\gamma_0\mathfrak{G}_0{}^{(\pm)}\mathfrak{X}_{\pm}u,\end{aligned}\quad(11.26)$$

the final expression resulting from use of (9.2), (9.14), and (10.19). By making use also of the identities

$$\mathfrak{G}_{\pm}\gamma R=R\hat{G}^{\pm}\tilde{\gamma},\quad(11.27)$$

$$(1+\mathfrak{G}_{0\pm}\mathfrak{X}_{\pm})R_0v=R(1+\hat{G}_0{}^{\pm}\hat{\mathfrak{X}}^{\pm})v,\quad(11.28)$$

which are proved, respectively, with the aid of (6.11), (9.35), (10.3), (10.6), and (9.40), (10.14), (10.18), we may recast expression (11.26) in the form

$$\begin{aligned}\delta\mathfrak{f}_{\pm}=R[&\hat{G}^{\pm}R^{\sim}\delta\gamma\,\mathfrak{f}_{\pm}\mp(1+\hat{G}_0{}^{\pm}\hat{\mathfrak{X}}^{\pm})\\&\times\hat{G}_0{}^{(+)}R_0{}^{\sim}\delta\gamma_0\mathfrak{G}_0{}^{(\pm)}\mathfrak{X}_{\pm}u].\end{aligned}\quad(11.29)$$

There remains only to show the invariance of I, V, and Λ under gauge transformations

$$\delta u=R_0\delta\zeta_0,\quad(11.30)$$

where $\delta\zeta_0$ satisfies $\hat{F}_0\delta\zeta_0=0$. This, however, is an almost immediate consequence of Eqs. (9.38b) and (10.18) and will be left to the reader. An explicit form for the change in the $\mathfrak{f}_{\pm}$'s can be obtained by first decomposing $\delta\zeta_0$ into the v's of Eq. (9.1);

$$\delta\zeta_0=v\delta\lambda,\quad(11.31)$$

where the $\delta\lambda$'s are certain coefficients. Use of (10.9b) and (10.14) then yields

$$\delta\mathfrak{f}_{\pm}=Rg^{\pm}\delta\lambda.\quad(11.32)$$

We note that Eqs. (11.29) and (11.32) both leave the validity of Eqs. (10.10) undisturbed.

12. VACUUM STATES RELATIVE TO THE BACKGROUND FIELD. ABANDONMENT OF THE STRICT OPERATOR FORMALISM. CHRONOLOGICAL PRODUCTS AND TREES

In order to build up the S matrix we begin with the vacuum. The vacuum state is customarily defined by the condition

$$\mathbf{a}^{\pm}|0\rangle=0,\quad(12.1)$$

where the $\mathbf{a}^{\pm}$'s are the annihilation operators of the decomposition (8.12). In this state no field quanta are present, and the background field itself vanishes (flat, empty space-time). It will be noted that no $\pm$ signs have been affixed to the symbol $|0\rangle$, thus implying that

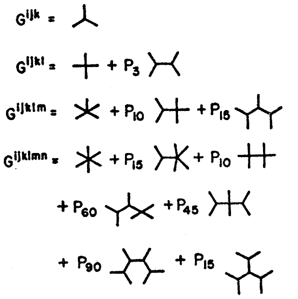

FIG. 1. Graphical representation of the bare n-point functions for $n=3$, 4, 5, 6. The symbol P indicates that the indices associated with the external lines are to be permuted just sufficiently to yield complete symmetry. The numerical subscript indicates the number of permutations required in each case.

if no quanta are initially present none will be produced in the course of time.[29]

Instead of working with $|0\rangle$ we shall find it convenient to work with *relative vacua* $|0,\pm\infty\rangle$, defined by

$$\boldsymbol{\alpha}^{\pm}|0,\pm\infty\rangle=0\,, \tag{12.2}$$

where the annihilation operators $\boldsymbol{\alpha}^{\pm}$ are based on a separation of the total field $\boldsymbol{\varphi}$ into a classical background φ, satisfying the classical field equations (2.2), and a quantum part $\boldsymbol{\phi}$ satisfying the same commutation relations as $\boldsymbol{\varphi}$. The classical background is always assumed to contain a finite amount of "energy" and hence it not only superposes linearly with $\boldsymbol{\phi}$ in the remote past and future, where *both* satisfy the asymptotic field equations (7.5), but it also disperses ultimately to a state of infinite weakness. We may therefore write[30]

$$\boldsymbol{\varphi}=\varphi+\boldsymbol{\phi}\,, \tag{12.3}$$

$$\boldsymbol{\varphi}^{\pm}=\varphi^{\pm}+\boldsymbol{\phi}^{\pm}\,, \tag{12.4}$$

$$\varphi^{\pm}=ua^{\pm}+u^{*}a^{\pm *}+R_0\zeta^{\pm}\,, \tag{12.5}$$

$$\boldsymbol{\phi}^{\pm}=u\boldsymbol{\alpha}^{\pm}+u^{*}\boldsymbol{\alpha}^{\pm *}+R_0(\boldsymbol{\zeta}^{\pm}-\zeta^{\pm})\,, \tag{12.6}$$

$$\mathbf{a}^{\pm}=a^{\pm}+\boldsymbol{\alpha}^{\pm}. \tag{12.7}$$

The states $|0,\pm\infty\rangle$ are functionals of the classical background. Because the background is capable of producing and absorbing pairs, triplets, quadruplets, etc. in individual elementary processes, any number of quanta may eventually be produced, and hence the two states are not identical. Our chief concern will be to study the response of the vacuum-to-vacuum amplitude $\langle 0,\infty|0,-\infty\rangle$ to variations in the background field. Schwinger[31] has used external sources for this purpose and has shown that all physical processes can be computed once the vacuum response itself is known. There is a well-known difficulty, however, in using sources when a non-Abelian invariance group is present, namely, group invariance requires the source to depend on the field. By working with a "free" background field we avoid this difficulty.

Suppose now the background field suffers an infinitesimal change $\delta\varphi$ which satisfies (6.1) so that the field equations are maintained. Since the total field operator $\boldsymbol{\varphi}$ does not depend on which classical field is chosen as the background the operator $\boldsymbol{\phi}$ must suffer (modulo an irrelevant group transformation) an opposite change:

$$\delta\boldsymbol{\phi}=-\delta\varphi\,,\quad \delta\boldsymbol{\alpha}^{\pm}=-\delta a^{\pm}. \tag{12.8}$$

This produces changes in the vacuum states satisfying

$$(\boldsymbol{\alpha}^{\pm}-\delta a^{\pm})(|0,\pm\infty\rangle+\delta|0,\pm\infty\rangle)=0 \tag{12.9}$$

or

$$\boldsymbol{\alpha}^{\pm}\delta|0,\pm\infty\rangle=\delta a^{\pm}|0,\pm\infty\rangle. \tag{12.10}$$

By making use of the orthonormality relations (8.13), (8.14) and the decompositions (12.5), (12.6), it is easy to see that the unitary transformation which yields (12.10) is

$$\delta|0,\pm\infty\rangle=-i\int_{\pm\infty}\boldsymbol{\phi}^{\pm\sim}\overleftrightarrow{s}_0{}^{\mu}\delta\varphi^{\pm}d\Sigma_{\mu}|0,\pm\infty\rangle. \tag{12.11}$$

Hence, remembering that $\boldsymbol{\phi}^{+}=\boldsymbol{\phi}$ and $\varphi^{+}=\varphi$ in the remote future, that $\boldsymbol{\phi}^{-}=\boldsymbol{\phi}$ and $\varphi^{-}=\varphi$ in the remote past, and that $\overleftrightarrow{s}_0{}^{\mu}=\overleftrightarrow{s}^{\mu}$ in both regions, we have

$$\begin{aligned}\delta\langle 0,\infty|0,-\infty\rangle&=i\langle 0,\infty|\left(\int_{\infty}-\int_{-\infty}\right)\boldsymbol{\phi}^{\sim}\overleftrightarrow{s}^{\mu}\delta\varphi d\Sigma_{\mu}|0,-\infty\rangle\\&=i\langle 0,\infty|\boldsymbol{\phi}^{\sim}(\overrightarrow{S}_2-\overleftarrow{S}_2)\delta\varphi|0,-\infty\rangle\\&=-i\delta\varphi^{i}\overrightarrow{S}_{,ij}\langle 0,\infty|\boldsymbol{\phi}^{j}|0,-\infty\rangle.\end{aligned} \tag{12.12}$$

We have now reached a critical point. From here on the description of the quantized Yang-Mills and gravitational fields in terms of operators must be dropped. No one knows (or at any rate no one has yet shown) how to develop a consistent operator language for these

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[29] When dealing with massless bare (i.e., unrenormalized) quanta one must be cautious in asserting that the vacuum is stable. For example, the Lagrangian $\mathfrak{L}=-\frac{1}{2}\varphi_{,\mu}\varphi_{,}{}^{\mu}-\frac{1}{6}\mu\varphi^3-(1/24)\lambda\varphi^4$ $(\lambda>0)$ *appears* to describe a self-coupled massless scalar field satisfying the usual condition $\langle\varphi\rangle=0$ in the vacuum. However, one finds in fact $\langle\varphi\rangle=-3\mu/\lambda$. The Lagrangian should be rewritten $\mathfrak{L}=-\frac{1}{2}(\phi_{,\mu}\phi_{,}{}^{\mu}+m^2\phi^2)+\frac{1}{3}\mu\phi^3-(1/24)\lambda\phi^4$, where $\phi=\varphi-\langle\varphi\rangle$ and $m^2=3\mu^2/2\lambda$, to display the fact that as long as $\mu\neq 0$ the actual quanta carry mass. This result shows up in another way if one attempts to compute the self-decay rate of the quanta on the assumption that they are massless. Because of the possibility of having the momenta of massless quanta all parallel, conservation arguments cannot be invoked to exclude the decay, and, contrary to a widespread impression, phase-space arguments do not suffice but must be investigated in detail. It turns out that the decay rate into softer quanta is infinite. The infinity arises from diagrams with internal lines. Such lines, when not in closed loops, are necessarily on the mass shell. Nevertheless, it is the φ^3 term of the Lagrangian which gives the trouble and *not* the φ^4 term. When $\mu=0$ phase-space limitations prevent the dangerous diagrams from contributing, and the decay rate then vanishes.

In the case of the gravitational field the work of Brill (see Ref. 13) and others shows that flat space-time really is the state of lowest energy, i.e., that bounded asymptotically vanishing deviations from flatness correspond to an *increase* in energy. Similarly the condition $\langle A_{\mu}{}^{\alpha}\rangle=0$ (*modulo* a gauge transformation) holds in the ground state of the Yang-Mills field, as follows from the positiveness of the term $\frac{1}{4}F_{\alpha ij}F^{\alpha ij}$ $(i,\,j=1,2,3)$ in the Hamiltonian when the generating group is compact. As for the stability of the quanta themselves, it turns out that in the graviton case the relevant matrix elements all vanish when the quanta have parallel momenta. The reason is that the coupling is always of the derivative type, which introduces momentum vectors having vanishing contractions with the polarization tensors. In the Yang-Mills case the cubic term in the Lagrangian yields a matrix element which likewise vanishes on account of derivative coupling. The matrix element of the quartic term does not vanish, but in this case phase space limitations prevent the decay. Gravitons and Yang-Mills quanta are therefore both stable.

[30] Because of the linearity of the group transformation law (4.1) ϕ transforms according to the homogeneous law $\delta\phi^i=R^i{}_{\alpha,j}\phi^j\delta\xi^{\alpha}$. From (7.7) it follows that the asymptotic fields transform according to $\delta\phi^{\pm}=-R_0\hat{G}_0{}^{\pm}R_0{}^{\sim}\gamma_0\delta\phi$.

[31] J. Schwinger, Proc. Nat. Acad. Sci. U. S. **37**, 452 (1951).

fields which is at the same time manifestly covariant and useful for calculations.[32]

What we shall do is to retain the operator language only for fields which possess no invariance groups. After developing the theory of such fields to the point at which all statements can be made in c-number language we shall then modify these statements in such a way as to become applicable to the Yang-Mills and gravitational fields.

To achieve maximum simplicity we shall assume not only that the field φ possesses no invariance group but also that its components *all commute with one another at the same space-time point*. In practice this limits us to scalar fields possessing vertex functions $S_{,ijk\cdots}$ which involve no derivative couplings. However, it in no way limits the *number* of scalar fields embraced by the symbol φ^i nor the algebraic complexity of their mutual couplings. Hence the abstract notation is still appropriate, and the combinatorial (i.e., diagrammatic) aspects of the theory are identical with what they will be for the fields of actual interest. It is by studying the combinatorics that we shall be led to a self-consistent general theory.

The chief advantages of the restriction to scalar fields and nonderivative couplings are that the ordering of factors in the field equations becomes immaterial,[33] chronological products can be defined unambiguously, and the operator $\tilde{\mathbf{G}}^{ij}$ which appears in the commutator

$$[\boldsymbol{\phi}^i,\boldsymbol{\phi}^j]=i\tilde{\mathbf{G}}^{ij} \tag{12.13}$$

reduces to the c-number function $\tilde{G}^{ij}$ of the background field when the space-time point associated with the index i is in the immediate vicinity of that associated with j. The latter simplification has the consequence that

$$-i\int_{\Sigma_i}[\boldsymbol{\phi}^i,\boldsymbol{\phi}^j]\overleftrightarrow{S}^{\mu}{}_{jk}\delta\varphi^k d\Sigma_\mu=\delta\varphi^i, \tag{12.14}$$

where Σ_i is any spacelike hypersurface containing the space-time point associated with i.

With these simplifications we are ready to obtain further variational formulas. We first compute

$$\begin{aligned}\delta\langle 0,\infty\,|\,\boldsymbol{\phi}^i|\,0,-\infty\rangle=&\langle 0,\infty\,|\,\delta\boldsymbol{\phi}^i|\,0,-\infty\rangle\\&+\sum(\delta\langle 0,\infty\,|\,\boldsymbol{\phi}'\rangle)\boldsymbol{\phi}'^i\langle\boldsymbol{\phi}'\,|\,0,-\infty\rangle\\&+\sum\langle 0,\infty\,|\,\boldsymbol{\phi}'\rangle\boldsymbol{\phi}'^i\delta\langle\boldsymbol{\phi}'\,|\,0,-\infty\rangle.\end{aligned} \tag{12.15}$$

Here the $|\,\boldsymbol{\phi}'\rangle$ are eigenvectors of the complete set of commuting operators $\boldsymbol{\phi}^j$, including $j=i$, taken over a hypersurface Σ_i, and the summation is to be extended over all the eigenvalues. If the variation (12.15) is due to a change in the background field we have

$$(\boldsymbol{\phi}^j-\delta\varphi^j)(|\,\boldsymbol{\phi}'\rangle+\delta\,|\,\boldsymbol{\phi}'\rangle)=\boldsymbol{\phi}'^j(|\,\boldsymbol{\phi}'\rangle+\delta\,|\,\boldsymbol{\phi}'\rangle), \tag{12.16}$$

or

$$(\boldsymbol{\phi}^j-\boldsymbol{\phi}'^j)\delta\,|\,\boldsymbol{\phi}'\rangle=\delta\varphi^j\,|\,\boldsymbol{\phi}'\rangle, \tag{12.17}$$

where $\boldsymbol{\phi}^j$ is restricted to Σ_i. In view of (12.14), the unitary transformation which yields this is

$$\delta\,|\,\boldsymbol{\phi}'\rangle=-i\int_{\Sigma_i}\boldsymbol{\phi}^{\sim}\overleftrightarrow{S}^{\mu}\delta\varphi d\Sigma_\mu\,|\,\boldsymbol{\phi}'\rangle. \tag{12.18}$$

Making use also of (12.8) and (12.11) we therefore get

$$\begin{aligned}\delta\langle 0,\infty\,|\,\boldsymbol{\phi}^i|\,0,-\infty\rangle=&-\delta\varphi^i\langle 0,\infty\,|\,0,-\infty\rangle-i\delta\varphi^j\vec{S}_{,jk}\\&\times\langle 0,\infty\,|\,T(\boldsymbol{\phi}^i\boldsymbol{\phi}^k)\,|\,0,-\infty\rangle,\end{aligned} \tag{12.19}$$

where T denotes the chronological product.

Since the field φ^i now has no invariance group the operator S_2 is nonsingular, and Eqs. (12.12) and (12.19) may be rewritten in the forms

$$\begin{aligned}\langle 0,\infty\,|\,\boldsymbol{\phi}^i|\,0,-\infty\rangle=&G^{ij}(\delta/i\delta\varphi^j)\\&\times\langle 0,\infty\,|\,0,-\infty\rangle,\end{aligned} \tag{12.20}$$

$$\begin{aligned}\langle 0,\infty\,|\,T(\boldsymbol{\phi}^i\boldsymbol{\phi}^j)\,|\,0,-\infty\rangle=&\left(-iG^{ij}+G^{ik}\frac{\delta}{i\delta\varphi^k}G^{jl}\frac{\delta}{i\delta\varphi^l}\right)\\&\times\langle 0,\infty\,|\,0,-\infty\rangle,\end{aligned} \tag{12.21}$$

the Feynman propagator being used because of the boundary conditions specified by the relative vacua. Continuing in this way we obtain an infinite set of equations, all of which are comprehended in the generating-functional formula

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{i^n}{n!}\lambda_{i_1}\cdots\lambda_{i_n}\langle 0,\infty\,|\,T(\boldsymbol{\phi}^{i_1}\cdots\boldsymbol{\phi}^{i_n})\,|\,0,-\infty\rangle\\=\exp\left(i\sum_{n=2}^{\infty}\frac{1}{n!}\lambda_{i_1}\cdots\lambda_{i_n}G^{i_1\cdots i_n}\right)\\\times\exp\left(\lambda_i G^{ij}\frac{\delta}{\delta\varphi^j}\right)\langle 0,\infty\,|\,0,-\infty\rangle,\end{aligned} \tag{12.22}$$

where the λ's are arbitrary variables and the $G^{i_1\cdots i_n}$ are defined by

$$G^{i_1\cdots i_n}\equiv G^{i_1j_1}\frac{\delta}{\delta\varphi^{j_1}}\cdots G^{i_{n-2}j_{n-2}}\frac{\delta}{\delta\varphi^{j_{n-2}}}G^{i_{n-1}i_n}. \tag{12.23}$$

[32] The most beautiful attempt at such a language is that of S. Mandelstam [Ann. Phys. (N.Y.) **19**, 25 (1962)]. By propagating local frames from infinity along intrinsically defined paths, Mandelstam is able to deal exclusively with operators which are coordinate-invariant and hence possessed of unique commutation relations. Mandelstam's formalism is on the borderline of being practical, but unfortunately becomes excessively complicated beyond all but the simplest calculations. A choice of paths is ultimately equivalent to construction of an explicit gauge, and the freedom to work with local (differential) rather than nonlocal (integral) gauge conditions, is to be preferred if at all attainable.

[33] Under these restrictions the usual practice of "normal ordering" is unnecessary as far as the formal theory is concerned. The residue obtained on converting from ordinary to normal ordering can always be lumped with vertices of lower order.

It is easy to verify that the operators $G^{ij}\delta/\delta\varphi^j$ commute with each other, and from this it follows that the $G^{i_1\cdots i_n}$ are completely symmetric in their indices. These functions, which are known as the *bare n-point functions*, have a well-known graphical representation which is illustrated in Fig. 1 for the cases $n=3, 4, 5, 6$. Feynman propagators are represented by lines and bare vertex functions S_m by vertices or forks with m prongs. The lines are joined together at vertices in the same ways that the propagators in the explicit expressions for the $G^{i_1\cdots i_n}$'s are coupled to vertex functions by dummy indices. It is easy to see that the diagrams making up $G^{i_1\cdots i_n}$ are obtained from those for $G^{i_1\cdots i_{n-1}}$ by inserting an additional external line in all possible ways. $G^{i_1\cdots i_n}$ is therefore expressible as the sum of all distinct trees having n branches, the indices attached to the latter being permuted just sufficiently to yield complete symmetry.

A *tree* is any diagram which has no disconnected parts but which is divided into two disconnected parts by cutting any line. A tree therefore possesses no closed loops. We shall see that the first factor on the right-hand side of (12.22) describes all the lowest-order or *bare* scattering processes. The radiative corrections, which involve closed loops, are all contained in the remaining factors.

13. DEFINITION OF THE S MATRIX. ITS STRUCTURE IN THE ABSENCE OF AN INVARIANCE GROUP

The S matrix, like the vacuum states, may be defined relative to the background field. It then has as elements the amplitudes

$$\langle A_1'\cdots A_n',\infty\,|A_1\cdots A_m,-\infty\rangle,$$

where

$$|A_1\cdots A_n,\pm\infty\rangle \equiv \mathbf{a}^{\pm}{}_{A_1}{}^*\cdots\mathbf{a}^{\pm}{}_{A_n}{}^*|0,\pm\infty\rangle. \tag{13.1}$$

If the possibility of stable composite structures is ignored, the above states form two complete orthogonal bases in the physical Hilbert space, and the scattering amplitudes may be regarded as the matrix elements, *with respect to either basis*, of the unitary operator

$$\mathbf{S}\equiv\sum_{n=0}^{\infty}|A_1\cdots A_n,-\infty\rangle\frac{1}{n!}\langle A_1\cdots A_n,\infty\,|\,. \tag{13.2}$$

Here an implicit summation-integration is to be understood over the repeated A's. It is easily verified that $\mathbf{S}$ satisfies

$$\mathbf{B}^+=\mathbf{S}^{-1}\mathbf{B}^-\mathbf{S}, \tag{13.3}$$

where the $\mathbf{B}$'s are any asymptotic invariants [cf. (7.10)];

$$\mathbf{B}^{\pm}\equiv I_B\boldsymbol{\phi}^{\pm}. \tag{13.4}$$

By the standard Lehmann-Symanzik-Zimmermann (LSZ) method one can show that the scattering amplitudes are given by

$$\langle A_1'\cdots A_n',\infty\,|A_1\cdots A_m,-\infty\rangle,$$
$$=\sum_{l=0}^{\infty}P_{(m,n;\,l)}\delta_{A_1'A_1}\cdots\delta_{A_l'A_l}$$
$$\times(A_{l+1}'\cdots A_n',\infty\,|A_{l+1}\cdots A_m,-\infty), \tag{13.5}$$

where[34]

$$\begin{aligned}(A_1'\cdots A_n',\infty\,|A_1\cdots A_m,-\infty)&\\ \equiv(-i)^{m+n}u^{j_1}{}_{A_1'}{}^*\cdots u^{j_n}{}_{A_n'}{}^*\overrightarrow{S}_{,j_1l_1}{}^0\cdots\overrightarrow{S}_{,j_nl_n}{}^0&\\ \times\langle0,\infty\,|T(\boldsymbol{\phi}^{l_1}\cdots\boldsymbol{\phi}^{l_n}\boldsymbol{\phi}^{k_1}\cdots\boldsymbol{\phi}^{k_m})|0,-\infty\rangle&\\ \times\overleftarrow{S}_{,k_1i_1}{}^0\cdots\overleftarrow{S}_{,k_mi_m}{}^0u^{i_1}{}_{A_1}\cdots u^{i_m}{}_{A_m},&\end{aligned} \tag{13.6}$$

and where the symbol P in (13.5) indicates that the expression following it is to be summed over all distinct permutations of the A's and A''s, the subscript

$$(m,n;l)\equiv\frac{m!n!}{(m-l)!(n-l)!} \tag{13.7}$$

denoting the number of permutations required in each case. It is important to realize that the LSZ method is formally applicable even when an invariance group is present, and hence Eqs. (13.5) and (13.6) hold in the general case. This is because the creation and annihilation operators, in virtue of (8.13), (8.14), and (12.6), are unambiguously defined by

$$\mathbf{a}^{\pm*}=i\int_{\pm\infty}u^{\sim}\overleftrightarrow{s_0}^{\mu}\boldsymbol{\phi}d\Sigma_\mu,\quad \mathbf{a}^{\pm}=-i\int_{\pm\infty}u^{\dagger}\overleftrightarrow{s_0}^{\mu}\boldsymbol{\phi}d\Sigma_\mu. \tag{13.8}$$

The only difficulty is that we do not yet know, in the general case, how to calculate the chronological products appearing in (13.6).

In the restricted case of scalar fields with nonderivative couplings Eqs. (12.22), (13.5), and (13.6) permit us to write the following compact expressions for the scattering operator:

$$\mathbf{S}=:\sum_{n=2}^{\infty}\frac{(-i)^n}{n!}\boldsymbol{\phi}^{\pm i_1}\cdots\boldsymbol{\phi}^{\pm i_n}\overrightarrow{S}_{,i_1j_1}{}^0\cdots\overrightarrow{S}_{,i_nj_n}{}^0$$
$$\times\langle0,\infty\,|T(\boldsymbol{\phi}^{j_1}\cdots\boldsymbol{\phi}^{j_n})|0,-\infty\rangle: \tag{13.9a}$$

$$=:\exp\left(i\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}\boldsymbol{\phi}^{\pm i_1}\cdots\boldsymbol{\phi}^{\pm i_n}\overrightarrow{S}_{,i_1j_1}{}^0\cdots\overrightarrow{S}_{,i_nj_n}{}^0G^{j_1\cdots j_n}\right)$$
$$\times\exp\left(-\boldsymbol{\phi}^{\pm i}\overrightarrow{S}_{,ij}{}^0G^{jk}\frac{\delta}{\delta\varphi^k}\right)\langle0,\infty\,|0,-\infty\rangle:. \tag{13.9b}$$

[34] In Eq. (13.6) the reader should remember to insert a renormalization factor $Z^{-1/2}$ with each wave function u. (See Ref. 22). An alternative procedure is to choose for the action functional S the "pre-renormalized" action, in which all "counter terms" have been inserted in advance. The operators $S_{,ij}{}^0$ and wave functions u will then automatically contain the necessary Z factors in (13.6) without the need for making them explicit.

The colons indicate that the creation and annihilation operators making up the $\phi^{\pm}$'s are to be normal-ordered. An alternative and very useful version of (13.9b) is

$$(A_1'\cdots A_n',\infty\,|A_1\cdots A_n,-\infty)$$
$$=\Bigg[\frac{\delta}{\delta\alpha_{A_1'}{}^*}\cdots\frac{\delta}{\delta\alpha_{A_n'}{}^*}\frac{\delta}{\delta\alpha_{A_1}}\cdots\frac{\delta}{\delta\alpha_{A_m}}$$
$$\times\exp\Big(\tfrac{1}{2}iX_{ij}\phi_{00}{}^i\phi_{00}{}^j+i\sum_{n=3}^{\infty}\frac{1}{n!}t_{i_1\cdots i_n}\phi_0{}^{i_1}\cdots\phi_0{}^{i_n}\Big)$$
$$\times\exp\Big(\phi_0{}^i\frac{\delta}{\delta\varphi^i}\Big)\langle 0,\infty\,|0,-\infty\rangle\Bigg]_{\alpha=\alpha^*=0},\quad(13.10)$$

where the t's, which will be called *tree functions*, are the bare n-point functions with their external lines removed:

$$t_{i_1\cdots i_n}\equiv(-1)^nS_{,i_1j_1}\cdots S_{,i_nj_n}G^{j_1\cdots j_n},\quad(13.11)$$

and where

$$\phi_{00}\equiv u\alpha+u^*\alpha^*,\quad(13.12)$$

$$\phi_0\equiv(1+G_0X)\phi_{00}\equiv f\alpha+\bar{f}\alpha^*,\quad(13.13)$$

the f's being the functions defined by (10.8) and the $\bar{f}$'s the corresponding functions with u replaced by u^*.

The above expressions can also be used to obtain the hierarchy of conditions on the scattering amplitudes which follow from the unitarity of the S matrix. Thus, inserting (13.9a) into $\mathsf{S}^\dagger\mathsf{S}=1$ and reordering operators into normal products, one finds

$$\sum_{n=0}^{\infty}\frac{1}{n!}\langle\phi^{i_1}\cdots\phi^{i_n}\rangle^*c_{i_1j_1}\cdots c_{i_nj_n}\langle\phi^{j_1}\cdots\phi^{j_n}\rangle=e^{-i(W-W^*)},\quad(13.14)$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{l=0}^{m}(-1)^lP_{(m;l)}\langle\phi^{i_1}\cdots\phi^{i_n}\phi^{k_1}\cdots\phi^{k_l}\rangle^*$$
$$\times c_{i_1j_1}\cdots c_{i_nj_n}\langle\phi^{j_1}\cdots\phi^{j_n}\phi^{k_{l+1}}\cdots\phi^{k_m}\rangle=0,$$
$$m=1,2,\cdots,\quad(13.15)$$

where the permutation sum $P_{(m;l)}$ is over the $(m;l)\equiv m!/(m-l)!l!$ distinct arrangements of the k's, and where

$$\langle\phi^{i_1}\cdots\phi^{i_n}\rangle\equiv\frac{\langle 0,\infty\,|T(\phi^{i_1}\cdots\phi^{i_n})|0,-\infty\rangle}{\langle 0,\infty\,|0,-\infty\rangle},\quad(13.16)$$

$$c\equiv\overleftarrow{S}_2{}^0uu^\dagger\overrightarrow{S}_2{}^0,\quad(13.17)$$

$$W\equiv-i\ln\langle 0,\infty\,|0,-\infty\rangle.\quad(13.18)$$

The equation $\mathsf{S}\mathsf{S}^\dagger=1$ leads to identical conditions.

We may note that Eqs. (13.15) are not independent of (13.14) but can be obtained from it by functional differentiation. For this reason it suffices to verify (13.14) alone in order to check that the formalism yields an S matrix which is unitary. This is one of the important advantages of working with an arbitrary background field.

14. THE S MATRIX IN THE PRESENCE OF AN INVARIANCE GROUP. THE TREE THEOREM

Consider the operator $\exp(\phi_0{}^i\delta/\delta\varphi^i)$ appearing in Eq. (13.10). By taking into account the fact that $\phi_0{}^i$ depends on the background field through its dependence on the f's of Eq. (13.13), it is not difficult to show that the effect of this operator, when acting on any functional of the background field φ, is to replace φ by $\varphi+\phi$ where ϕ is obtained by iteration of

$$\phi^i=\phi_0{}^i+G^{ij}\sum_{n=2}^{\infty}\frac{1}{n!}S_{,ji_1\cdots i_n}\phi^{i_1}\cdots\phi^{i_n}.\quad(14.1)$$

Equation (14.1) has three remarkable properties. First, its iterated solution yields, as coefficients, all of the tree functions:

$$\phi^i=\phi_0{}^i+G^{ij}\sum_{n=2}^{\infty}\frac{1}{n!}t_{ji_1\cdots i_n}\phi_0{}^{i_1}\cdots\phi_0{}^{i_n}.\quad(14.2)$$

Second, if φ satisfies the classical field equations then so does $\varphi+\phi$. Third, and most important, the second property holds *even in the presence of an invariance group*, provided the definition (13.13) is generalized to

$$\phi_0\equiv\mathfrak{f}_\pm\alpha+\mathfrak{f}_\pm{}^{(*)}\alpha^*+R\zeta,\quad(14.3)$$

where the $\mathfrak{f}_\pm$ are the functions (10.9), the $\mathfrak{f}_\pm{}^{(*)}$ are obtained from these by replacing u by u^*, and ζ is arbitrary.[35]

The first property may be verified by straightforward iteration and term-by-term comparison. The second property is obvious; the third, however, requires special discussion.

We first rewrite Eq. (14.1) in the form

$$\phi=\phi_0+G[\varphi](S_1[\varphi+\phi]-S_2[\varphi]\phi),\quad(14.4)$$

in which the functional dependence of the various factors is made explicit. Then we note that the relations

$$0=S_2[\varphi]\phi_0,\quad(14.5)$$

$$0=S_2[\varphi]R[\varphi],\quad(14.6)$$

$$0\equiv R^\sim[\varphi]S_1[\varphi],\quad(14.7)$$

$$0\equiv R^\sim[\varphi+\phi]S_1[\varphi+\phi]$$
$$\equiv(R^\sim[\varphi]+R_1{}^\sim\phi)S_1[\varphi+\phi],\quad(14.8)$$

[35] Since $\mathfrak{f}_+$ and $\mathfrak{f}_-$ differ from one another by a gauge transformation [Eq. (10.23)] we do not bother to put $\pm$ signs on ϕ_0. The difference can always be absorbed into the term $R\zeta$.

permit us to write

$$\begin{aligned}0&=S_2[\varphi]\{\phi-\phi_0-G[\varphi](S_1[\varphi+\phi]-S_2[\varphi]\phi)\}\\&=S_2[\varphi]\phi-(F[\varphi]-\gamma[\varphi]R[\varphi]\tilde{\gamma}^{-1}[\varphi]R^{\sim}[\varphi]\gamma[\varphi])\\&\qquad\times G[\varphi](S_1[\varphi+\phi]-S_2[\varphi]\phi)\\&=S_1[\varphi+\phi]+\gamma[\varphi]R[\varphi]\hat{G}[\varphi]R^{\sim}[\varphi]S_1[\varphi+\phi]\\&=(1-\gamma[\varphi]R[\varphi]\hat{G}[\varphi]R_1{}^{\sim}\phi)S_1[\varphi+\phi],\end{aligned}\tag{14.9}$$

in which the analog of (6.11), with $G^{\pm}$ replaced by G, has been used. The factor in parentheses in the final expression is generally nonsingular. Hence it may be removed, yielding the desired result

$$S_1[\varphi+\phi]=0.\tag{14.10}$$

It is to be emphasized that this result depends in no way on the choice of γ's used in the definition of the Green's function G. In fact we can show that a change in the γ's produces only a group transformation of the ϕ's, of the form

$$\delta\phi=R[\varphi+\phi]\delta\xi.\tag{14.11}$$

We first take the variation of Eq. (14.4) and rearrange the result in the form

$$\begin{aligned}\delta\phi_0=\delta\phi-\delta G[\varphi](S_1[\varphi+\phi]-S_2[\varphi]\phi)\\-G[\varphi](S_2[\varphi+\phi]-S_2[\varphi])\delta\phi.\end{aligned}\tag{14.12}$$

We then insert (11.21) into (9.19) and make use of the analog of (6.11) to obtain

$$\delta G=G\delta\gamma R\hat{G}R^{\sim}-R\hat{G}\delta\tilde{\gamma}\hat{G}R^{\sim}+R\hat{G}R^{\sim}\delta\gamma G.\tag{14.13}$$

Next we remember that

$$S_2[\varphi+\phi]R[\varphi+\phi]=0,\tag{14.14}$$

which results from functional differentiation of (14.8) and use of (14.10). Finally we note that the operator in the final parentheses in (14.12) can act in either direction. This is because of the fundamental assumption which is always implicit in the use of decompositions of the form (14.3), namely that the α's are such as to give ϕ_0 the character of a wave packet. The difference between $S_2[\varphi+\phi]$ and $S_2[\varphi]$ therefore vanishes sufficiently rapidly at infinity to make reversal possible.

Writing $R[\varphi+\phi]=R+R_1\phi$, and making use of (14.4), (14.6), (14.11), (14.13), and (14.14), we now have

$$\begin{aligned}\delta\phi_0&=\delta\phi-R\hat{G}R^{\sim}\delta\gamma G(S_1[\varphi+\phi]-S_2\phi)\\&\qquad-G(\overleftarrow{S}_2[\varphi+\phi]-\overleftarrow{S}_2)\delta\phi\\&=R\delta\xi+R_1\phi\delta\xi-R\hat{G}R^{\sim}\delta\gamma(\phi-\phi_0)\\&\qquad+G(\overleftarrow{F}-\gamma R\tilde{\gamma}^{-1}R^{\sim}\gamma)R_1\phi\delta\xi\\&=R(1-\hat{G}R^{\sim}\gamma R_1\phi)\delta\xi-R\hat{G}R^{\sim}\delta\gamma(\phi-\phi_0).\end{aligned}\tag{14.15}$$

But from (11.29) and (14.3) we have

$$\delta\phi_0=R(\hat{G}\pm\hat{G}^{(\pm)})R^{\sim}\delta\gamma(\phi_0-R\zeta)+R\delta\zeta',\tag{14.16a}$$

$$\begin{aligned}\delta\zeta'\equiv\delta\zeta\mp(1+\hat{G}_0{}^{\pm}\hat{X}^{\pm})[\hat{G}_0{}^{(+)}R_0{}^{\sim}\delta\gamma_0\mathfrak{G}_0{}^{(\pm)}\mathfrak{X}_{\pm}u\alpha\\+\hat{G}_0{}^{(-)}R_0{}^{\sim}\delta\gamma_0\mathfrak{G}_0{}^{(\mp)}\mathfrak{X}_{\pm}{}^*u^*\alpha^*],\end{aligned}\tag{14.16b}$$

where $\delta\zeta$ is any change in ζ which one may wish to include along with the change in γ. Equating the right-hand sides of (14.15) and (14.16) we therefore get

$$\begin{aligned}\delta\xi=(1-\hat{G}R^{\sim}\gamma R_1\phi)^{-1}(\hat{G}R^{\sim}\delta\gamma\phi\pm\hat{G}^{(\pm)}R^{\sim}\delta\gamma\phi_0\\-\hat{G}^{\pm}R^{\sim}\delta\gamma R\zeta+\delta\zeta').\end{aligned}\tag{14.17}$$

It is straightforward to show in a similar manner that the gauge transformation (11.30) in the u's also produces a change in ϕ of the form (14.11), with $\delta\xi$ given in this case by

$$\delta\xi=(1-\hat{G}R^{\sim}\gamma R_1\phi)^{-1}(g^{\pm}\delta\lambda\alpha+g^{\pm*}\delta\lambda^*\alpha^*).\tag{14.18}$$

In both cases we can rewrite (14.11) in the form

$$\delta(\varphi+\phi)=R[\varphi+\phi]\delta\xi,\tag{14.19}$$

since the background field remains unaffected.

We may ask what happens if the background field itself suffers a group transformation. Here it is convenient to assume that the ζ of (14.3) transforms according to the adjoint representation of the group; any portion of it which does not transform in this way can be lumped with the $\delta\zeta$ of (14.17). It then follows from (4.9) and (11.15) that ϕ_0 suffers the transformation

$$\delta\phi_0{}^i=R^i{}_{\alpha,j}\phi_0{}^j\delta\xi^\alpha.\tag{14.20}$$

The tree functions, on the other hand, transform in a contragredient fashion, i.e., in precisely the manner indicated by the downward position of their indices. This is because they are built from Feynman propagators and bare vertex functions by simple contractions of indices, and because we have taken care to construct the propagators in a manifestly covariant way. From this and Eq. (14.2) it follows that ϕ transforms like ϕ_0:

$$\delta\phi^i=R^i{}_{\alpha,j}\phi^j\delta\xi^\alpha.\tag{14.21}$$

Hence

$$\delta(\varphi^i+\phi^i)=R^i{}_\alpha\delta\xi^\alpha+R^i{}_{\alpha,j}\phi^j\delta\xi^\alpha=R^i{}_\alpha[\varphi+\phi]\delta\xi^\alpha,\tag{14.22}$$

which has again the form (14.19).

We now have the following lemma: *If $A[\varphi]$ is any invariant functional of the background field then $A[\varphi+\phi]$ remains completely unchanged under all the invariance transformations of the theory.*

The above results suggest that when an invariance group is present Eq. (13.10) for the S-matrix amplitudes may be generalized to

$$\begin{aligned}(A_1{}'\cdots A_n{}',\infty\,|A_1\cdots A_m,-\infty)=\Bigg[\frac{\delta}{\delta\alpha_{A_1{}'}{}^*}\cdots\frac{\delta}{\delta\alpha_{A_n{}'}{}^*}\\\times\frac{\delta}{\delta\alpha_{A_1}}\cdots\frac{\delta}{\delta\alpha_{A_m}}\exp\Big(\tfrac{1}{2}i\mathfrak{X}_{\pm ij}\phi_{00}{}^i\phi_{00}{}^j+i\sum_{n=3}^{\infty}\frac{1}{n!}\\\times t_{i_1\cdots i_n}\phi_0{}^{i_1}\cdots\phi_0{}^{i_n}+iW[\varphi+\phi]\Big)\Bigg]_{\alpha=\alpha^*=0}\mathfrak{F},\end{aligned}\tag{14.23}$$

where W is defined by (13.18) and where

$$\phi_{00} \equiv u\alpha + u^*\alpha^* + R_0\zeta_0, \qquad (14.24)$$

ζ_0 being an arbitrary gauge parameter satisfying $\hat{F}_0\zeta_0 = 0$.

The demonstration that this is indeed the case is a task which falls into two parts. First, we must obtain an explicit form for $W[\varphi]$ which, because the vacuum-to-vacuum amplitude is a physical observable, must be invariant under changes in the γ's as well as under group transformations of φ. Second, we must verify the group invariance of (14.23) itself. The first of these tasks is the most difficult and will be carried out in subsequent sections. Here we accomplish the second.

In view of the lemma stated above, group invariance of the term $W[\varphi+\phi]$ in (14.23) follows from the invariance of $W[\varphi]$ itself. Invariance of the term in $\mathfrak{X}_\pm$ follows from the invariance of the amplitudes I, V, and Λ, which has been proved earlier. Only the terms involving the tree functions require further investigation.

These terms are manifestly invariant under group transformations of the background field. We may remark that because the tree functions are obtained by iteration of Eq. (14.1), which involves the ordinary Feynman propagator, it is the ordinary Feynman propagator which is used for the internal lines of the tree diagrams. However, because of the transformation law (11.12) the invariance of the tree terms would not be spoiled if the functions $\mathfrak{G}_\pm$ were substituted for G. As a matter of fact, it can be shown that this substitution leaves the tree terms unaffected, and that although the propagator G is the most convenient one to use in practical calculations, the propagator $\mathfrak{G}_\pm$ *could* be used for the internal lines instead.[36]

In order to show that the tree terms are also invariant under changes in ϕ and ϕ_0 of the form (14.11), (14.16), etc., we observe that in virtue of (14.2) and (14.4) we may write

$$\sum_{n=3}^{\infty} \frac{1}{(n-1)!} t_{i_1\cdots i_n}\phi_0^{i_1}\cdots\phi_0^{i_n} = -\phi_0{}^\sim \vec{F}(\phi-\phi_0)$$
$$= \phi_0{}^\sim (S_1[\varphi+\phi] - S_2\phi). \qquad (14.25)$$

The variation of the right-hand side of this equation has the form

$$\delta\phi_0{}^\sim(S_1[\varphi+\phi]-S_2\phi) + \phi_0{}^\sim(\overleftarrow{S}_2[\varphi+\phi] - \overleftarrow{S}_2)\delta\varphi,$$

in which use has once again been made of the reversibility of the operator $S_2[\varphi+\phi]-S_2[\varphi]$. Using Eqs. (14.10) and (14.14), as well as the equations $\phi_0{}^\sim\overleftarrow{S}_2 = 0$ and $\delta\phi_0{}^\sim S_2 = 0$, the latter of which holds under (11.25) and (11.29), we see that this variation vanishes. Since the α's in the decomposition (14.3) are completely arbitrary it follows the *every term in the sum on the left of* (14.25) *is fully group-invariant*:

$$\delta(t_{i_1\cdots i_n}\phi_0^{i_1}\cdots\phi_0^{i_n}) = 0. \qquad (14.26)$$

This result is known as the *tree theorem*.[37]

The tree theorem provides a very useful check on the accuracy of lowest-order scattering calculations. One simply replaces any one of the external-line wave functions by R and looks to see if the resulting amplitude vanishes. Since scattering calculations involve lengthy algebraic expressions, mistakes are often discovered in this way. In applying the test it is important to remember that *all* the diagrams which go to make up a given tree amplitude must be added together. They are not individually invariant.[38]

15. LORENTZ INVARIANCE. INVARIANCE UNDER CHANGE OF VARIABLES. QUANTUM VERSUS CLASSICAL SCATTERING

Space-time in S-matrix theory is assumed to be asymptotically flat. A flat space-time has group-theoretical properties not possessed by a general manifold, namely Lorentz invariance. In S-matrix theory the Poincaré group must appear as an asymptotic invariance group.[39]

If the zero point of the gravitational field were chosen differently in this paper—corresponding to a manifold with some other group of isometries—then the formalism would have a different appearance, since the pertinent physical questions to be asked would not involve the scattering of plane waves but something else instead. It would still be necessary to make an independent check of the theory for invariance with respect to the underlying isometry group, because the origin of such a group—in particular, of the Poincaré group—is distinct from general coordinate invariance.

It is quite easy to verify the Lorentz invariance of the present theory, much easier than it would be to check invariance under any other asymptotic or underlying symmetry group. This is because, with the usual choices of field variables (Table I), Lorentz invariance is

[36] Since the functions $\mathfrak{G}_\pm$ are not symmetric there is a problem of relative orientation of the internal lines. The orientation must be that which results from the iteration of (14.1) with G replaced by $\mathfrak{G}_\pm$. (See Ref. 46).

[37] R. P. Feynman, Acta Phys. Polonica **24**, 697 (1963).

[38] The test is usually carried out in momentum space. Since the wave-packet assumption is implicit in the Fourier transformation process it is then no longer necessary to worry about conditions of reversibility of the order of various operations. In fact, the whole test reduces to an algebraic exercise.

[39] It has been pointed out by Sachs that the asymptotic invariance group of gravity is actually much bigger than the Poincaré group. [See R. K. Sachs, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach Science Publishers, Inc., New York, 1964).] We make no attempt here to investigate this larger group, the existence of which seems to be related to certain conformal invariance properties of the theory. We remark, however, that such an investigation might yield important new insights into the properties of S-matrix amplitudes.

manifest in both the Yang-Mills and gravitational cases. The only point which really needs checking is the invariance of the theory under changes in the time-like unit vector n_μ which is used to define the bicharacteristic $\bar{p}_\mu$ of Eq. (8.25) and the asymptotic wave functions u. But we have already seen from Eq. (8.26) that changes in n_μ lead to changes in the u's which are compounded of (1) gauge transformations of the form (11.27), which have previously been shown to leave the theory invariant, and (2) phase transformations. The phase transformations alter the scattering amplitudes only by phase factors and leave the probabilities themselves unchanged. Therefore, as long as we use helicity assignments for the initial and final states the theory is indeed Lorentz-invariant.

The following question, however, arises: Suppose we were to replace the basic field variables of the theory by arbitrary nonlinear functions (or local functionals) of themselves. Would we then still arrive at the same quantum theory by the methods outlined here, even though such a change of variables would generally destroy the manifest covariance? In particular, would the scattering amplitudes remain unchanged?

We must remark that not all nonlinear transformations destroy manifest covariance. For example, in the case of gravity the change of variables $\varphi_{\mu\nu}\to\varphi'_{\mu\nu}$ or $\varphi_{\mu\nu}\to\varphi'^{\mu\nu}$, where $\varphi'_{\mu\nu}=g^{-s}g_{\mu\nu}-\eta_{\mu\nu}$, $\varphi'^{\mu\nu}=g^s g^{\mu\nu}-\eta^{\mu\nu}$, $s\neq\frac{1}{4}$, affects neither the manifest Lorentz invariance nor the linearity of the general coordinate transformation laws. However, we need not consider these cases separately, as it is just as easy to consider the general case directly.

It is not difficult to see that a change $\varphi^i\to\varphi'^i$ from one set of basic field variables to another produces the following changes in the various quantities appearing in the theory[40]:

$$S_{,i}'\equiv S_{,j}\frac{\delta\varphi^j}{\delta\varphi'^i}=0\,, \tag{15.1}$$

$$S_{,ij}'\equiv S_{,kl}\frac{\delta\varphi^k}{\delta\varphi'^i}\frac{\delta\varphi^l}{\delta\varphi'^j}+S_{,k}\frac{\delta^2\varphi^k}{\delta\varphi'^i\delta\varphi'^j}=S_{,kl}\frac{\delta\varphi^k}{\delta\varphi'^i}\frac{\delta\varphi^l}{\delta\varphi'^j}\,, \tag{15.2}$$

$$R'^i{}_\alpha=\frac{\delta\varphi'^i}{\delta\varphi^j}R^j{}_\alpha\,, \tag{15.3}$$

$$u'^i{}_A=\left(\frac{\delta\varphi'^i}{\delta\varphi^j}\right)_0 u^j{}_A\,, \tag{15.4}$$

$$\phi'^i=\frac{\delta\varphi'^i}{\delta\varphi^j}\phi^j+\frac{1}{2!}\frac{\delta^2\varphi'^i}{\delta\varphi^j\delta\varphi^k}\phi^j\phi^k+\cdots. \tag{15.5}$$

[40] For convenience it will be assumed that $\varphi'^i=0$ when $\varphi^i=0$ and that the transformation is one-to-one analytic at the zero point so that series such as (15.5) have a nonvanishing domain of convergence.

That these changes must leave invariant the term in $\mathfrak{X}_\pm$ of the amplitude (14.23) follows from the fact that $\mathfrak{X}_\pm$ refers to disturbances which propagate without mutual interaction. The theory of such disturbances is identical with that of infinitesimal disturbances on the background field, and it does not matter what background variables are chosen to represent them. This reasoning also leads to the simple transformation laws

$$\mathfrak{f}_\pm'^i{}_A=\frac{\delta\varphi'^i}{\delta\varphi^j}\mathfrak{f}_\pm{}^j{}_A\,, \tag{15.6}$$

$$\phi_0'^i=\frac{\delta\varphi'^i}{\delta\varphi^j}\phi_0{}^j\,, \tag{15.7}$$

provided we require γ_{ij} to transform like $S_{,ij}$ [Eq. (15.2)] so that the Feynman propagator suffers the change[41]

$$G'^{ij}=\frac{\delta\varphi'^i}{\delta\varphi^k}\frac{\delta\varphi'^j}{\delta\varphi^l}G^{kl}. \tag{15.8}$$

Less obvious is the invariance of the tree terms. This is because the bare vertex functions, and hence the tree functions, unlike $\phi_0{}^i$, do not transform in a simple fashion. [See Eqs. (19.29), (19.30), and (19.31).] It is nevertheless true that when the tree functions are multiplied by ϕ_0's, as in (14.23) or (14.25), the result is invariant. To see this we note that the right-hand side of Eq. (14.25), in terms of the new variables, becomes

$$\phi_0'^{\sim}(S_1'[\varphi'+\phi']-S_2'\phi')=-\phi^{0\sim\prime}S_2'\phi'$$
$$=-\phi_0{}^i\overleftrightarrow{S}_{,ij}\left(\phi^j+\frac{1}{2!}\frac{\delta\varphi^j}{\delta\varphi'^k}\frac{\delta^2\varphi'^k}{\delta\varphi^l\delta\varphi^m}\phi^l\phi^m+\cdots\right). \tag{15.9}$$

Since the wave-packet assumption is always implicit, the nonlinear terms in the ϕ's inside the parentheses vanish at infinity rapidly enough so that *for them* the arrow on $S_{,ij}$ may be reversed. Expression (15.9) therefore reduces immediately to (14.25), and we have, for all $n\geq 3$,

$$t'_{i_1\cdots i_n}\phi_0'^{i_1}\cdots\phi_0'^{i_n}=t_{i_1\cdots i_n}\phi_0{}^{i_1}\cdots\phi_0{}^{i_n}, \tag{15.10}$$

a result which, in each individual case, can also be verified by a straightforward but nontrivial computation.

There remains to be discussed only the term in W. Since W is a physical observable its value must remain unaffected by changes in the mode of description of the field. Its functional form must therefore adjust in such a way that

$$W'[\varphi']=W[\varphi], \tag{15.11a}$$

which, together with (15.5), implies

$$W'[\varphi'+\phi']=W[\varphi+\phi]. \tag{15.11b}$$

[41] Any other transformation law for γ_{ij} would simply add a gauge term to (15.6) and (15.7). The γ invariance of the theory has already been demonstrated.

We cannot, however, give a *proof* of this since we do not yet possess a formal prescription for constructing W out of the basic building blocks of the theory, viz., the bare vertex functions and Green's functions. What we shall in fact do is *use* (15.11) as one of several interlocking requirements which will ultimately serve to define W in a unique manner. It turns out that (15.11) leads to a rather interesting and previously unknown result which can be translated into the q-number language as follows: When operator field equations exist (e.g., when no invariance group is present) they must necessarily contain *nonlocal* terms, which vanish in the classical limit $\hbar\to 0$, in order that the theory be invariant under changes of variables. We shall discuss later the reasons why such terms are not normally considered.

The reader will have noted the ease with which fundamental theorems may be proved now that the theory has been expressed completely in c-number language. The c-number language has also the effect of emphasizing similarities between the classical and quantum theories of wave scattering. From a classical point of view the function ϕ represents a finite disturbance on a background φ, and the tree functions describe the self-scattering which it suffers. The differences between the classical and quantum theories arise from the existence, in the latter, of the radiative correction term $W[\varphi+\phi]$, which has no counterpart in the classical theory, and from the fact that it is not the retarded or advanced Green's function which is used but the Feynman propagator, with the result that ϕ is complex instead of real.

16. FIRST APPROXIMATION TO THE VACUUM-TO-VACUUM AMPLITUDE. PROOF OF ITS GROUP INVARIANCE

We come now to the most difficult part of the theory; the determination of the functional W which describes all radiative corrections or so-called vacuum processes. We do this first for a fictitious system defined by the action functional $\frac{1}{2}S_{,ij}\phi^i\phi^j$ and then later extend the results to the real system. It is clear, from the point of view of perturbation theory, that the fictitious system provides a first approximation to the real system. This approximation will be denoted by the subscript (1).

Since the quanta of the fictitious system do not interact with one another the tree functions all vanish, and the scattering operator reduces to

$$\mathbf{S}_{(1)}=:\exp(iW_{(1)}+\tfrac{1}{2}i\boldsymbol{\phi}^{\pm\sim}\mathfrak{X}_{\pm}\boldsymbol{\phi}^{\pm}): \qquad (16.1\text{a})$$

$$=:\exp(iW_{(1)}+i\boldsymbol{\alpha}^{\pm\dagger}I\boldsymbol{\alpha}^{\pm}+\tfrac{1}{2}i\boldsymbol{\alpha}^{\pm\dagger}V\boldsymbol{\alpha}^{\pm *}+\tfrac{1}{2}i\boldsymbol{\alpha}^{\pm\sim}\Lambda\boldsymbol{\alpha}^{\pm}):, \qquad (16.1\text{b})$$

which is obtained by reexpressing (14.23) in the format of (13.9b). The functional $W_{(1)}$ will be determined by the requirement that $\mathbf{S}_{(1)}$ be unitary.

The $\pm$ signs in (16.1b) are irrelevant and may be dropped. Introducing right and left eigenvectors of the $\boldsymbol{\alpha}$'s and $\boldsymbol{\alpha}^*$'s, respectively, we may then write (assuming $\langle 0|0\rangle=1$)

$$\langle\alpha^\dagger|\mathbf{S}_{(1)}{}^\dagger\mathbf{S}_{(1)}|\alpha\rangle=e^{i(W_{(1)}-W_{(1)}{}^*)}\langle\alpha^\dagger|\alpha\rangle\langle 0|\mathbf{F}|0\rangle, \qquad (16.2)$$

where α and $\alpha^\dagger$ are the eigenvalues and where

$$\mathbf{F}\equiv\exp[-i\boldsymbol{\alpha}^\dagger I^\dagger(\boldsymbol{\alpha}+\alpha)-\tfrac{1}{2}i(\boldsymbol{\alpha}^\sim+\alpha^\sim)V^\dagger(\boldsymbol{\alpha}+\alpha)-\tfrac{1}{2}i\boldsymbol{\alpha}^\dagger\Lambda^\dagger\boldsymbol{\alpha}^*]\exp[i(\boldsymbol{\alpha}^\dagger+\alpha^\dagger)I\alpha+\tfrac{1}{2}i(\boldsymbol{\alpha}^\dagger+\alpha^\dagger)\times V(\boldsymbol{\alpha}^*+\alpha^*)+\tfrac{1}{2}i\alpha^\sim\Lambda\alpha]. \qquad (16.3)$$

Unitarity requires

$$2\,\mathrm{Im}W_{(1)}=\ln\langle 0|\mathbf{F}|0\rangle. \qquad (16.4)$$

Since $W_{(1)}$ is independent of the eigenvalues α and $\alpha^\dagger$ it should be possible to simplify the right-hand side of this equation by setting these eigenvalues equal to zero. To show that this is indeed the case we first compute the commutators

$$[\boldsymbol{\alpha},\mathbf{F}]=\mathbf{F}[iI\alpha+iV(\boldsymbol{\alpha}^*+\alpha^*)], \qquad (16.5)$$

$$[\mathbf{F},\boldsymbol{\alpha}^\dagger]=[-i\alpha^\dagger I^\dagger-i(\boldsymbol{\alpha}^\sim+\alpha^\sim)V^\dagger]\mathbf{F}. \qquad (16.6)$$

Each of these commutators may be used to reexpress the other in the form

$$[\boldsymbol{\alpha},\mathbf{F}]=[iI\alpha+iV(\boldsymbol{\alpha}^*+\alpha^*)+VI^*\alpha^*+VV^\dagger(\boldsymbol{\alpha}+\alpha)]\mathbf{F}, \qquad (16.7)$$

$$[\mathbf{F},\boldsymbol{\alpha}^\dagger]=\mathbf{F}[-i\alpha^\dagger I^\dagger-i(\boldsymbol{\alpha}^\sim+\alpha^\sim)V^\dagger+\alpha^\sim I^\sim V^\dagger+(\boldsymbol{\alpha}^\dagger+\alpha^\dagger)VV^\dagger], \qquad (16.8)$$

from which, with the aid of the optical theorems (11.6), (11.7), and (11.8), we obtain

$$\mathbf{F}\alpha+iV\boldsymbol{\alpha}^*\mathbf{F}=(1+iI)[(1-iI^\dagger)\boldsymbol{\alpha}-iI^\dagger\alpha-i\Lambda^\dagger\boldsymbol{\alpha}^*]\mathbf{F}, \qquad (16.9)$$

$$\boldsymbol{\alpha}^\dagger\mathbf{F}-i\mathbf{F}\boldsymbol{\alpha}^\sim V^\dagger=\mathbf{F}[\boldsymbol{\alpha}^\dagger(1+iI)+i\alpha^\dagger I+i\alpha^\sim\Lambda]\times(1-iI^\dagger). \qquad (16.10)$$

From these equations it follows, after factoring out the $(1+iI)$ and $(1-iI^\dagger)$, that

$$\begin{aligned}0&=\langle 0|(1-iI^\dagger)\boldsymbol{\alpha}-iI^\dagger\alpha-i\Lambda^\dagger\boldsymbol{\alpha}^*]\mathbf{F}|0\rangle\\&=\langle 0|\{[-iI^\dagger(\boldsymbol{\alpha}+\alpha)-i\Lambda^\dagger\boldsymbol{\alpha}^*]\mathbf{F}+[\boldsymbol{\alpha},\mathbf{F}]\}|0\rangle\\&=\delta\langle 0|\mathbf{F}|0\rangle/\delta\alpha^\dagger,\end{aligned} \qquad (16.11)$$

and

$$\begin{aligned}0&=\langle 0|\mathbf{F}[\boldsymbol{\alpha}^\dagger(1+iI)+i\alpha^\dagger I+i\alpha^\sim\Lambda]|0\rangle\\&=\langle 0|\{\mathbf{F}[i(\boldsymbol{\alpha}^\dagger+\alpha^\dagger)I+i\alpha^\sim\Lambda]+[\mathbf{F},\boldsymbol{\alpha}^\dagger]\}|0\rangle\\&=\delta\langle 0|\mathbf{F}|0\rangle/\delta\alpha,\end{aligned} \qquad (16.12)$$

which is the desired result.

With the eigenvalues α and $\alpha^\dagger$ set equal to zero Eqs. (16.7) and (16.8) become

$$(1-VV^\dagger)\boldsymbol{\alpha}\mathbf{F}_0=\mathbf{F}_0\boldsymbol{\alpha}+iV\boldsymbol{\alpha}^*\mathbf{F}_0, \qquad (16.13)$$

$$\mathbf{F}_0\boldsymbol{\alpha}^\dagger(1-VV^\dagger)=\boldsymbol{\alpha}^\dagger\mathbf{F}_0-i\mathbf{F}_0\boldsymbol{\alpha}^\sim V^\dagger, \qquad (16.14)$$

$$\mathbf{F}_0\equiv\exp(-\tfrac{1}{2}i\boldsymbol{\alpha}^\sim V^\dagger\boldsymbol{\alpha})\times\exp(\tfrac{1}{2}i\boldsymbol{\alpha}^\dagger V\boldsymbol{\alpha}^*), \qquad (16.15)$$

whence

$$\delta\langle 0|\mathbf{F}_0|0\rangle/\delta V^\dagger = -\tfrac{1}{2}i\langle 0|\boldsymbol{\alpha}\boldsymbol{\alpha}^\sim \mathbf{F}_0|0\rangle$$
$$= -\tfrac{1}{2}i\langle 0|\boldsymbol{\alpha}(\mathbf{F}_0\boldsymbol{\alpha}^\sim + i\boldsymbol{\alpha}^\dagger V\mathbf{F}_0)|0\rangle \times (1-V^\dagger V)^{-1}$$
$$= \tfrac{1}{2}V(1-V^\dagger V)^{-1}\langle 0|\mathbf{F}_0|0\rangle, \qquad (16.16)$$

$$\delta\langle 0|\mathbf{F}_0|0\rangle/\delta V = \tfrac{1}{2}i\langle 0|\mathbf{F}_0\boldsymbol{\alpha}^*\boldsymbol{\alpha}^\dagger|0\rangle$$
$$= \tfrac{1}{2}i(1-V^\dagger V)^{-1}\langle 0|\boldsymbol{\alpha}^*\mathbf{F}_0 - i\mathbf{F}_0V^\dagger\boldsymbol{\alpha})\boldsymbol{\alpha}^\dagger|0\rangle$$
$$= \tfrac{1}{2}(1-V^\dagger V)^{-1}V^\dagger\langle 0|\mathbf{F}_0|0\rangle. \qquad (16.17)$$

Under variation of V and $V^\dagger$ (caused, for example, by a variation in the background field) we therefore have

$$2\,\mathrm{Im}\delta W_{(1)} = \delta\ln\langle 0|\mathbf{F}_0|0\rangle$$
$$= \tfrac{1}{2}\,\mathrm{tr}[V(1-V^\dagger V)^{-1}\delta V^\dagger + (1-V^\dagger V)^{-1}V^\dagger\delta V]$$
$$= -\tfrac{1}{2}\delta\,\mathrm{tr}\ln(1-V^\dagger V)$$
$$= -\tfrac{1}{2}\delta\ln\det(1-V^\dagger V), \qquad (16.18)$$

which, with the boundary condition: $W_{(1)}=0$ when $V=0$, may be integrated to yield

$$2\,\mathrm{Im}W_{(1)} = -\tfrac{1}{2}\ln\det(1-V^\dagger V). \qquad (16.19)$$

If we had used the condition $\mathbf{S}_{(1)}\mathbf{S}_{(1)}{}^\dagger = 1$ we would have arrived at the result

$$2\,\mathrm{Im}W_{(1)} = -\tfrac{1}{2}\ln\det(1-\Lambda\Lambda^\dagger) \qquad (16.20)$$

instead. The two results are, however, identical in view of the transposition invariance of the determinant and the identity

$$\det(1-VV^\dagger) = \det(1+iI)\det(1-iI^\dagger)$$
$$= \det(1-\Lambda^\dagger\Lambda), \qquad (16.21)$$

which follows from (11.6). We note that these results insure that the vacuum-to-vacuum probability lies between 0 and 1:

$$0 \le |\langle 0,\infty|0,-\infty\rangle_{(1)}|^2 = e^{-2\,\mathrm{Im}W_{(1)}}$$
$$= \det(1-V^\dagger V)^{1/2} \le 1. \qquad (16.22)$$

We note that they also permit, with a suitable choice of phase, the complete identification

$$W_{(1)} = -\tfrac{1}{2}i\ln\det(1+iI). \qquad (16.23)$$

In order to compute lowest-order radiative corrections it is necessary to perform functional differentiations on $W_{(1)}$. For this purpose it is convenient to re-express $W_{(1)}$ in a different form. We first recall that a formal determinant like (16.23) may be expanded by the Fredholm method in terms of traces. Remembering the cyclic invariance of the trace and making use of (9.10) and (9.39) we may therefore write

$$\det(1+iI) = \det(1+iu^\dagger\mathfrak{X}_+u) = \det(1-\mathfrak{X}_+\mathfrak{G}_0{}^{(+)})$$
$$= \det(1+X^+\mathfrak{G}_0{}^{(+)})^{-1}. \qquad (16.24)$$

We next compare this determinant with

$$\det(1+X^+G_0{}^{(+)})^{-1},$$

which contains the effects of both physical and nonphysical quanta. Using the canonical decomposition (9.7), the fundamental lemma (10.17), and Eqs. (9.2), (9.13), (9.14), and (10.18), we have

$$\det(1+X^+G_0{}^{(+)}) = \det[1-iX^+(uu^\dagger + R_0vN^{-1}v^\dagger\bar{R}_0{}^\sim + \bar{R}_0vN^{-1\sim}v^\dagger R_0{}^\sim)]$$
$$= \det\begin{pmatrix} 1-iu^\dagger X^+u & -iu^\dagger X^+R_0v & -iu^\dagger X^+\bar{R}_0{}^\sim vN^{-1\sim} \\ -iN^{-1}v^\dagger\bar{R}_0{}^\sim X^+u & 1-iN^{-1}v^\dagger\bar{R}_0{}^\sim X^+R_0v & -iN^{-1}v^\dagger\bar{R}_0{}^\sim X^+\bar{R}_0vN^{-1\sim} \\ -iv^\dagger R_0{}^\sim X^+u & -iv^\dagger R_0{}^\sim X^+R_0v & 1-iv^\dagger R_0{}^\sim X^+\bar{R}_0vN^{-1\sim} \end{pmatrix}$$
$$= \det\begin{pmatrix} 1-iu^\dagger X^+u & 0 & -iu^\dagger X^+\bar{R}_0{}^\sim vN^{-1\sim} \\ -iN^{-1}v^\dagger\bar{R}_0{}^\sim X^+u & 1-iM^{-1}v^\dagger\hat{X}^+v & -iN^{-1}v^\dagger\bar{R}_0{}^\sim X^+\bar{R}_0vN^{-1\sim} \\ 0 & 0 & 1-iv^\dagger\hat{X}^+vM^{-1} \end{pmatrix}$$
$$= \det(1+X^+\mathfrak{G}_0{}^{(+)})\det(1+\hat{X}^+\hat{G}_0{}^{(+)})^2. \qquad (16.25)$$

From this we obtain at once

$$W_{(1)} = \tfrac{1}{2}i\ln\det(1+X^+G_0{}^{(+)})$$
$$-i\ln\det(1+\hat{X}^+\hat{G}_0{}^{(+)}). \qquad (16.26)$$

Other forms for $W_{(1)}$ may be obtained by making use of Eqs. (9.27), (9.28) and their analogs for $\hat{G}_0{}^+$, $\hat{G}_0$, etc., namely

$$W_{(1)} = -\tfrac{1}{2}i\ln\frac{\det(1+XG_0)}{\det(1+X^+G_0{}^+)}$$
$$+i\ln\frac{\det(1+\hat{X}\hat{G}_0)}{\det(1+\hat{X}^+\hat{G}_0{}^+)} \qquad (16.27a)$$

$$= \tfrac{1}{2}i\ln\frac{\det(1-UG_0)}{\det(1-UG_0{}^+)} - i\ln\frac{\det(1-\hat{U}\hat{G}_0)}{\det(1-\hat{U}\hat{G}_0{}^+)} \qquad (16.27b)$$

$$= -\tfrac{1}{2}i\ln\frac{\det G\,\det\hat{G}_0{}^2\,\det G_0{}^+\,\det\hat{G}^{+2}}{\det G_0\,\det\hat{G}^2\,\det G^+\,\det\hat{G}_0{}^{+2}}. \qquad (16.27c)$$

The last expression must be used only formally, as the determinants of the Green's functions themselves do not really exist.[42]

[42] The determinants $\det(1-UG_0)$, $\det(1-UG_0{}^+)$, etc. do not exist either. However, the divergences which they contain are removable by renormalization procedures. These divergences may be shown to contribute only to the real part of $W_{(1)}$ and hence do not affect the vacuum-to-vacuum probability (16.22).

Since the matrix I is group-invariant $W_{(1)}$, as given by (16.23), is invariant. This invariance must also hold for the forms (16.26) and (16.27), and this provides us with a useful consistency check on our results. Expressions (16.26) and (16.27) are manifestly invariant under gauge transformations of the u's, since they do not even depend on the u's. Their invariance under changes in the γ's may be verified with the aid of (14.13) and

$$\delta\hat{F}=R^{\sim}\delta\gamma R. \tag{16.28}$$

Thus we have

$$\begin{aligned}\delta\ln\frac{\det G}{\det\hat{G}^2}&=-\mathrm{tr}(F\delta G)-2\,\mathrm{tr}(\hat{G}\delta\hat{F})\\&=-\mathrm{tr}(FG\delta\gamma R\hat{G}R^{\sim}-\mathrm{F}R\hat{G}\delta\tilde{\gamma}\hat{G}R^{\sim}\\&\qquad+FR\hat{G}R^{\sim}\delta\gamma G+2\hat{G}R^{\sim}\delta\gamma R)\\&=-\mathrm{tr}(\gamma R\tilde{\gamma}^{-1}\delta\tilde{\gamma}\hat{G}R^{\sim})=\mathrm{tr}(\tilde{\gamma}^{-1}\delta\tilde{\gamma}),\end{aligned} \tag{16.29}$$

and similarly

$$\delta\ln\frac{\det\hat{G}^{+2}}{\det G^{+}}=-\mathrm{tr}(\tilde{\gamma}^{-1}\delta\tilde{\gamma}), \tag{16.30}$$

with corresponding expressions for the zero-point quantities, whence $\delta W_{(1)}=0$.

To verify invariance under group transformations of the background field we use (11.9), obtaining

$$\begin{aligned}\delta\ln\det G^{\pm}&=-\mathrm{tr}(F\delta G^{\pm})\\&=-F_{ij}(R^i{}_{\alpha,k}G^{\pm kj}+R^j{}_{\alpha,k}G^{\pm ik})\delta\xi^\alpha\\&=2R^i{}_{\alpha,i}\delta\xi^\alpha.\end{aligned} \tag{16.31}$$

Similarly,

$$\begin{aligned}\delta\ln\det\hat{G}^{\pm}&=-\mathrm{tr}(\hat{F}\delta\hat{G}^{\pm})\\&=-\hat{F}_{\alpha\beta}(c^\alpha{}_{\gamma\delta}\hat{G}^{\pm\delta\beta}+c^\beta{}_{\gamma\delta}\hat{G}^{\pm\alpha\delta})\delta\xi^\gamma\\&=2c^\alpha{}_{\gamma\alpha}\delta\xi^\gamma.\end{aligned} \tag{16.32}$$

We may now either use (4.10) or else note that these variations will be exactly cancelled by identical expressions coming from the G's and $\hat{G}$'s of (16.27c). In either case we have $\delta W_{(1)}=0$, which completes the consistency check.

We remark that no special significance is to be attached to the use of the advanced Green's functions in Eqs. (16.27). Because of the transposition invariance of the determinant the retarded Green's functions could be used just as well.

17. SINGLE QUANTUM PRODUCTION. FICTITIOUS VIRTUAL QUANTA

The simplest example of a physical process which can be classed as a radiative correction or a closed-loop effect is the production of a single quantum by the background field. In this process the background field first produces two or more virtual quanta which, after various interactions with each other and with the background field (involving scatterings both forwards and backwards in time) proceed to coalesce into a single quantum via elementary vertex interactions. From Eq. (14.23) it is easy to see that the amplitude for this process is

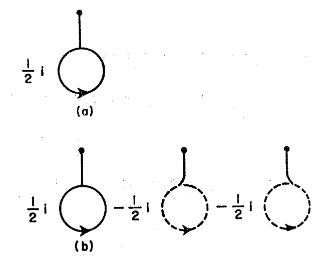

FIG. 2. Lowest-order diagrams for the single-quantum-production amplitude. Diagrams (a) and (b) refer to the cases in which infinite-dimensional invariance groups are, respectively, absent and present. Lines terminating in dots represent external-line wave functions. Lines bearing arrows represent virtual quanta on the mass shell. Dashed lines represent fictitious quanta. The asymmetry of the vertices from which fictitious quanta emanate is indicated by the obliquity of the angle at which the solid lines are attached.

$$\langle A,\infty\,|\,0,-\infty\rangle=ie^{iW}W_{,i}\mathfrak{f}_{\pm}{}^{(*)i}{}_A. \tag{17.1}$$

For simplicity we ignore the vacuum processes described by the exponential and replace $\mathfrak{f}_{\pm}{}^{(*)}$ by the full wave packet ϕ_0; we may regain individual amplitudes by functional differentiation with respect to the α's when desired. In lowest-order perturbation theory the amplitude then becomes

$$\begin{aligned}\phi_0{}^iW_{(1),i}&=-\tfrac{1}{2}i(G^{jk}-G^{\pm jk})F_{jk,i}\phi_0{}^i\\&\qquad+i(\hat{G}^{\alpha\beta}-\hat{G}^{\pm\alpha\beta})\hat{F}_{\alpha\beta,i}\phi_0{}^i\\&=\pm\tfrac{1}{2}i\phi_0{}^iS_{,ijk}G^{(\pm)jk}\\&\qquad\mp\tfrac{1}{2}i\phi_0{}^i(V_{(\alpha i)\beta}+V_{(\beta i)\alpha})\hat{G}^{(\pm)\alpha\beta},\end{aligned} \tag{17.2}$$

where

$$V_{(\alpha i)\beta}\equiv R^j{}_{\alpha,i}R_{j\beta}. \tag{17.3}$$

When no invariance group is present the second term on the right of Eq. (17.2) is absent, and the amplitude may be given the graphical representation depicted in Fig. 2(a). The line terminating in a dot represents the external-line wave function, and the solid line bearing an arrow represents the function $G^{(+)jk}$. The arrow may be assumed oriented in the direction "k to j"[43] and serves as a reminder that the virtual particles associated with it are *on the mass shell*, as follows from the fact that $G^{(+)}$ satisfies the homogeneous equation $FG^{(+)}=0$.

When an invariance group is present the function $G^{(+)}$ propagates nonphysical as well as physical quanta, and the second term on the right of (17.2) appears in order to compensate for the unwanted quanta. Feynman, who was the first to call attention to the need for this

[43] Unlike the Feynman propagator the function $G^{(+)jk}$ is not symmetric in its indices.

extra term, has referred to the auxiliary propagator $\hat{G}^{(\pm)}$ which occurs in it as *the propagator for fictitious quanta.*[44] In the case of the Yang-Mills field the fictitious quanta constitute a set of massless scalar particles which transform among themselves according to the adjoint representation of the group. In the case of gravity the fictitious quanta are massless vector particles.

It is to be noted that the fictitious quanta are needed only when the invariance group is non-Abelian. In the Abelian case the vertices $V_{(\alpha i)\beta}$ to which they are coupled vanish. This is one of the reasons why quantum electrodynamics, with its Abelian gauge group, fails to provide a satisfactory training ground for studies in quantum gravidynamics. Another peculiarity of the vertices $V_{(\alpha i)\beta}$ is their lack of symmetry with respect to the group indices. Although they appear in a symmetric combination in (17.2) they do not always appear thus in more complicated processes. Their asymmetry is indicated in Fig. 2(b) by making the solid lines attached to them join the dotted lines at an oblique angle. The dotted lines represent fictitious quanta and the presence of the arrows indicates that the propagator $\hat{G}^{(+)}$ rather than $\hat{G}$ is to be employed. The sum of the three diagrams appearing in Fig. 2(b) gives the full production amplitude.

Explicit calculation of the amplitude leads to divergences which must be handled by the methods of renormalization theory. For this reason use of the manifestly covariant functions $G^{(\pm)}$ and $\hat{G}^{(\pm)}$ is essential. From a purely formal standpoint, however, the functions $\mathfrak{G}^{(\pm)}$, which propagate only physical quanta, suggest themselves as natural replacements for $G^{(\pm)}$; they should in principle permit one to avoid dealing with the fictitious quanta. That is, we expect that it should be possible to rewrite (17.2) formally in the simpler form

$$\phi_0{}^i W_{(1),i} = \pm\tfrac{1}{2} i\phi_0{}^i S_{,ijk}\mathfrak{G}^{(\pm)jk}, \tag{17.4}$$

in which the propagators $\hat{G}^{(\pm)}$ no longer appear.

Equation (17.4) can in fact be shown to follow from (16.23). We shall here show its equivalence to (17.2) directly. For this purpose we must first assemble a number of fundamental identities.

We begin with Eqs. (9.20), (9.21), (9.37), and (9.38), which, after a certain amount of algebraic manipulation, yield

$$X-\mathfrak{X}_\pm = X(G_0-\mathfrak{G}_{0\pm})\mathfrak{X}_\pm = \mathfrak{X}_\pm(G_0-\mathfrak{G}_{0\pm})X \tag{17.5a}$$

$$= \mathfrak{X}_\pm(G_0-\mathfrak{G}_{0\pm})[1-\mathfrak{X}_\pm(G_0-\mathfrak{G}_{0\pm})]^{-1}\mathfrak{X}_\pm, \tag{17.5b}$$

$$\begin{aligned} G-\mathfrak{G}_\pm &= \mp G^{(\pm)} \pm \mathfrak{G}^{(\pm)} \\ &= (1+G_0X)(G_0-\mathfrak{G}_{0\pm})(1+\mathfrak{X}_\pm\mathfrak{G}_{0\pm}) \\ &= (1+\mathfrak{G}_{0\pm}\mathfrak{X}_\pm)(G_0-\mathfrak{G}_{0\pm})(1+XG_0) \end{aligned} \tag{17.6a}$$

$$= (1+\mathfrak{G}_{0\pm}\mathfrak{X}_\pm)(G_0-\mathfrak{G}_{0\pm})[1-\mathfrak{X}_\pm(G_0-\mathfrak{G}_{0\pm})]^{-1} \times(1+\mathfrak{X}_\pm\mathfrak{G}_{0\pm}). \tag{17.6b}$$

Further reduction of these expressions is most easily carried out by direct formal expansion of the bracketed factors. Using

$$\begin{aligned} G_0-\mathfrak{G}_{0+} &= -G_0{}^{(+)}+\mathfrak{G}_0{}^{(+)} \\ &= iR_0vN^{-1}v^\dagger\bar{R}_0{}^{\sim}+i\bar{R}_0vN^{-1\sim}v^\dagger R_0{}^{\sim}, \end{aligned} \tag{17.7a}$$

$$\begin{aligned} G_0-\mathfrak{G}_{0-} &= G_0{}^{(-)}-\mathfrak{G}_0{}^{(-)} \\ &= iR_0v^*N^{-1}v^{\sim}\bar{R}_0{}^{\sim}+i\bar{R}_0v^*N^{-1\sim}v^{\sim}R_0{}^{\sim}, \end{aligned} \tag{17.7b}$$

which follow from (9.7) and (9.10), and

$$\mathfrak{X}_\pm R_0v = X^\pm R_0v, \tag{17.8}$$

which follows from (9.38b) and (10.18), we find, with the aid of (9.13), (9.14), and the lemma (10.17),

$$\begin{aligned} &(G_0-\mathfrak{G}_{0+})[1-\mathfrak{X}_+(G_0-\mathfrak{G}_{0+})]^{-1} \\ &\quad = i[R_0(1+\hat{G}_0{}^{(+)}\hat{X}^+)^{-1}vN^{-1}v^\dagger\bar{R}_0{}^{\sim}+R_0vN^{-1\sim}v^\dagger \\ &\qquad \times(1+\hat{X}^+\hat{G}_0{}^{(+)})R_0{}^{\sim}]-R_0(1+\hat{G}_0{}^{(+)}\hat{X}^+)^{-1}vN^{-1} \\ &\qquad \times v^\dagger\bar{R}_0{}^{\sim}\mathfrak{X}_+\bar{R}_0vN^{-1\sim}v^\dagger(1+\hat{X}^+\hat{G}_0{}^{(+)})R_0{}^{\sim}. \end{aligned} \tag{17.9}$$

(The corresponding formula with $+$ signs replaced by $-$ signs can be obtained from this by transposition.) Inserting this into (17.6b) and using the analog for the functions $\hat{G}_0$, $\hat{G}$, etc. of Eq. (9.27), together with

$$(1+\mathfrak{G}_{0\pm}\mathfrak{X}_\pm)R_0v = R(1+\hat{G}_0{}^\pm\hat{X}^\pm)v, \tag{17.10}$$

which follows from (9.40), (10.14), and (10.18), we obtain

$$\begin{aligned} G-\mathfrak{G}_+ &= -G^{(+)}+\mathfrak{G}^{(+)} \\ &= iRQ_+ + iQ_-{}^{\sim}R^{\sim} - RP_+R^{\sim}, \end{aligned} \tag{17.11a}$$

$$\begin{aligned} G-\mathfrak{G}_- &= G^{(-)}-\mathfrak{G}^{(-)} \\ &= iRQ_- + iQ_+{}^{\sim}R^{\sim} - RP_-R^{\sim}, \end{aligned} \tag{17.11b}$$

where

$$Q_+ \equiv (1+\hat{G}_0\hat{X})vN^{-1}v^\dagger\bar{R}_0{}^{\sim}(1+\mathfrak{X}_+\mathfrak{G}_{0+}), \tag{17.12a}$$

$$Q_- \equiv (1+\hat{G}_0\hat{X})v^*N^{-1}v^{\sim}\bar{R}_0{}^{\sim}(1+\mathfrak{X}_-\mathfrak{G}_{0-}), \tag{17.12b}$$

$$P_+ \equiv P_-{}^{\sim} \equiv (1+\hat{G}_0\hat{X})vN^{-1}v^\dagger\bar{R}_0{}^{\sim}\mathfrak{X}_+\bar{R}_0vN^{-1\sim}v^\dagger \times(1+\hat{X}\hat{G}_0). \tag{17.13}$$

Before Eqs. (17.11) can be used to compute the effect of replacing $G^{(\pm)}$ by $\mathfrak{G}^{(\pm)}$ some special properties of the functions $Q_\pm$, $P_\pm$ must be derived. First we note that Eq. (9.2) permits us to write

$$F(1+G_0{}^\pm X^\pm)\bar{R}_0v = -\vec{F}G^\pm\overleftarrow{F}_0\bar{R}_0v = F_0\bar{R}v = 0. \tag{17.14}$$

From this, together with (4.7) and (5.11), it follows that

$$\begin{aligned} &\hat{F}\tilde{\gamma}^{-1}R^{\sim}\gamma(1+G_0{}^\pm X^\pm)\bar{R}_0v \\ &\qquad = R^{\sim}F(1+G_0{}^\pm X^\pm)\bar{R}_0v = 0. \end{aligned} \tag{17.15}$$

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[44] R. P. Feynman, *Proceedings of the 1962 Warsaw Conference on the Theory of Gravitation* (PWN-Editions Scientifiques de Pologne, Warszawa, 1964).

By the laws of propagation, taking into account boundary conditions in the remote past and future, this in turn implies

$$\tilde{\gamma}^{-1}R^{\sim}\gamma(1+G_0{}^{\pm}X^{\pm})\bar{R}_0v=(1+\hat{G}_0{}^{\pm}\hat{X}^{\pm})\tilde{\gamma}_0{}^{-1}R_0{}^{\sim}\gamma_0\bar{R}_0v =(1+\hat{G}_0{}^{\pm}\hat{X}^{\pm})vM^{-1}N^{\sim}, \quad (17.16)$$

in which (9.13) has been used in obtaining the second form. Analogous reasoning, combined with (8.18), leads to

$$\tilde{\gamma}^{-1}R^{\sim}\gamma(1+G_0{}^{\pm}X^{\pm})u=(1+\hat{G}_0{}^{\pm}\hat{X}^{\pm})\tilde{\gamma}_0{}^{-1}R_0{}^{\sim}\gamma_0u, \quad (17.17)$$

which also follows from (10.3) and (10.6). Equations (17.16) and (17.17), together with (9.40), then yield

$$\tilde{\gamma}^{-1}R^{\sim}\gamma(1+\mathfrak{G}_{0\pm}\mathfrak{X}_{\pm})\bar{R}_0v=(1+\hat{G}_0{}^{\pm}\hat{X}^{\pm})vM^{-1}N^{\sim}. \quad (17.18)$$

From this, with the aid of (9.14) and the analogs for $\hat{G}_0$, $\hat{G}$, etc. of (9.27) and (9.30a), we obtain

$$Q_+\gamma R\tilde{\gamma}^{-1}=\tilde{\gamma}^{-1}R^{\sim}\gamma Q_-{}^{\sim}=i\hat{G}^{(+)}, \quad (17.19a)$$

$$Q_-\gamma R\tilde{\gamma}^{-1}=\tilde{\gamma}^{-1}R^{\sim}\gamma Q_+{}^{\sim}=-i\hat{G}^{(-)}. \quad (17.19b)$$

We also have the equations

$$FQ_{\pm}{}^{\sim}=0, \quad (17.20)$$

$$\hat{F}Q_{\pm}=0, \quad (17.21)$$

$$\hat{F}P_{\pm}=0, \quad (17.22)$$

which are immediate consequences of

$$F(1+\mathfrak{G}_{0\pm}\mathfrak{X}_{\pm})\bar{R}_0v=-\vec{F}\mathfrak{G}_{\pm}\overleftarrow{F}_0\bar{R}_0v=F_0\bar{R}_0v=0, \quad (17.23)$$

$$\hat{F}(1+\hat{G}_0\hat{X})v=-\vec{\hat{F}}\hat{G}\overleftarrow{\hat{F}}_0v=\hat{F}_0v=0. \quad (17.24)$$

These equations, combined with (5.11) and (17.19), give us

$$Q_{\pm}\overleftarrow{S}_2=\mp i\hat{G}^{(\pm)}R^{\sim}\gamma. \quad (17.25)$$

For completeness we record here also the following useful and readily verified identities:

$$S_2G=-1-\gamma R\hat{G}R^{\sim}, \quad (17.26)$$

$$S_2G^{(\pm)}=-\gamma R\hat{G}^{(\pm)}R^{\sim}, \quad (17.27)$$

$$\mathfrak{G}^{(\pm)}\gamma R=0, \quad (17.28)$$

$$S_2\mathfrak{G}^{(\pm)}=0, \quad (17.29)$$

$$\mathfrak{G}^{(+)}=-\mathfrak{G}^{(-)\sim}=-i\mathfrak{f}_+(1+iI)^{-1}\mathfrak{f}_-{}^{(*)\sim}. \quad (17.30)$$

The last identity, which is obtained with the aid of (9.10), (9.42), (10.9a), and (11.2), shows explicitly that the functions $\mathfrak{G}^{(\pm)}$ propagate real quanta on the mass shell only.

We are now ready to employ (17.11) in the verification of (17.4). In this, as well as in many similar but more complicated derivations later to be stated without proof, repeated use is made not only of (4.7), (14.5), (17.25), and the other identities collected above, but also of a hierarchy of identities following from (4.8), namely

$$S_{,i_1\cdots i_nj}R^j{}_\alpha =-S_{,ji_2\cdots i_n}R^j{}_{\alpha,i_1}-\cdots-S_{,i_1\cdots i_{n-1}j}R^j{}_{\alpha,i_n}, \quad (17.31)$$

which relate bare vertex functions differing in order by unity. We give the steps of the present derivation without comment:

$$\begin{aligned}\tfrac{1}{2}i\phi_0{}^iS_{,ijk}\mathfrak{G}^{(+)jk} &=\tfrac{1}{2}i\phi_0{}^iS_{,ijk}(G^{(+)jk}+iR^j{}_\alpha Q_+{}^{\alpha k}+iQ_-{}^{\alpha j}R^k{}_\alpha - R^j{}_\alpha P_+{}^{\alpha\beta}R^k{}_\beta)\\ &=\tfrac{1}{2}i\phi_0{}^iS_{,ijk}G^{(+)jk}+\tfrac{1}{2}\phi_0{}^i(S_{,jk}R^j{}_{\alpha,i}+S_{,ij}R^j{}_{\alpha,k})Q_+{}^{\alpha k}\\ &\quad+\tfrac{1}{2}\phi_0{}^i(S_{,kj}R^k{}_{\alpha,i}+S_{,ik}R^k{}_{\alpha,j})Q_-{}^{\alpha j}\\ &\quad+\tfrac{1}{2}i\phi_0{}^i(S_{,jk}R^j{}_{\alpha,i}+S_{,ij}R^j{}_{\alpha,k})P_+{}^{\alpha\beta}R^k{}_\beta\\ &=\tfrac{1}{2}i\phi_0{}^iS_{,ijk}G^{(+)jk}-\tfrac{1}{2}i\phi_0{}^iR_{j\beta}R^j{}_{\alpha,i}\hat{G}^{(+)\alpha\beta}\\ &\quad+\tfrac{1}{2}i\phi_0{}^iR_{k\beta}R^k{}_{\alpha,i}\hat{G}^{(-)\alpha\beta}. \end{aligned}\quad (17.32)$$

In view of the symmetry relation $\hat{G}^{(\pm)\sim}=-\hat{G}^{(\mp)}$ [cf. (9.6)] the last line reduces immediately to the right-hand side of (17.2).

Aside from eliminating the fictitious quanta, Eq. (17.4) has the important advantage of yielding an immediate formal proof of the group invariance of the amplitude $\phi_0{}^iW_{(1),i}$. To see this we note that $S_{,ijk}$ is identical with the tree function t_{ijk}. Therefore, in view of (17.30) the right-hand side of (17.4) appears as *a sum over tree amplitudes* in which all of the external-line wave functions refer to physical quanta on the mass shell. Group invariance of the total amplitude follows immediately from the tree theorem. This possibility, namely of reducing all amplitudes to sums over tree amplitudes so that group invariance is assumed by the tree theorem, was first suggested by Feynman.[37] We shall now see how it works in more complicated processes.

18. MULTIQUANTUM PROCESSES. FEYNMAN BASKETS

Next in order of complexity are the lowest-order radiative corrections to the amplitudes for scattering, pair production, and pair annihilation by the background field.[45] These are obtained by functionally differentiating the amplitudes of Fig. 2 and using the variational law

$$\delta G^{(\pm)}=G^{(\pm)}\delta FG+G\delta FG^{(\pm)}\pm G^{(\pm)}\delta FG^{(\pm)}, \quad (18.1)$$

which follows from (6.19), (9.19), and (9.29). When no invariance group is present the result is

$$\phi_0{}^i\phi_0{}^j(W_{(1),ij}+S_{,ijk}G^{kl}W_{(1),l})=\tfrac{1}{2}i\phi_0{}^i\phi_0{}^jt_{ijkl}G^{(+)kl} +\tfrac{1}{2}i\phi_0{}^it_{ikl}G^{(+)km}G^{(+)nl}t_{mnj}\phi_0{}^j, \quad (18.2)$$

which has the graphical representation shown in Fig. 3(a). We see immediately that Feynman's idea works;

[45] When the background field vanishes these reduce to the self-energy corrections to the 1-quantum propagator.

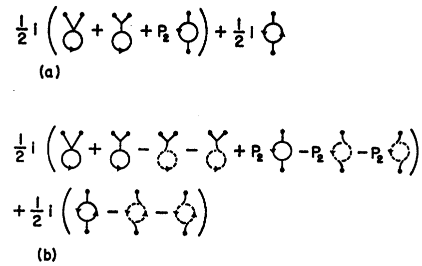

FIG. 3. Lowest-order radiative corrections to the 2-quantum amplitude. (a) Invariance group absent. (b) Invariance group present.

the total amplitude appears as a sum of products of tree amplitudes. In the figure the diagrams have been grouped into sets corresponding to the tree structure, i.e., to the two terms of Eq. (18.2). These sets are known as *Feynman baskets*. The key method in developing the general theory of radiative corrections of arbitrarily high order will be to take diagrams having a given topological structure and reassemble them into Feynman baskets.

The corresponding amplitude when an invariance group is present may be obtained by three distinct methods: (1) functional differentiation of (17.2); (2) functional differentiation of (17.4); and (3) replacement of $G^{(+)}$ by $\mathfrak{G}^{(+)}$ in (18.2) and use of the identities (17.11), (17.19), (17.27), (17.31), etc. All yield the same result, namely

$$\begin{aligned}\phi_0{}^i\phi_0{}^j(W_{(1),ij}+S_{,ijk}G^{kl}W_{(1),l})&=\tfrac{1}{2}i\phi_0{}^i\phi_0{}^j t_{ijkl}\mathfrak{G}^{(+)kl}\\&\quad+\tfrac{1}{2}i\phi_0{}^i t_{ikl}\mathfrak{G}^{(+)km}\mathfrak{G}^{(+)nl}t_{mnj}\phi_0{}^j \qquad (18.3\text{a})\\&=\tfrac{1}{2}i\phi_0{}^i\phi_0{}^j[t_{ijkl}G^{(+)kl}+t_{ikl}G^{(+)km}G^{(+)nl}t_{mnj}\\&\quad-2V_{(\alpha i)\beta}\hat{G}^{\alpha\gamma}\hat{G}^{(+)\beta\delta}V_{(\delta j)\gamma}-2V_{(\beta i)\alpha}\hat{G}^{\alpha\gamma}\hat{G}^{(+)\beta\delta}V_{(\gamma j)\delta}\\&\quad-V_{(\alpha i)\beta}\hat{G}^{(+)\gamma\alpha}\hat{G}^{(+)\beta\delta}V_{(\delta j)\gamma}-V_{(\beta i)\alpha}\hat{G}^{(+)\gamma\alpha}\hat{G}^{(+)\beta\delta}V_{(\gamma j)\delta}\\&\quad-S_{,ijk}G^{kl}(V_{(\alpha l)\beta}+V_{(\beta l)\alpha})\hat{G}^{(+)\alpha\beta}], \qquad (18.3\text{b})\end{aligned}$$

which has the graphical representation shown in Fig. 3(b). In each case the derivation is straightforward but tedious. Obviously, the amount of computational labor involved in converting from $\mathfrak{G}^{(+)}$ to the functions $G^{(+)}$ and $\hat{G}^{(+)}$ mounts rapidly as the complexity of the underlying tree diagrams increases.

In functionally differentiating either the external-line wave functions or the physical propagators it is necessary to have a variational law for $\mathfrak{G}^{(\pm)}$ analogous to (18.1). This is obtained by inserting (6.19) and (11.10) into (11.11) and then using (9.23) and (9.35), which yields

$$\begin{aligned}\delta\mathfrak{G}^{(\pm)}&=G^{\pm}\delta F\mathfrak{G}^{(\pm)}+\mathfrak{G}^{(\pm)}\delta FG^{\pm}\mp\mathfrak{G}^{(\pm)}\delta F\mathfrak{G}^{(\pm)}\\&=\mathfrak{G}_{\pm}\delta F\mathfrak{G}^{(\pm)}+\mathfrak{G}^{(\pm)}\delta F\mathfrak{G}_{\pm}\pm\mathfrak{G}^{(\pm)}\delta F\mathfrak{G}^{(\pm)}, \qquad (18.4)\end{aligned}$$

and, incidentally,

$$\delta\mathfrak{G}_{\pm}=\mathfrak{G}_{\pm}\delta F\mathfrak{G}_{\pm}. \qquad (18.5)$$

From Eq. (17.28) and the explicit form

$$\delta F=\delta S_2+\delta(\gamma R\tilde{\gamma}^{-1}R^{\sim}\gamma), \qquad (18.6)$$

it then follows that

$$\begin{aligned}\delta\mathfrak{G}^{(\pm)}&=\mathfrak{G}_{\pm}(\delta S_2+\gamma R\tilde{\gamma}^{-1}\delta R^{\sim}\gamma+\gamma R\tilde{\gamma}^{-1}R^{\sim}\delta\gamma)\mathfrak{G}^{(\pm)}\\&\quad+\mathfrak{G}^{(\pm)}(\delta S_2+\delta\gamma R\tilde{\gamma}^{-1}R^{\sim}\gamma+\gamma\delta R\tilde{\gamma}^{-1}R^{\sim}\gamma)\mathfrak{G}_{\pm}\\&\quad\pm\mathfrak{G}^{(\pm)}\delta S_2\mathfrak{G}^{(\pm)}\\&=\mathfrak{G}_{\pm}\delta S_2\mathfrak{G}^{(\pm)}+\mathfrak{G}^{(\pm)}\delta S_2\mathfrak{G}_{\pm}\pm\mathfrak{G}^{(\pm)}\delta S_2\mathfrak{G}^{(\pm)}\\&\quad+R\hat{G}^{\pm}(\delta R^{\sim}\gamma+R^{\sim}\delta\gamma)\mathfrak{G}^{(\pm)}\\&\quad+\mathfrak{G}^{(\pm)}(\delta\gamma R+\gamma\delta R)\hat{G}^{\pm}R^{\sim}. \qquad (18.7)\end{aligned}$$

When this law is used for the purpose of generating Feynman baskets it turns out that the last two terms never contribute anything owing to the presence of the R's. One finds also that the $\mathfrak{G}_{\pm}$'s in the first two terms may always be replaced by G's, a result which is directly related to the previously mentioned possibility of using G or $\mathfrak{G}_{\pm}$ interchangeably for the internal lines of tree diagrams.[46]

Another fact which is useful in computations is that in performing functional differentiations one may skip over any γ's which occur. Terms involving functional derivatives of γ's conspire mutually to cancel in any observable amplitude. This is a consequence of the γ invariance of the theory.

There is, however, one possible source of worry which needs to be disposed of. In passing from an expression like (17.2), say, to the expression (17.4), one makes use of (6.11) and many other identities which depend on the background field equations being satisfied. The right-hand sides of (17.2) and (17.4) are therefore not identical but are equal only modulo the field equations. They differ by an expression of the form $a^iS_{,i}$. One may ask what happens to this difference when *it* gets differentiated. The answer is that, in the passage from the one-quantum amplitude to the n-quantum amplitude, the combinations in which the functional derivatives

[46] With the identities which we now have at our disposal it is straightforward to show that the theory of tree functions may be based on $\mathfrak{G}_{\pm}$ rather than G. One replaces Eq. (14.4) by

$$\phi_{\pm}=\phi_0+\mathfrak{G}_{\pm}(S_1[\varphi+\phi_{\pm}]-S_2\phi_{\pm})$$

and obtains $S_1[\varphi+\phi_{\pm}]=0$, in complete analogy with (14.10). The corresponding tree amplitudes are then obtained from (14.25) by replacing ϕ with $\phi_{\pm}$. To show that this replacement leaves the tree amplitudes unaffected we write

$$\phi-\phi_{\pm}=-G\vec{S}_2\phi+\mathfrak{G}_{\pm}\vec{S}_2\phi_{\pm}=-G\vec{S}_2(\phi-\phi_{\pm})-(G-\mathfrak{G}_{\pm})\vec{S}_2\phi_{\pm}.$$

In view of (4.7), (5.11), (10.10), (14.3), and (17.11) this equation is solved by

$$\phi-\phi_{\pm}=-iRQ_{\pm}(F-\gamma R\tilde{\gamma}^{-1}R^{\sim}\gamma)(\phi_{\pm}-\phi_0).$$

The quantity $\phi_{\pm}-\phi_0$ vanishes at infinity rapidly enough so that the operator in parenthesis can be reversed. From (17.19) and (17.20) we therefore get

$$\phi-\phi_{\pm}=\mp R\hat{G}^{(\pm)}R^{\sim}\gamma(\phi_{\pm}-\phi_0)=\mp R\hat{G}^{(\pm)}R^{\sim}\gamma(\phi-\phi_0),$$

from which the invariance of (14.25) immediately follows.

of this difference occur always add up to zero. These combinations, in order of increasing complexity, are

$$(a^i{}_{,j}S_{,i}+a^iS_{,ij})\phi_0{}^j,$$

$$[a^i{}_{,jk}S_{,i}+2a^i{}_{,j}S_{,ik}+a^iS_{,ijk} + (a^i{}_{,l}S_{,i}+a^iS_{,il})G^{lm}S_{,mjk}]\phi_0{}^j\phi_0{}^k, \quad (18.8)$$

etc. By making use of (2.2), (10.10), (17.26), and (17.31) one readily verifies in each case that these combinations vanish. It is not hard to show, in fact, that this is to be expected as a corollary of (14.10).

We close this section by recording the contributions of $W_{(1)}$ to the three- and four-quantum amplitudes:

$$\phi_0{}^i\phi_0{}^j\phi_0{}^k(W_{(1),ijk}+3t_{ijl}G^{lm}W_{(1),mk}+t_{ijkl}G^{lm}W_{(1),m})=\tfrac{1}{2}i\phi_0{}^i\phi_0{}^j\phi_0{}^k(t_{ijklm}\mathfrak{G}^{(+)lm}+3t_{ijlm}\mathfrak{G}^{(+)lp}\mathfrak{G}^{(+)qm}t_{pqk} +2t_{ilm}\mathfrak{G}^{(+)mp}t_{pqj}\mathfrak{G}^{(+)qr}t_{rsk}\mathfrak{G}^{(+)sl}), \quad (18.9)$$

$$\phi_0{}^i\phi_0{}^j\phi_0{}^k\phi_0{}^l(W_{(1),ijkl}+6t_{ijm}G^{mn}W_{(1),nkl}+4t_{ijkm}G^{mn}W_{(1),nl}+3t_{ijm}t_{kln}G^{mp}G^{nq}W_{(1),pq}+t_{ijklm}G^{mn}W_{(1),n})$$
$$=\tfrac{1}{2}i\phi_0{}^i\phi_0{}^j\phi_0{}^k\phi_0{}^l(t_{ijklmn}\mathfrak{G}^{(+)mn}+4t_{ijkmn}\mathfrak{G}^{(+)mp}\mathfrak{G}^{(+)qn}t_{pql}+3t_{ijm}\mathfrak{G}^{(+)mp}\mathfrak{G}^{(+)qn}t_{pqkl}+12t_{ijmn}\mathfrak{G}^{(+)np}t_{pqk}\mathfrak{G}^{(+)qr}t_{rsl}\mathfrak{G}^{(+)rm} +6t_{imn}\mathfrak{G}^{(+)np}t_{pqj}\mathfrak{G}^{(+)qr}t_{rsk}\mathfrak{G}^{(+)st}t_{tul}\mathfrak{G}^{(+)um}). \quad (18.10)$$

The corresponding diagrams when no invariance group is present are shown in Fig. 4. The grouping of the amplitudes into Feynman baskets is again evident. The task of reexpressing (18.9) and (18.10) in the general case in terms of G, $\hat{G}$, $G^{(+)}$, $\hat{G}^{(+)}$ will be left to the reader as a (rather lengthy) exercise. The reader may also enjoy discovering the simple rules of differentiation which lead in a step by step fashion from Eq. (17.4), through Eqs. (18.3a), (18.9), and (18.10), to the lowest-order radiative correction to the general n-quantum amplitude.

19. HIGHER-ORDER RADIATIVE CORRECTIONS. USE OF THE FEYNMAN FUNCTIONAL INTEGRAL TO CONSTRUCT A CONSISTENT THEORY

The functional derivatives of $W_{(1)}$ are represented by diagrams each of which has only a single closed loop.

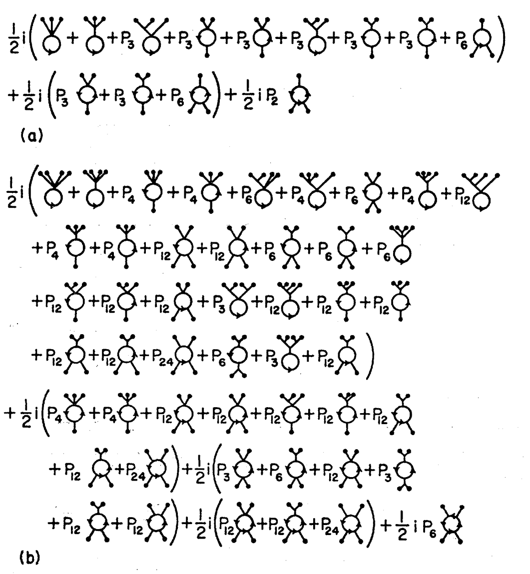

FIG. 4. Lowest-order radiative corrections to (a) the three-quantum amplitude and (b) the four-quantum amplitude in the absence of an invariance group. When an invariance group is present all the mass-shell propagators $G^{(+)}$are replaced by $\mathfrak{G}^{(+)}$.

Connected diagrams having two or more closed loops correspond to higher-order radiative corrections. A diagram having n-independent closed loops is said to be of the nth order.

Consider the set of all connected nth-order diagrams which contribute to a given scattering amplitude. By repeated functional integration one may remove the external lines. The resulting *vacuum diagrams* represent the nth-order contribution to W, which will be denoted by $W_{(n)}$.

The basic topology of the vacuum diagrams and the numerical coefficients to be attached to them are the same for all field theories. For purposes of orientation we begin with the case in which no invariance group is present. The Feynman functional integral may then be used as a convenient formal expression for the vacuum-to-vacuum amplitude:

$$\langle 0,\infty|0,-\infty\rangle=e^{iW[\varphi]}, \quad (19.1)$$

$$W[\varphi]=w[\varphi]-w[0], \quad (19.2)$$

$$e^{iw[\varphi]}=\int \exp i(S[\varphi+\phi]-S[\varphi]-S_{,i}[\varphi]\varphi^i) \times\Delta[\varphi+\phi]d\phi \quad (19.3)$$

$$d\phi\equiv\prod_i d\phi^i. \quad (19.4)$$

Here Δ is a density which serves to define the functional volume element. It will be chosen in such a way as to maintain invariance of the theory under the variable transformation (15.5).

It is not difficult to show that Eq. (19.3), when supplemented by the statement

$$\langle 0,\infty|T(A[\phi])|0,-\infty\rangle=e^{-iw[0]}\int A[\phi] \times\exp i(S[\varphi+\phi]-S[\varphi]-S_{,i}[\varphi]^i) \times\Delta[\varphi+\phi]d\phi, \quad (19.5)$$

yields the hierarchy of Eqs. (12.20), (12.21), (12.22).

Thus,

$$G^{ij}\frac{\delta}{i\delta\varphi^i}\langle 0,\infty|0,-\infty\rangle$$

$$=e^{-iw[0]}G^{ij}\int\bigg(\frac{\delta}{i\delta\phi^j}\{\exp i(S[\varphi+\phi]-S[\varphi]-S_{,l}[\varphi]\phi^l)\Delta[\varphi+\phi]\}-S_{,jk}[\varphi]\phi^k\exp i(S[\varphi+\phi]-S[\varphi]-S_{,l}[\varphi]\phi^l)\Delta[\varphi+\phi]\bigg)d\phi$$

$$=\langle 0,\infty|\phi^i|0,-\infty\rangle, \quad (19.6)$$

$$G^{ik}\frac{\delta}{i\delta\varphi^k}G^{jl}\frac{\delta}{i\delta\varphi^l}\langle 0,\infty|0,-\infty\rangle$$

$$=G^{ik}\frac{\delta}{i\delta\varphi^k}\langle 0,\infty|\phi^j|0,-\infty\rangle$$

$$=e^{-iw[0]}G^{ik}\int\bigg(\frac{\delta}{i\delta\phi^k}\{\phi^j\exp i(S[\varphi+\phi]-S[\varphi]-S_{,l}[\varphi]\phi^l)\Delta[\varphi+\phi]\}+\{i\delta^j{}_k-S_{,kl}[\varphi]\phi^l\phi^j\}\times\exp i(S[\varphi+\phi]-S[\varphi]-S_{,m}[\varphi]\phi^m)\Delta[\varphi+\phi]\bigg)d\phi$$

$$=iG^{ij}\langle 0,\infty|0,-\infty\rangle+\langle 0,\infty|T(\phi^i\phi^j)|0,-\infty\rangle, \quad (19.7)$$

etc., in which functional integrals of total functional derivatives are set formally equal to zero. In fact, Eqs. (12.20) *et al.*, can be used to *derive* Eqs. (19.3) and (19.5), showing once again that the technique of varying the background field is completely equivalent to (but of wider applicability than) more familiar methods employing external sources.

The formal identity

$$\int\frac{\delta}{\delta\phi^i}\{\exp i(S[\varphi+\phi]-S[\varphi]-S_{,i}[\varphi]\phi^i)\times\Delta[\varphi+\phi]\}d\phi\equiv 0, \quad (19.8)$$

combined with the condition $S_{,i}[\varphi]=0$ on the background field, suggests that the operator field equations of the theory may be written in the form

$$T(S_{,i}[\varphi+\boldsymbol{\phi}]-i\{\ln\Delta[\varphi+\boldsymbol{\phi}]\}_{,i})=0. \quad (19.9)$$

On the other hand we expect that they may also be expressed in the simpler "classical" form

$$0=S_{,i}[\varphi+\boldsymbol{\phi}]=S_{,ij}\boldsymbol{\phi}^j+\frac{1}{2!}S_{,ijk}\boldsymbol{\phi}^j\boldsymbol{\phi}^k+\frac{1}{3!}S_{,ijkl}\boldsymbol{\phi}^j\boldsymbol{\phi}^k\boldsymbol{\phi}^l+\cdots, \quad (19.10)$$

the manifest Hermiticity of which follows from the symmetry of the coefficients (bare vertex functions). Equation (19.10) will, in fact, turn out to be not quite right; it cannot be reexpressed in the form (19.9) and, moreover, it is not form-invariant under transformation of variables. However, we shall adopt it tentatively and then correct it later.

The term in Δ in Eq. (19.9) may be regarded as arising from the process of converting from ordinary to chronological products, and may be computed on this basis. In rearranging factor sequences we need to know the commutator $[\boldsymbol{\phi}^i,\boldsymbol{\phi}^j]$.[47] For this purpose we take the commutator of (19.10) with $\boldsymbol{\phi}^k$ and find that the result is solved by

$$[\boldsymbol{\phi}^i,\boldsymbol{\phi}^j]\equiv i\tilde{\mathbf{G}}^{ij}, \quad (19.11)$$

$$\tilde{\mathbf{G}}^{ij}=\tilde{G}^{ij}+\tilde{G}^{ij}{}_{,k}\boldsymbol{\phi}^k+(1/2!)\tilde{G}^{ij}{}_{,kl}\boldsymbol{\phi}^k\boldsymbol{\phi}^l+\cdots. \quad (19.12)$$

The algebra is straightforward. Here we work only up to the order needed in discussing $W_{(2)}$; more efficient methods of procedure will be given in the next section.

Further algebra yields

$$\boldsymbol{\phi}^i\boldsymbol{\phi}^j-T(\boldsymbol{\phi}^i\boldsymbol{\phi}^j)=-i\theta(j,i)\tilde{\mathbf{G}}^{ji}\equiv i\mathbf{G}^{+ij}=iG^{+ij}+iG^{+ij}{}_{,k}\boldsymbol{\phi}^k+\tfrac{1}{2}iG^{+ij}{}_{,kl}T(\boldsymbol{\phi}^k\boldsymbol{\phi}^l)-\tfrac{1}{2}G^{ij}{}_{,kl}G^{+kl}+\cdots, \quad (19.13)$$

$$\boldsymbol{\phi}^i\boldsymbol{\phi}^j\boldsymbol{\phi}^k-T(\boldsymbol{\phi}^i\boldsymbol{\phi}^j\boldsymbol{\phi}^k)=i\theta(j,k)\mathbf{G}^{+ij}\boldsymbol{\phi}^k+i\theta(k,j)\boldsymbol{\phi}^k\mathbf{G}^{+ij}+i\theta(k,j)\mathbf{G}^{+ik}\boldsymbol{\phi}^j+i\theta(j,k)\boldsymbol{\phi}^j\mathbf{G}^{+ik}+i\boldsymbol{\phi}^i\mathbf{G}^{+jk}$$
$$=iG^{+ij}\boldsymbol{\phi}^k+iG^{+ik}\boldsymbol{\phi}^j+iG^{+jk}\boldsymbol{\phi}^i+iG^{+ij}{}_{,l}T(\boldsymbol{\phi}^l\boldsymbol{\phi}^k)+iG^{+ik}{}_{,l}T(\boldsymbol{\phi}^l\boldsymbol{\phi}^j)+iG^{+jk}{}_{,l}T(\boldsymbol{\phi}^l\boldsymbol{\phi}^i)-G^{+ij}{}_{,l}G^{+lk}-G^{+jk}{}_{,l}G^{-li}+\cdots, \quad (19.14)$$

$$\boldsymbol{\phi}^i\boldsymbol{\phi}^j\boldsymbol{\phi}^k\boldsymbol{\phi}^l-T(\boldsymbol{\phi}^i\boldsymbol{\phi}^j\boldsymbol{\phi}^k\boldsymbol{\phi}^l)=iG^{+ij}T(\boldsymbol{\phi}^k\boldsymbol{\phi}^l)+iG^{+kl}T(\boldsymbol{\phi}^i\boldsymbol{\phi}^j)-G^{+ij}G^{+kl}+iG^{+ik}T(\boldsymbol{\phi}^j\boldsymbol{\phi}^l)+iG^{+jl}T(\boldsymbol{\phi}^i\boldsymbol{\phi}^k)-G^{+ik}G^{+jl}+iG^{+il}T(\boldsymbol{\phi}^j\boldsymbol{\phi}^k)+iG^{+jk}T(\boldsymbol{\phi}^i\boldsymbol{\phi}^l)-G^{+il}G^{+jk}+\cdots, \quad (19.15)$$

[47] Here we proceed purely formally and ignore the fact that in a local theory all the ϕ's of Eq. (19.10) are evaluated at the same space-time point.

where $\theta(i,j)$ is the temporal step function[48]

$$\theta(i,j)=1 \quad \text{for} \quad i>j$$
$$=0 \quad \text{for} \quad j>i. \tag{19.16}$$

These results permit us to reexpress Eq. (19.10) in the form

$$0=T(S_{,i}[\varphi+\phi])+\tfrac{1}{2}S_{,ijk}\{iG^{+jk}+iG^{+ja}S_{,abc}G^{+bk}\phi^c+\tfrac{1}{2}iG^{+ja}S_{,abcd}G^{+bk}[T(\phi^c\phi^d)+iG^{+cd}]$$
$$+\tfrac{1}{2}iG^{+ja}S_{,abc}G^{+be}S_{,efd}G^{+fk}[2T(\phi^c\phi^d)+iG^{+cd}+iG^{-cd}]+\cdots\}+\tfrac{1}{6}S_{,ijkl}\{3iG^{+jk}\phi^l+3iG^{+ja}S_{,abc}G^{+bk}T(\phi^c\phi^l)$$
$$-G^{+ja}S_{,abc}G^{+bk}(G^{+cl}+G^{-cl})+\cdots\}+\tfrac{1}{4}iS_{,ijklm}G^{+jk}T(\phi^l\phi^m)-\tfrac{1}{8}S_{,ijklm}G^{+jk}G^{+lm}+\cdots. \tag{19.17}$$

The terms following $T(S_{,i}[\varphi+\phi])$ in this equation are *almost*, but not quite, expressible in the form $-iT(\{\ln\Delta[\varphi+\phi]\}_{,i})$ of Eq. (19.9). What is missing is a term having the following structure:

$$\tfrac{1}{12}S_{,ijk}G^{+ja}S_{,abc}G^{+be}S_{,efd}G^{+fk}\tilde{G}^{cd}. \tag{19.18}$$

If this term is added to Eqs. (19.10) and (19.17) we find

$$\Delta[\varphi]=(\det G^+)^{-1/2}\exp(-\tfrac{1}{6}iS_{,ijk}G^{+il}G^{+jm}G^{-kn}S_{,lmn}$$
$$-\tfrac{1}{8}iS_{,ijkl}G^{+ij}G^{+kl}+\cdots). \tag{19.19}$$

$$\Delta[\varphi+\phi]=(\det G^+)^{-1/2}\exp(-\tfrac{1}{2}G^{+ij}S_{,ijk}\phi^k$$
$$-\tfrac{1}{4}G^{+ia}S_{,abk}G^{+bj}S_{,ijl}\phi^k\phi^l-\tfrac{1}{4}G^{+ij}S_{,ijkl}\phi^k\phi^l$$
$$-\cdots-\tfrac{1}{6}iS_{,ijk}G^{+il}G^{+jm}G^{-kn}S_{,lmn}$$
$$-\tfrac{1}{8}iS_{,ijkl}G^{+ij}G^{+kl}+\cdots). \tag{19.20}$$

Expression (19.18) is the first of an infinite sequence of correction terms, which must be discovered by laborious computation. These terms maintain the formal Hermiticity of the field equations [e.g., (19.18) is real] but are not mathematically well defined. Like the terms of Δ they involve Green's functions with coincident arguments and hence cannot be properly discussed apart from renormalization theory. However, they may be regarded as possessed of certain formal properties. Owing to the kinematics of the Green's functions they depend only locally on the fields, and in the case of scalar fields with nonderivative couplings they may be regarded as vanishing by virtue of the commutativity of field components at the same space-time point.

We are now ready to compute $W_{(2)}$. We first reexpress Eq. (19.3) in the form

$$e^{iw[\varphi]}=(\exp iw_{(1)}[0])\langle 0,\infty|T(\exp i[(1/3!)S_{,ijk}\phi^i\phi^j\phi^k$$
$$+(1/4!)S_{,ijkl}\phi^k\phi^j\phi^k\phi^l+\cdots]$$
$$\times\Delta'[\varphi,\phi])|0,-\infty\rangle_{(1)}. \tag{19.21}$$

Here $\Delta'[\varphi,\phi]$ is $\Delta[\varphi+\phi]$ with the factor $(\det G^+)^{-1/2}$ removed, and

$$\langle 0,\infty|T(A[\phi])|0,-\infty\rangle_{(1)}$$
$$\equiv[\exp(-iw_{(1)}[0])]\int A[\phi]\exp(i\tfrac{1}{2}S_{,ij}\phi^i\phi^j)$$
$$\times(\det G^+)^{-1/2}d\phi, \tag{19.22}$$

$$\exp(iw_{(1)}[\varphi])$$
$$\equiv\int\exp(i\tfrac{1}{2}S_{,ij}\phi^i\phi^j)(\det G^+)^{-1/2}d\phi$$
$$=Z\frac{(\det G)^{1/2}}{(\det G^+)^{1/2}}, \tag{19.23}$$

where Z is a numerical constant determined by the lattice spacing used in the definition of the functional integral, but independent of the background field. The Green's function G makes its appearance in (19.23) owing to the Feynman boundary conditions assumed in the Gaussian integral.

Writing

$$W=\sum_{n=1}^{\infty}W_{(n)},\quad w=\sum_{n=1}^{\infty}w_{(n)}, \tag{19.24}$$

$$W_{(n)}[\varphi]=w_{(n)}[\varphi]-w_{(n)}[0], \tag{19.25}$$

and making use of (19.20), (19.21) and the hierarchy of equations generated by

$$\langle 0,\infty|T(\exp i\lambda_i\phi^i)|0,-\infty\rangle_{(1)}$$
$$=\exp(iW_{(1)}+\tfrac{1}{2}i\lambda_i\lambda_jG^{ij}), \tag{19.26}$$

we find

$$W_{(1)}=-\tfrac{1}{2}i\ln\frac{\det G\,\det G_0{}^+}{\det G_0\,\det G^+}, \tag{19.27}$$

$$W_{(2)}=-\tfrac{1}{12}S_{,ijk}(G^{il}G^{jm}G^{kn}+2G^{+il}G^{+jm}G^{-kn}$$
$$-3G^{il}G^{+jm}G^{-kn})S_{,lmn}-\tfrac{1}{8}(G^{ij}-G^{+ij})S_{,ijk}G^{kl}S_{,lmn}$$
$$\times(G^{mn}-G^{+mn})-\tfrac{1}{8}S_{,ijkl}(G^{ij}-G^{+ij})(G^{kl}-G^{+kl})$$
$$\text{minus the same terms evaluated with } \varphi=0. \tag{19.28}$$

Equation (19.27) is observed to agree with (16.27c) for the case in which no invariance group is present.

Expression (19.28) for $W_{(2)}$, on the other hand, is still not quite right; it fails to be invariant under the variable transformation (15.5). To find out how it

[48] The step function need not be defined for spacelike separations of i and j but must be handled with care when the two space-time points coincide. Fortunately it disappears in the final forms of Eqs. (19.13), (19.14), and (19.15).

changes we make use of the transformation laws

$$S_{,ij}{}' = S_{,ab}\frac{\delta\varphi^a}{\delta\varphi'^i}\frac{\delta\varphi^b}{\delta\varphi'^j}, \tag{19.29}$$

$$S_{,ijk}{}' = S_{,abc}\frac{\delta\varphi^a}{\delta\varphi'^i}\frac{\delta\varphi^b}{\delta\varphi'^j}\frac{\delta\varphi^c}{\delta\varphi'^k} + S_{,ab}\left(\frac{\delta\varphi^a}{\delta\varphi'^i}\frac{\delta^2\varphi^b}{\delta\varphi'^j\delta\varphi'^k} + \frac{\delta\varphi^a}{\delta\varphi'^j}\frac{\delta^2\varphi^b}{\delta\varphi'^k\delta\varphi'^i} + \frac{\delta\varphi^a}{\delta\varphi'^k}\frac{\delta^2\varphi^b}{\delta\varphi'^i\delta\varphi'^j}\right), \tag{19.30}$$

$$\begin{aligned} S_{,ijkl}{}' = {} & S_{,abcd}\frac{\delta\varphi^a}{\delta\varphi'^i}\frac{\delta\varphi^b}{\delta\varphi'^j}\frac{\delta\varphi^c}{\delta\varphi'^k}\frac{\delta\varphi^d}{\delta\varphi'^l} + S_{,abc}\left(\frac{\delta\varphi^a}{\delta\varphi'^i}\frac{\delta\varphi^b}{\delta\varphi'^j}\frac{\delta^2\varphi^c}{\delta\varphi'^k\delta\varphi'^l} + \frac{\delta\varphi^a}{\delta\varphi'^k}\frac{\delta\varphi^b}{\delta\varphi'^l}\frac{\delta^2\varphi^c}{\delta\varphi'^i\delta\varphi'^j} + \frac{\delta\varphi^a}{\delta\varphi'^i}\frac{\delta\varphi^b}{\delta\varphi'^k}\frac{\delta^2\varphi^c}{\delta\varphi'^j\delta\varphi'^l}\right. \\ & \left. + \frac{\delta\varphi^a}{\delta\varphi'^j}\frac{\delta\varphi^b}{\delta\varphi'^l}\frac{\delta^2\varphi^c}{\delta\varphi'^i\delta\varphi'^k} + \frac{\delta\varphi^a}{\delta\varphi'^i}\frac{\delta\varphi^b}{\delta\varphi'^l}\frac{\delta^2\varphi^c}{\delta\varphi'^j\delta\varphi'^k} + \frac{\delta\varphi^a}{\delta\varphi'^j}\frac{\delta\varphi^b}{\delta\varphi'^k}\frac{\delta^2\varphi^c}{\delta\varphi'^i\delta\varphi'^l}\right) + S_{,ab}\left(\frac{\delta^2\varphi^a}{\delta\varphi'^i\delta\varphi'^j}\frac{\delta^2\varphi^b}{\delta\varphi'^k\delta\varphi'^l} + \frac{\delta^2\varphi^a}{\delta\varphi'^i\delta\varphi'^k}\frac{\delta^2\varphi^b}{\delta\varphi'^i\delta\varphi'^l}\right. \\ & \left. + \frac{\delta^2\varphi^a}{\delta\varphi'^i\delta\varphi'^l}\frac{\delta^2\varphi^b}{\delta\varphi'^j\delta\varphi'^k} \frac{\delta\varphi^a}{\delta\varphi'^i}\frac{\delta^3\varphi^b}{\delta\varphi'^j\delta\varphi'^k\delta\varphi'^l} + \frac{\delta\varphi^a}{\delta\varphi'^j}\frac{\delta^3\varphi^b}{\delta\varphi'^k\delta\varphi'^l\delta\varphi'^i} + \frac{\delta\varphi^a}{\delta\varphi'^k}\frac{\delta^3\varphi^b}{\delta\varphi'^l\delta\varphi'^i\delta\varphi'^j} + \frac{\delta\varphi^a}{\delta\varphi'^l}\frac{\delta^3\varphi^b}{\delta\varphi'^i\delta\varphi'^j\delta\varphi'^k}\right), \end{aligned} \tag{19.31}$$

from which terms in $S_{,a}$ have been omitted owing to the fact that the background field obeys the classical field equations. These laws permit us to infer [cf. (15.8)]

$$G'^{ij} = \frac{\delta\varphi'^i}{\delta\varphi^k}\frac{\delta\varphi'^j}{\delta\varphi^l}G^{kl}, \quad G^{+\prime ij} = \frac{\delta\varphi'^i}{\delta\varphi^k}\frac{\delta\varphi'^j}{\delta\varphi^l}G^{+kl}, \tag{19.32}$$

whence it follows that $W_{(1)}$ *is* invariant. For $W_{(2)}$, on the other hand, we find, by a straightforward but tedious calculation,

$$\begin{aligned} W_{(2)}{}'[\varphi'] - W_{(2)}[\varphi] = {} & \frac{1}{24}S_{,abc}\tilde{G}^{ad}\tilde{G}^{be}\frac{\delta^2\varphi^c}{\delta\varphi'^i\delta\varphi'^j}\frac{\delta\varphi'^i}{\delta\varphi^d}\frac{\delta\varphi'^j}{\delta\varphi^e} \\ & + \frac{1}{48}S_{,ab}\frac{\delta^2\varphi^a}{\delta\varphi'^i\delta\varphi'^j}\frac{\delta^2\varphi^b}{\delta\varphi'^k\delta\varphi'^l}\frac{\delta\varphi'^i}{\delta\varphi^c}\frac{\delta\varphi'^j}{\delta\varphi^d}\frac{\delta\varphi'^k}{\delta\varphi^e}\frac{\delta\varphi'^l}{\delta\varphi^f}\tilde{G}^{ce}\tilde{G}^{df} \\ & \text{minus the same terms evaluated with } \varphi = 0, \end{aligned} \tag{19.33}$$

showing that Eq. (15.11a) is violated.

The violation, however, is not very great. Relative to the large number of terms involved in the calculation and the large amount of cancellation which takes place between them, expression (19.33) represents a very small residue. One suspects that it can be easily eliminated by the addition of a suitable term to (19.28). The desired term should be real, so as not to disturb the vacuum-to-vacuum *probability*, and should be built out of quantities, such as Green's functions and bare vertex functions, which already exist in the classical theory. It is not difficult to verify that there is only one second-order expression with the necessary properties, namely

$$\begin{aligned} Y_{(2)} = {} & (1/48)S_{,ijk}\tilde{G}^{il}\tilde{G}^{jm}G^{+kn}S_{,lmn} \\ & - (1/48)S_{,ijk}{}^0\tilde{G}_0{}^{il}\tilde{G}_0{}^{jm}\tilde{G}_0{}^{+kn}S_{,lmn}{}^0. \end{aligned} \tag{19.34}$$

We therefore conclude that the final correct forms for $W_{(2)}$ and Δ (to second order) are

$$\begin{aligned} W_{(2)} = {} & -\tfrac{1}{12}S_{,ijk}(G^{il}G^{jm}G^{kn} + 2G^{+il}G^{+jm}G^{-kn} \\ & - 3G^{il}G^{+jm}G^{-kn} - \tfrac{1}{4}\tilde{G}^{il}\tilde{G}^{jm}G^{+kn})S_{,lmn} \\ & - \tfrac{1}{8}(G^{ij} - G^{+ij})S_{,ijk}G^{kl}S_{,lmn}(G^{mn} - G^{+mn}) \\ & - \tfrac{1}{8}S_{,ijkl}(G^{ij} - G^{+ij})(G^{kl} - G^{+kl}) \text{ minus the} \\ & \text{same terms evaluated with } \varphi = 0, \end{aligned} \tag{19.35}$$

$$\begin{aligned} \Delta[\varphi] = {} & (\det G^+)^{-1/2}\exp(-\tfrac{1}{6}iS_{,ijk}G^{+il}G^{+jm}G^{-kn}S_{,lmn} \\ & + (1/48)iS_{,ijk}\tilde{G}^{il}\tilde{G}^{jm}G^{+kn}S_{,lmn} \\ & - \tfrac{1}{8}iS_{,ijkl}G^{+ij}G^{+kl} + \cdots). \end{aligned} \tag{19.36}$$

The introduction of the term (19.34) brings a qualitatively new element into the theory. It adds to the operator field equations (19.10) a term of the form $T(Y_{(2),i}[\varphi+\phi])$ which, unlike (19.18), depends *nonlocally* on the fields and is nonvanishing even for scalar fields with nonderivative coupling. This implies that within the framework of local field theory there exists no covariant ordering of the factors of the operator field equations which maintains form-invariance of the theory under arbitrary (local) transformations of variables. Such a conclusion, however, presupposes a definition of "locality" which, because of its formality, is perhaps not very useful. Of greater importance are the conditions of analyticity on scattering amplitudes which ought to hold if certain conditions of causality (conventionally assumed to follow from "locality") are to be valid. The "derivations" of this section are purely heuristic (since one is dealing with the unrenormalized fields) and there is evidence that the surgery effected by standard renormalization techniques (which, when applicable, is implicit also in dispersion theory) removes from the theory precisely the formal nonlocality represented by $Y_{(2)}$. We shall return briefly to this question in the next section, where alternative, more systematic methods for treating the higher-order radiative corrections are discussed.

20. NONCAUSAL CHAINS. FEYNMAN BASKETS FOR OVERLAPPING LOOPS. GENERAL ALGORITHM FOR OBTAINING THE PRIMARY DIAGRAMS TO ALL ORDERS

If, in Eq. (19.3), the density functional Δ is set equal to unity then all the terms drop out of Eqs. (19.27) and (19.35) save those which involve the Feynman propagator G only. The resulting functional will be denoted by $\bar{W}$:

$$\bar{W} = -\tfrac{1}{2}i \ln \det G - \tfrac{1}{12} S_{,ijk}G^{il}G^{jm}G^{kn}S_{,lmn} - \tfrac{1}{8}G^{ij}S_{,ijk}G^{kl}S_{,lmn}G^{mn} - \tfrac{1}{8}S_{,ijkl}G^{ij}G^{kl} + \cdots$$

minus the same terms evaluated with $\varphi = 0$. (20.1)

The basic topology of vacuum diagrams is already contained in the terms of this series. Each term corresponds to what will be called a *primary diagram*, composed of bare vertices and Feynman propagators only. The primary diagrams of orders 1 through 3 are shown in Fig. 5. In these diagrams the terms with $\varphi = 0$ are to be understood as already having been subtracted out. In most applications one is not interested in the vacuum-to-vacuum amplitude itself but only in its functional derivatives, which yield the radiative corrections to scattering amplitudes. The terms with $\varphi = 0$ make no contribution to these amplitudes, being essentially constants of integration. Therefore, no attempt has been made to represent them pictorially.

The terms of Eqs. (19.27) and (19.35) which are missing from (20.1) are topologically similar to the primary diagrams. They differ only in the replacement of various Feynman propagators by G^+, G^-, and $\tilde{G}$. The question which presents itself is how these replacements are to be made in the general case and with what coefficients.

It is evident from the analysis of the preceding section that the diagrams which cause the most trouble are those which contain overlapping loops (just as in renormalization theory). Let us therefore consider first the simpler diagrams in which no loop touches any other loop in more than a single point. By referring to Eq. (19.35) and to Figs. 2, 3, and 4 it is not difficult to see that, as far as these diagrams are concerned, the correct expression for W is obtained from that for $\bar{W}$ simply by removing the *noncausal chains* from all loops. By "noncausal chain" we mean any cyclic product of advanced (or retarded) Green's functions connecting a sequence of points of which the last is equal to the first. Such cyclic products necessarily vanish except when all the points coincide, and hence they depend only locally on the background field. In the case of scalar fields with nonderivative coupling they may be formally set equal to zero. In the general case they must be explicitly removed.[49]

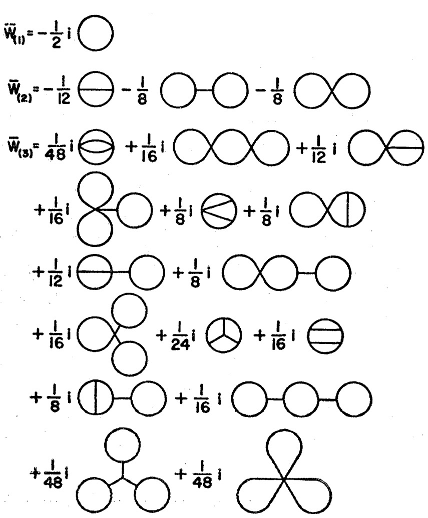

FIG. 5. Topology of higher-order radiative corrections. Primary diagrams of orders 1 through 3. No invariance group present.

The diagrams with overlapping loops cannot be treated so simply. Here the difficulty is twofold. First, the noncausal chains enter in a more complicated way and, except in the case of $W_{(2)}$, there is no unique way of removing them. Second, the removal of noncausal chains by itself does not suffice to lead to invariant amplitudes.

The situation may be described more fully thus: At a certain point in the process of removing noncausal chains from a given primary diagram one must stop; no further noncausal chains remain. At this point the diagram no longer contains closed loops composed of Feynman propagators only. At least one segment of every loop consists of a "free" propagator $G^{(+)}$ or $G^{(-)}$. That is to say, the removal of the noncausal chains "breaks open" all the closed loops, and the result is representable as a sum over tree diagrams with all external lines on the mass shell. However, the particular trees which are obtained, and the coefficients attached to them, generally depend on which noncausal chains are removed first and on what *orientation* one chooses to assign to them. In the more complicated diagrams there is not even a unique way of averaging over orientations.

One may nevertheless ask whether there is a "correct" way of removing noncausal chains. The answer is yes, but it must be determined separately in each individual case by a computation which is as complicated as those of the preceding section; no simple general algorithm has so far been found. Moreover, even when the noncausal chains have been properly removed the resulting tree diagrams cannot yet be assembled into Feynman baskets. "Nonlocal terms" beginning in lowest order with $Y_{(2)}$, have also to be discovered and added.

To gain an appreciation of the complexities which arise the reader may try his hand at decomposing $\bar{W}_{(3)}$, remembering to take into account the contribution which $Y_{(2)}$ makes in this order, through its presence in

[49] For $W_{(1)}$ this means subtracting $\ln \det G^+$ from $\ln \det G$; the latter is represented by the simple circle in Fig. 5.

the density functional Δ. We shall content ourselves here with the decomposition of $\bar{W}_{(2)}$.

In this case it turns out that although the removal of noncausal chains can be carried out in various ways the end result is always the same. Thus the three propagators of the first diagram for $\bar{W}_{(2)}$ in Fig. 5 may, with the aid of Eqs. (9.17), be decomposed as follows:

$$G\otimes G\otimes G = G\otimes G^{+}\otimes G^{-} + G^{-}\otimes G^{+}\otimes G + G^{+}\otimes G\otimes G^{-} - G^{-}\otimes G^{+}\otimes G^{-} - G^{+}\otimes G^{+}\otimes G^{-} + G^{(-)}\otimes G\otimes G^{(-)} + G^{(+)}\otimes G^{(+)}\otimes G - G\otimes G^{(+)}\otimes G^{(-)} + G^{(-)}\otimes G^{(+)}\otimes G^{(-)} - G^{(+)}\otimes G^{(+)}\otimes G^{(-)}. \quad (20.2)$$

The first five terms on the right of this equation yield noncausal chains. If they are subtracted one obtains the first term on the right of Eq. (19.28). We have already seen that this expression is not quite right; we must add the quantity $Y_{(2)}$, obtaining (19.35) as the correct full expression for $W_{(2)}$. It is then straightforward to verify that $W_{(2)}$ has the following decomposition into Feynman baskets:

$$W_{(2)} = -\tfrac{1}{8}t_{ijkl}G^{(+)ij}G^{(+)kl} + (1/48)t_{ijk}G^{(+)il}G^{(+)jm} \times G^{(+)kn}t_{lmn} - \tfrac{3}{16}t_{ijk}G^{(+)il}G^{(+)jm}G^{(+)nk}t_{lmn} \text{ minus the same terms evaluated with } \varphi = 0, \quad (20.3)$$

a result which admits of immediate extension to the case in which an invariance group is present:

$$W_{(2)} = -\tfrac{1}{8}t_{ijkl}\mathfrak{G}^{(+)ij}\mathfrak{G}^{(+)kl} + (1/48)t_{ijk}\mathfrak{G}^{(+)il}\mathfrak{G}^{(+)jm} \times \mathfrak{G}^{(+)kn}t_{lmn} - \tfrac{3}{16}t_{ijk}\mathfrak{G}^{(+)il}\mathfrak{G}^{(+)jm}\mathfrak{G}^{(+)jnk}t_{lmn} \text{ minus the same terms evaluated with } \varphi = 0. \quad (20.4)$$

Several observations may now be made. First, the possibility of decomposing the vacuum-to-vacuum functionals into Feynman baskets is closely related to unitarity of the S matrix; unitarity statements such as Eq. (13.14) involve sums over tree amplitudes of precisely the form (20.4).[50] Second, although the requirement that the theory be invariant under transformations of variables has led us to functionals which decompose into Feynman baskets, it is clear from the tree theorem (14.26) and the invariance statement (15.10) that we could instead have started from decomposability itself as a criterion for the discovery of "correction" terms such as $Y_{(2)}$, and thereby obtained vacuum-to-vacuum amplitudes which are not only invariant under transformations of variables but group-invariant as well. Evidently the various consistency requirements of the theory fit together in an interlocking fashion, and it appears that the imposition of one will yield the others also. This makes it possible to consider alternative approaches to the theory of radiative corrections.

One such approach is arrived at by reexpressing (20.4) in terms of the manifestly covariant propagators $G, G^{(+)}, \hat{G}, \hat{G}^{(+)}$. Using Eqs. (4.2), (5.6), (17.11), (17.19), (17.26), (17.27), and (17.31) one finds, by rather intricate and tedious algebra, that (20.4) appears as

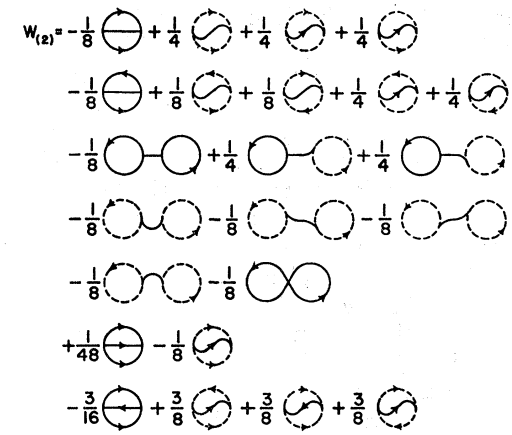

FIG. 6. Second-order vacuum diagrams when an invariance group is present.

the sum of the 23 terms which are depicted in Fig. 6. *It turns out that these terms can alternatively be obtained from the primary diagrams of Fig. 7* by removing noncausal chains and adding the nonlocal "correction"

$$Y_{(2)} = (1/48)S_{,ijk}\tilde{G}^{il}\tilde{G}^{jm}G^{+kn}S_{,lmn} - (1/24)V_{(\alpha i)\beta} \times(\hat{G}^{(+)\alpha\delta} + \hat{G}^{(-)\alpha\delta})(\hat{G}^{(+)\beta\gamma} + \hat{G}^{(-)\beta\gamma})G^{+ij}V_{(\gamma j)\delta} - (1/24)V_{(\alpha i)\beta}(\hat{G}^{(+)\alpha\delta} + \hat{G}^{(-)\alpha\delta})\hat{G}^{+\beta\gamma}\tilde{G}^{ij}V_{(\gamma j)\delta} - (1/24)V_{(\alpha i)\beta}\hat{G}^{+\alpha\delta}(\hat{G}^{(+)\beta\gamma} + \hat{G}^{(-)\beta\gamma})\tilde{G}^{ij}V_{(\gamma j)\delta}, \quad (20.5)$$

which is a generalization of (19.34). In this case the noncausal chains must be removed in a maximally symmetric manner which gives equal weight to both dotted and solid lines and to the various distinct orientations of the diagrams.

Now it is a remarkable fact that $\bar{W}_{(2)}$, as given by the primary diagrams of Fig. 7, is already group-invariant as it stands. It is not only invariant under group transformations of the background field, which is obvious from its manifestly covariant construction, but it is also γ-invariant as well. The latter assertion may be verified by a straightforward but tedious computation which makes use of Eqs. (4.2), (4.3), (4.10),[51] (6.11), (14.13), (16.28), (17.26), and (17.31).

This result shows that combinations of tree amplitudes are not the only group-invariant quantities in the theory and suggests that the method of decomposing diagrams into Feynman baskets, and the formal com-

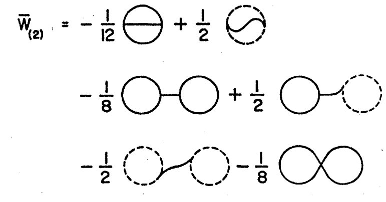

FIG. 7. Primary diagrams of order 2 when an invariance group is present. (Dashed lines without arrows represent Feynman propagators $\hat{G}$ for fictitious quanta.)

[50] An explicit verification that (20.4) satisfies unitarity has been carried out (unpublished).

[51] Equations (4.10) are never needed except when dealing with primary diagrams.

plications which go with it, can be avoided. Indeed in conventional field theory one works with the primary diagrams from the beginning and never bothers to remove the noncausal chains. In the case of nonoverlapping loops it is easy to see why one nevertheless gets correct results. It is a standard procedure in momentum-space calculations, after all terms of an integrand have been brought to a common denominator of the form $(k^2+2p\cdot k+\Delta-i0)^n$, to perform a rotation through 90° in the k^0 plane and thereby to convert from Minkowski space to Euclidean space for the subsequent evaluation. When the integral is convergent the procedure is legitimate, but when the integral diverges a part—the arc at infinity—is lost which can be shown to correspond exactly to a noncausal chain. Moreover, since the arc is at infinity in momentum space its contribution is necessarily "local" in space-time and would in any case be removed by renormalization, e.g., with the use of regulators.

In the case of overlapping loops *nonlocal* renormalizations, i.e., renormalizations *within* momentum subintegrations, must be performed in order to get rid of the well-known overlapping divergences. Although a complete analysis of the overlapping case remains to be carried out there is considerable evidence that here too renormalization absorbs the "correction" terms, which now include not only noncausal chains but also the "nonlocal" quantities $Y_{(2)}$, etc. One expects that the decomposition of radiative corrections into Feynman baskets is in effect replaced by analyticity statements, and that the unitarity of the S matrix is secured by the famous Cutkosky rules.[52]

As a working procedure we shall therefore assume, just as in conventional field theory, that it suffices to deal with the primary diagrams alone. Although much work remains to be done to establish this assumption with complete rigor, it is then quite easy to construct a manifestly covariant quantum theory of gravity (and/or the Yang-Mills field) which is unique to all orders of perturbation theory. One has only to discover what diagrams have to be added to those of Fig. 5, etc., in order to obtain γ-invariant vacuum-to-vacuum amplitudes, and this problem has been completely solved.

The solution of the problem for the case of $\bar{W}_{(2)}$ is given in Fig. 7. The diagrams of this figure can be discovered in the following way.[53] One adds to the $\bar{W}_{(2)}$ diagrams of Fig. 5 other topologically similar diagrams, involving the fictitious quanta in all possible ways, each with an arbitrary coefficient, and then adjusts the coefficients so that the total expression becomes invariant under changes in the γ's. In the process one discovers the following facts, which hold to all orders: (1) The fictitious quanta always occur in closed loops; they never begin or end on solid lines. (2) In addition, to the bare vertices S_n the only vertex which is needed is $V_{(\alpha i)\beta}$. Vertices such as $R^k{}_{\alpha,i}\gamma_{kl}R^l{}_{\beta,j}$ at which more than one solid line meets a dotted line never occur. (We shall see later that they do not even occur when external lines are inserted into the vacuum diagrams.) (3) The solid lines which enter a given fictitious quantum loop all do so with the same orientation around the loop. (Remember they enter obliquely.) This means, for example, that the combination $V_{(\alpha i)\beta}\hat{G}^{\alpha\gamma}\hat{G}^{\beta\delta}G^{ij}V_{(\gamma j)\delta}$ does not appear in Fig. 7.

It is remarkable that the condition of γ invariance alone suffices to determine all the higher-order radiative corrections. By going through the computation for $\bar{W}_{(2)}$ one is easily convinced that the same procedure gives unique results to all orders, with no ambiguity about coefficients. However, it is extremely tedious to carry out the computations required, order by order, and one naturally asks whether or not a short cut can be found. Fortunately it can.

One introduces a fictitious system described by the action functional $\frac{1}{2}F_{ij}\phi^i\phi^j+F_{\alpha\beta}\psi^{*\alpha}\psi^\beta$, where the field ϕ^i is of the commuting type and the fields ψ^α and $\psi^{*\alpha}$, which create and annihilate the fictitious quanta, are of *anticommuting* type. One then computes $\bar{W}$ from the formulas

$$\bar{W}[\varphi]=\bar{w}[\varphi]-\bar{w}[0], \tag{20.6}$$

$$\begin{aligned}\exp i\bar{w}[\varphi]=&(\exp i\bar{w}_{(1)}[0])(\det\tilde{\gamma})^{-1/2}\\ &\times\langle 0,\infty\,|\,T(\exp i[V_{(\alpha i)\beta}\psi^{*\alpha}\phi^i\psi^\beta+(1/3!)S_{,ijk}\phi^i\phi^j\phi^k\\ &+(1/4!)S_{,ijkl}\phi^i\phi^j\phi^k\phi^l+\cdots])\,|\,0,-\infty\rangle_{(1)},\end{aligned} \tag{20.7}$$

where the subscript (1) indicates that the evaluation is to be carried out with reference to the fictitious system. The vacuum-to-vacuum amplitude of the fictitious system itself is to be understood as defined without the removal of acausal chains. Thus

$$\langle 0,\infty\,|\,0,-\infty\rangle_{(1)}=\frac{(\det\tilde{\gamma})^{1/2}}{(\det\tilde{\gamma}_0)^{1/2}}\exp i\bar{W}_{(1)}[\varphi], \tag{20.8}$$

$$\bar{W}_{(1)}[\varphi]=\bar{w}_{(1)}[\varphi]-\bar{w}_{(1)}[0], \tag{20.9}$$

$$\exp i\bar{w}_{(1)}[\varphi]=\frac{(\det G)^{1/2}(\det\tilde{\gamma})^{-1/2}}{\det\hat{G}}, \tag{20.10}$$

with no factor $(\det\hat{G}^+)/(\det G^+)^{1/2}$ appearing in (20.10). The factor $(\det\tilde{\gamma})^{-1/2}$ or its inverse is inserted into Eqs. (20.7), (20.8), and (20.10) so as to make $\bar{w}_{(1)}$ $\tilde{\gamma}$-invariant [see Eq. (16.29)].

The anticommuting character of the fields ψ^α and $\psi^{*\alpha}$ implies that *the fictitious quanta are fermions*.[54] It is this property which enables them to play a compensatory role in the theory. For example, it is what causes $\det\hat{G}$ to appear in the denominator rather than

[52] R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960). When the Cutkosky rules are applied to divergent diagrams it is always assumed that the divergence is first removed by regulators. The nonacausal chains are therefore automatically excluded.

[53] B. S. DeWitt, Phys. Rev. Letters **12**, 742 (1964).

[54] The usual relation between spin and statistics obviously need not apply to these quanta.

the numerator of Eq. (20.10).[55] It is also what restricts the fictitious quanta to appearing only in closed loops.

The explicit evaluation of expression (20.7) is carried out with the aid of the hierarchy of equations generated by

$$\langle 0,\infty|T\,(\exp(i\lambda_i\phi^i+i\lambda^*{}_\alpha\psi^\alpha+i\psi^{*\alpha}\lambda_\alpha))|0,-\infty\rangle_{(1)}\frac{(\det\tilde{\gamma}_0)^{1/2}}{(\det\tilde{\gamma})^{1/2}}$$
$$=\exp(i\bar{W}_{(1)}+\tfrac{1}{2}i\lambda_i\lambda_jG^{ij}+i\lambda^*{}_\alpha\lambda_\beta\hat{G}^{\alpha\beta}),\quad(20.11)$$

where the λ_i are ordinary c-number variables and the λ_α, $\lambda_\alpha{}^*$ are variables from an anticommuting number system.[56] The determination of the higher-order primary diagrams then becomes a straightforward exercise. In Fig. 8 we show, for example, the diagrams which must be added to those of Fig. 5 in the case of $\bar{W}_{(3)}$.

The proof that formula (20.7) yields γ-invariant vacuum amplitudes to all orders may be carried out by first rewriting it in the form

$$\exp i\bar{w}[\varphi]=Z'\int\frac{\exp i(\frac{1}{2}\tilde{\gamma}_{\alpha\beta}\chi^\alpha\chi^\beta+\frac{1}{2}F_{ij}\phi^i\phi^j+(1/3!)S_{,ijk}\phi^i\phi^j\phi^k+\cdots)}{\int\exp(i\mathfrak{F}_{\alpha\beta}\psi^{*\alpha}\psi^\beta)d\psi d\psi^*}d\chi d\phi$$
$$=Z'\int\frac{\exp i(S[\varphi+\phi]-S[\varphi]-S_{,i}[\varphi]\phi^i+\frac{1}{2}\tilde{\gamma}_{\alpha\beta}\chi^\alpha\chi^\beta+\frac{1}{2}R_{i\alpha}R_j{}^\alpha\phi^i\phi^j)}{\int\exp(i\mathfrak{F}_{\alpha\beta}\psi^{*\alpha}\psi^\beta)d\psi d\psi^*}d\chi d\phi,\quad(20.12)$$

where the factor Z' is a normalizing constant for the functional integrals, the field χ_α is introduced in order to generate the necessary factor $(\det\tilde{\gamma})^{-1/2}$, and $\mathfrak{F}$ is a non-self-adjoint operator defined by

$$\mathfrak{F}_{\alpha\beta}=\mathfrak{R}^i{}_\alpha R_{i\beta}\equiv\hat{F}_{\alpha\beta}+V_{(\alpha i)\beta}\phi^i,\quad(20.13)$$

$$\mathfrak{R}^i{}_\alpha\equiv R^i{}_\alpha+R^i{}_{\alpha,j}\phi^j\equiv R^i{}_\alpha[\varphi+\phi].\quad(20.14)$$

We next place primes on all the field symbols χ, ϕ, ψ, ψ^* in (20.12) and (20.13). This is purely a formal step which changes nothing. However, we may regard it as corresponding to actual changes in the integration variables;

$$\chi'^\alpha=\chi^\alpha+\delta\chi^\alpha,\quad\phi'^i=\phi^i+\delta\phi^i,\quad\psi'^\alpha=\psi^\alpha+\delta\psi^\alpha,$$
$$\psi^{*\prime\alpha}=\psi^{*\alpha}+\delta\psi^{*\alpha}.\quad(20.15)$$

We then choose these changes in the following way:

$$\delta\chi^\alpha=\tfrac{1}{2}\tilde{\gamma}^{-1\alpha\beta}\delta\tilde{\gamma}_{\beta\gamma}\chi^\gamma,\quad(20.16)$$

$$\delta\phi^i=\mathfrak{R}^i{}_\alpha\delta\xi^\alpha,\quad(20.17)$$

$$\delta\psi^\alpha=-\mathcal{G}^{\alpha\beta}\mathfrak{R}^i{}_\beta(\delta\gamma_{ij}R^j{}_\gamma-V_{(\delta i)\gamma}\delta\xi^\delta)\psi^\gamma,\quad(20.18)$$

$$\delta\psi^{*\alpha}=c^\alpha{}_{\beta\gamma}\psi^{*\gamma}\delta\xi^\beta,\quad(20.19)$$

$$\delta\xi^\alpha=-\phi^i(\tfrac{1}{2}R_{i\delta}\delta\tilde{\gamma}^{-1\delta\gamma}\tilde{\gamma}_{\gamma\beta}+\delta\gamma_{ij}R^j{}_\beta)\mathcal{G}^{\beta\alpha},\quad(20.20)$$

where $\mathcal{G}$ is the Feynman propagator[57] for the operator $\mathfrak{F}$, and the $\delta\gamma_{ij}$, $\delta\tilde{\gamma}_{\alpha\beta}$ are to be regarded as arbitrary infinitesimals.

We now compute the effect which these changes produce on the numerator and denominator of the big integrand in (20.12). By making use of the identity

$$V_{(\alpha i)\beta}\mathfrak{R}^i{}_\gamma-V_{(\gamma i)\beta}\mathfrak{R}^i{}_\alpha=c^\delta{}_{\alpha\gamma}\mathfrak{F}_{\delta\beta}\quad(20.21)$$

and the fact that $S[\varphi+\phi]$ is invariant under (20.17), we find

$$\delta(\tfrac{1}{2}\tilde{\gamma}_{\alpha\beta}\chi^\alpha\chi^\beta+\tfrac{1}{2}F_{ij}\phi^i\phi^j+(1/3!)S_{,ijk}\phi^i\phi^j\phi^k+\cdots)$$
$$=\tilde{\gamma}_{\alpha\beta}\chi^\alpha\delta\chi^\beta+R_i{}^\alpha R_{j\alpha}\phi^i\delta\phi^j=\tfrac{1}{2}\chi^\alpha\delta\tilde{\gamma}_{\alpha\beta}\chi^\beta+\phi^iR_i{}^\alpha\delta\xi^\beta\mathfrak{F}_{\beta\alpha}.$$
$$=\tfrac{1}{2}\chi^\alpha\delta\tilde{\gamma}_{\alpha\beta}\chi^\beta+\tfrac{1}{2}\phi^iR_{i\alpha}\delta\tilde{\gamma}^{-1\alpha\beta}R_{j\beta}\phi^j+\phi^iR_i{}^\alpha R^k{}_\alpha\delta\gamma_{kj}\phi^j,\quad(20.22)$$

$$\delta(\mathfrak{F}_{\alpha\beta}\psi^{*\alpha}\psi^\beta)$$
$$=\mathfrak{F}_{\alpha\beta}\psi^{*\alpha}\psi^\beta+\mathfrak{F}_{\alpha\beta}\psi^{*\alpha}\delta\psi^\beta+\psi^{*\alpha}V_{(\alpha i)\beta}\psi^\beta\delta\phi^i$$
$$=\psi^{*\alpha}\mathfrak{R}^i{}_\alpha\delta\gamma_{ij}R^j{}_\beta\psi^\beta.\quad(20.23)$$

But these are just the changes which these quantities would suffer under the changes $\delta\gamma_{ij}$, $\delta\tilde{\gamma}_{\alpha\beta}$ in the γ's.

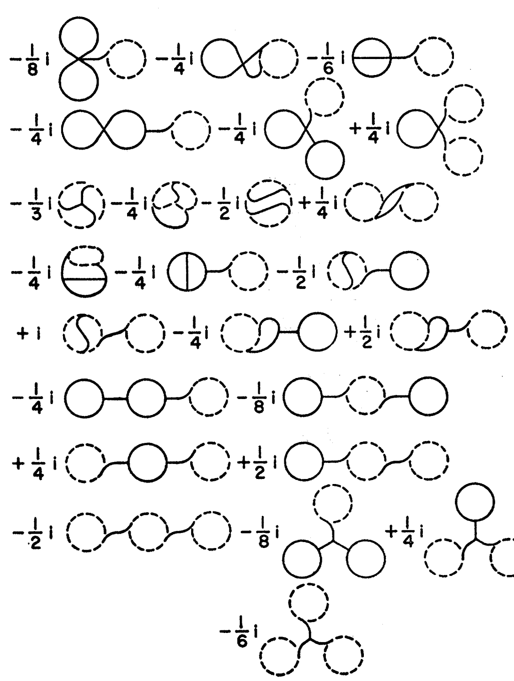

FIG. 8. Diagrams which must be added to $\bar{W}_{(3)}$ of Fig. 5 when an invariance group is present.

[55] That ln det$G$ appears in $\bar{W}$ with opposite signs for bosons and fermions was first pointed out long ago by Feynman. It is a consequence of the familiar minus sign which goes with closed fermion loops [R. P. Feynman, Phys. Rev. 76, 749 (1949)].

[56] See J. Schwinger, Proc. Nat. Acad. Sci. U. S. 48, 603 (1962).

[57] The presence of the Feynman propagators in Eqs. (20.18) and (20.20) indicates that we are dealing here with a special class of *nonlocal* transformations of variables. Transformations of this class are permissible because of the Feynman boundary conditions implicit in the functional integral.

Therefore, if we can show that the functional Jacobians of the transformations (20.16), (20.17), (20.18), and (20.19) cancel in (20.12), it follows that (20.12) is γ-invariant.

These Jacobians are obtained by first computing

$$\frac{\delta\chi'^\alpha}{\delta\chi^\beta}=\delta^\alpha{}_\beta+\tfrac{1}{2}\tilde{\gamma}^{-1\alpha\gamma}\delta\tilde{\gamma}_{\gamma\beta}, \tag{20.24}$$

$$\frac{\delta\phi'^i}{\delta\phi^j}=\delta^i{}_j+R^i{}_{\alpha,j}\delta\xi^\alpha+\mathfrak{R}^i{}_\alpha\frac{\delta(\delta\xi^\alpha)}{\delta\phi^j}, \tag{20.25}$$

$$\frac{\delta\psi'^\alpha}{\delta\psi^\beta}=\delta^\alpha{}_\beta-\mathcal{G}^{\alpha\gamma}\mathfrak{R}^i{}_\gamma(\delta\gamma_{ij}R^j{}_\beta-V_{(\delta i)\beta}\delta\xi^\delta), \tag{20.26}$$

$$\frac{\delta\psi^{*\prime\alpha}}{\delta\psi^{*\beta}}=\delta^\alpha{}_\beta+c^\alpha{}_{\gamma\beta}\delta\xi^\gamma, \tag{20.27}$$

$$\frac{\delta(\delta\xi^\alpha)}{\delta\phi^j}=-\tfrac{1}{2}R_{j\delta}\delta\tilde{\gamma}^{-1\delta\gamma}\tilde{\gamma}_{\gamma\beta}\mathcal{G}^{\beta\alpha}-\delta\gamma_{jk}R^k{}_\beta\mathcal{G}^{\beta\alpha}+\delta\xi^\beta V_{(\beta j)\gamma}\mathcal{G}^{\gamma\alpha}. \tag{20.28}$$

Invoking Eqs. (4.10) we then get

$$\begin{aligned}\frac{\delta(\chi',\phi')}{\delta(\chi,\phi)}&=1+\tfrac{1}{2}\operatorname{tr}(\tilde{\gamma}^{-1}\delta\tilde{\gamma})+R^i{}_{\alpha,i}\delta\xi^\alpha+\mathfrak{R}^i{}_\alpha\frac{\delta(\delta\xi^\alpha)}{\delta\phi^i}\\&=1-\mathfrak{R}^i{}_\alpha\delta\gamma_{ij}R^j{}_\beta\mathcal{G}^{\beta\alpha}+\mathfrak{R}^i{}_\alpha\delta\xi^\beta V_{(\beta i)\gamma}\mathcal{G}^{\gamma\alpha},\end{aligned} \tag{20.29}$$

$$\frac{\delta(\psi',\psi^{*\prime})}{\delta(\psi,\psi^*)}=1-\mathcal{G}^{\alpha\beta}\mathfrak{R}^i{}_\beta(\delta\gamma_{ij}R^j{}_\alpha-V_{(\gamma i)\alpha}\delta\xi^\gamma)+c^\alpha{}_{\gamma\alpha}\delta\xi^\gamma=\frac{\delta(\chi',\phi')}{\delta(\chi,\phi)}. \tag{20.30}$$

Since the latter Jacobian is independent of ψ^α and $\psi^{*\alpha}$ it can be removed from the integral in the denominator of (20.12), whereupon it cancels with the Jacobian for the χ and ϕ integrations. The γ invariance of formulas (20.7) and (20.12) is thus proved.

This establishes, in particular, the legitimacy of the scale transformation

$$\gamma_{ij}\to\lambda\gamma_{ij},\quad \tilde{\gamma}_{\alpha\beta}\to\lambda\tilde{\gamma}_{\alpha\beta}. \tag{20.31}$$

Under this transformation we have

$$R_{i\alpha}R_j{}^\alpha\to\lambda R_{i\alpha}R_j{}^\alpha,\quad \mathfrak{F}_{\alpha\beta}\to\lambda\mathfrak{F}_{\alpha\beta}, \tag{20.32}$$

and in the limit $\lambda\to 0$ Eq. (20.12) reduces to the conventional but ambiguous formula

$$\exp i\bar{w}[\varphi]=Z\int\exp i\left(\frac{1}{2!}S_{,ij}\phi^i\phi^j+\frac{1}{3!}S_{,ijk}\phi^i\phi^j\phi^k+\cdots\right)d\phi, \tag{20.33}$$

where

$$Z=Z'\int d\chi\bigg/\int d\psi\int d\psi^*. \tag{20.34}$$

Expression (20.12) evidently removes the ambiguity and may be regarded as the *definition* of the integral (20.33) when an invariance group is present.

There remains only the question how to insert external lines into the primary vacuum diagrams. When we dealt with W instead of $\bar{W}$ the insertion was accomplished by simple functional differentiation. Now that the noncausal chains are left unremoved and no correction terms are added we must proceed somewhat more carefully.

We have remarked earlier that expression (13.6) for the S matrix in terms of chronological products holds even when an invariance group is present. We were nevertheless forced to use it in a very circuitous manner, by restricting it to the case in which no group is present and then generalizing its c-number consequences, because we previously had no direct way of calculating the chronological products. Now we have.

Following the example of Eq. (19.5) (but ignoring the density functional Δ since we are here dealing with primary diagrams) we may set

$$\langle 0,\infty|T(A[\phi])|0,-\infty\rangle=\exp(-i\bar{w}[0])Z'\int A[\phi]\frac{\exp i(\frac{1}{2}\tilde{\gamma}_{\alpha\beta}\chi^\alpha\chi^\beta+\frac{1}{2}F_{ij}\phi^i\phi^j+(1/3!)S_{,ijk}\phi^i\phi^j\phi^k+\cdots)}{\int\exp(i\mathfrak{F}_{\alpha\beta}\psi^{*\alpha}\psi^\beta)d\psi d\psi^*}d\chi d\phi. \tag{20.35}$$

Then Eq. (19.6) is replaced by

$$\langle 0,\infty|\phi^i|0,-\infty\rangle=\exp(-i\bar{w}[0])Z'\mathfrak{G}_\pm{}^{ij}\int\left[\left(-F_{jk}\phi^k+\frac{\delta}{i\delta\phi^j}\right)\times\frac{\exp i(\frac{1}{2}\tilde{\gamma}_{\alpha\beta}\chi^\alpha\chi^\beta+\frac{1}{2}F_{lm}\phi^l\phi^m+(1/3!)S_{,lmn}\phi^l\phi^m\phi^n+\cdots)}{\int\exp(i\mathfrak{F}_{\alpha\beta}\psi^{*\alpha}\psi^\beta)d\psi d\psi^*}\right]d\chi d\phi \tag{20.36a}$$

$$\begin{aligned}=\exp(-i\bar{w}[0])Z'\mathfrak{G}_\pm{}^{ij}\int\Bigg[&\frac{1}{2!}S_{,jkl}\phi^k\phi^l+\frac{1}{3!}S_{,jklm}\phi^k\phi^l\phi^m+\cdots-\frac{\int V_{(\alpha i)\beta}\psi^{*\alpha}\psi^\beta\exp(i\mathfrak{F}_{\gamma\delta}\psi^{*\gamma}\psi^\delta)d\psi d\psi^*}{\int\exp(i\mathfrak{F}_{\alpha\beta}\psi^{*\alpha}\psi^\beta)d\psi d\psi^*}\Bigg]\\&\times\frac{\exp i(\frac{1}{2}\tilde{\gamma}_{\alpha\beta}\chi^\alpha\chi^\beta+\frac{1}{2}F_{rs}\phi^r\phi^s+(1/3!)S_{,rst}\phi^r\phi^s\phi^t+\cdots)}{\int\exp(i\mathfrak{F}_{\epsilon\zeta}\psi^{*\epsilon}\psi^\zeta)d\psi d\psi^*}d\chi d\phi,\end{aligned} \tag{20.36b}$$

where we now use the propagator $\mathfrak{G}_\pm$ in place of G in front of the integral so as to obtain correct external-line wave functions for the S matrix.

The only vertices which get inserted by the factor in square brackets in (20.36b) are the bare vertices S_n and $V_{(\alpha i)\beta}$. Therefore $\langle 0,\infty|\phi^i|0,-\infty\rangle$ may be expressed in the compact form (12.20), but with G replaced by $\mathfrak{G}_\pm$, provided the symbol $\delta/\delta\varphi$ is no longer taken literally but is understood to yield GS_3G when acting on G and S_{n+1} when acting on S_n, to have no effect on $V_{(\alpha i)\beta}$, and to insert (in all possible ways) into any fictitious quantum loop merely one more vertex $V_{(\alpha i)\beta}$ *having the same orientation as all the other vertices already in the loop.*[58] With this understanding it is easy to see that (20.35) then yields also Eqs. (12.21), (12.22), and (12.23), with the modification $G\rightarrow\mathfrak{G}_\pm$ applied to all external lines. Chronological product amplitudes defined in this way may be used directly in (13.6) to calculate the S matrix.

The consistency of these simple rules with previously obtained results is readily checked. For example, if noncausal chains are reinserted into Figs. 2(b) and 3(b) the resulting primary diagrams for the lowest-order radiative corrections to the one- and two-quantum amplitudes are precisely those obtained by the present prescription. We note in particular the sufficiency of the vertices S_n and $V_{(\alpha i)\beta}$ and the uniform orientation of the latter around any fictitious quantum loop.

[58] It will be noted that the operators $\delta/\delta\varphi^i$, when redefined in this way, are still commutative.

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Quantum Theory of Gravity. III. Applications of the Covariant Theory*

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The basic momentum-space propagators and vertices (including those for the fictitious quanta) are given for both the Yang-Mills and gravitational fields. These propagators are used to obtain the cross sections for gravitational scattering of two scalar particles, scattering of gravitons by scalar particles, graviton-graviton scattering, two-graviton annihilation of scalar-particle pairs, and graviton bremsstrahlung. Special features of these cross sections are noted. Problems arising in renormalization theory and the role of the Planck length are discussed. The gravitational Ward identity is derived, and the structure of the radiatively corrected 1-graviton vertex for a scalar particle is displayed. The Ward identity is only one of an infinity of identities relating the many-graviton vertex functions of the theory. The need for such identities may be eliminated in principle by computing radiative corrections directly in coordinate space, using the theory of manifestly covariant Green's functions. As an example of such a calculation, the contribution of conformal metric fluctuations to the vacuum-to-vacuum amplitude is summed to all orders. The physical significance of the renormalization terms is discussed. Finally, Weinberg's treatment of the infrared problem is examined. It is not difficult to show that the fictitious quanta contribute negligibly to infrared amplitudes, and hence that Weinberg's use of the DeDonder gauge is justified. His proof that the infrared problem in gravidynamics can be handled just as in electrodynamics is thereby made rigorous.

1. INTRODUCTION

In the first two papers of this series[1] two distinct mathematical approaches to the quantum theory of gravity were developed, one based on the so-called canonical or Hamiltonian theory and the other on the manifestly covariant theory of propagators and diagrams. So far no rigorous mathematical link between the two has been established. In part this is due to the kinds of questions each asks. The canonical theory leads almost unavoidably to speculations about the meaning of "amplitudes for different 3-geometries" or "the wave function of the universe." The covariant theory, on the other hand, concerns itself with "microprocesses" such as scattering, vacuum polarization, etc. Some of the questions raised by the canonical theory were explored in I. In this third and final paper of the series we examine some of the consequences of the covariant theory.

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[1] B. S. DeWitt, Phys. Rev. **160**, 1113 (1967); preceding paper, *ibid.* **162**, 1195 (1967). These papers will be referred to as I and II, respectively. The notation of the present paper is the same as that of II, which should be consulted for the definition of unfamiliar symbols, e.g., S_n for the n-pronged bare vertex and $V_{(\alpha i)\beta}$ for the asymmetric vertex coupling real and fictitious quanta.