Three-Body Problem with Charged Particles

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The formal modification, by long-range potentials, of the scattering equations describing three particles interacting through short-range potentials is discussed in terms of the exact Coulomb Green's function, whose mathematical properties are known. This Green's function can be explicitly written down in several special cases; approximate forms of the modified theory are described for these instances. The application of the Coulomb-modified three-particle scattering theory to the α-cluster model of C^{12} and to deuteron-induced nuclear reactions is described in some detail. For the three-α model, an heuristic generalization of the local plane-wave approximation is used to obtain a simple approximate form for the Coulomb Green's function. This new theory may have some relevance to the problem of helium burning in stars. Finally, in the Appendix a discussion is given of why the methods of multiparticle collision theory fail in the problem of calculating the general three-body Coulomb Green's function at positive, real energies. A possibly useful, mathematically well-defined method for constructing this Green's function at negative real energies is also given in the Appendix.

I. INTRODUCTION

THE Coulomb force plays an ambivalent role in nuclear physics. On one hand, it is regarded as a nuisance when it obscures the effect of the strong interactions in experiments performed with charged particles. On the other hand, because the Coulomb force is well understood, it can elucidate subtle features of nuclear structure. Clearly Coulomb effects are not always negligible: In certain circumstances incorrect results have been obtained through ignoring the physical consequences of their long range for particles with long wavelengths.[1]

It is the aim of this paper to show that the treatment of the Coulomb force is formally no harder in the three-body problem than in the two-body problem (although from the computational standpoint it is *much* harder). One merely constructs the Coulomb Green's function (which has well-defined mathematical properties,[2] although we only know how to explicitly construct it in special cases) and substitutes it everywhere for the free-particle Green's function. In situations where there might be uncertainty as to how to continue off the energy shell or where the use of a complex energy variable might lead to ambiguities,[3] we adopt the following physical convention. We treat the Coulomb potentials as though they were cut off at a distance much larger than the other distances in the problem. We then perform all necessary manipulations, express all physical quantities in terms of matrix elements involving the cutoff "Coulomb" wave functions, and employ the completeness relation for these functions wherever possible. Finally, all physically relevant expressions are analytically continued to real energy, and *then* the cutoff radius is allowed to become arbitrarily large. Performing the limiting processes in this order resolves all ambiguities and is clearly the way to arrive at the physical answer.

The organization of the paper is as follows: In Sec. II we shall see how to modify the Faddeev equations, and how to construct the generalized $3 \rightarrow 3$ amplitude. The derivation follows the physically rigorous procedure outlined above, which is in many ways reminiscent of the treatment of Goldberger and Watson,[4] except for the fact that it deals with three-particle scattering. The third section discusses the construction of the exact Coulomb Green's function in some important special cases. Section IV concludes by indicating how the general formalism may be applied to specific problems in nuclear physics, and discusses some possibly useful approximations. The Appendix shows how the Coulomb Green's function for an arbitrary three-particle system can be calculated for negative energies; this might be of some use in discussing the bound-state problem.

II. MODIFICATION OF THE FADDEEV EQUATIONS

The three-body system is assumed to be described by the Hamiltonian

$$H=H_0+V+U, \tag{1}$$

where H_0 is the kinetic-energy operator, V is the sum of short-range interactions, and U is the sum of the Coulomb interactions, which are taken to be screened with a range $\mathfrak{R}$ which is much larger than any of the other ranges in the problem. Let us define the Green's functions

$$G(W)=(W-H)^{-1}, \tag{2}$$

$$g(W)=(W-H_0-U)^{-1}. \tag{3}$$

In terms of $g(W)$, we may write

$$G(W)=g(W)+g(W)VG(W) \tag{4}$$

* Supported in part by the National Science Foundation.
[1] J. V. Noble, Phys. Rev. 148, 1528 (1966).
[2] W. Hunziker, Helv. Phys. Acta 39, 451 (1966).
[3] W. Ford, J. Math. Phys. 7, 626 (1966).
[4] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964), p. 259 ff.

using the usual identity. Following Faddeev,[5] we suppose that the solution of (4) may be written in the form

$$G(W)=g(W)+g(W)X(W)g(W), \tag{5}$$

where

$$X(W)=\sum_{\alpha=1}^{3} X_\alpha(W), \tag{6}$$

$$X_\alpha(W)=\tilde{t}_\alpha(W)+\tilde{t}_\alpha(W)g(W)\sum_{\beta\neq\alpha} X_\beta(W), \tag{7}$$

and, as usual,

$$\tilde{t}_\alpha(W)=V_\alpha+V_\alpha g(W)\tilde{t}_\alpha(W). \tag{8}$$

Equations (6)–(8) follow the by-now standard cyclic labeling convention, with $V_1\equiv V(r_{23})$, and so on; only pairwise interactions are assumed. By keeping the range of the Coulomb forces large but finite, all formal difficulties in Eqs. (5)–(8) are circumvented, and all of the operators are in principle well defined.[5]

Let us now consider the expression for the transition operator describing collisions in which both the initial and final states contain three free particles. Defining $|\chi_0\rangle$ to be the product of plane waves, i.e.,

$$(H_0-E)|\chi_0\rangle=0, \tag{9}$$

the amplitude for $3\rightarrow 3$ may be written

$$\lim_{\eta\rightarrow 0}\langle\chi_{0'}|[U+V+(U+V)G(E+i\eta)(U+V)]|\chi_0\rangle. \tag{10}$$

If we define the Coulomb-distorted plane waves by

$$|\phi_0^{\pm}\rangle=|\chi_0\rangle+g(E\pm i\eta)U|\chi_0\rangle, \tag{11}$$

we see that expression (10) may be rewritten.

$$T_{0'0}^{+}(E)=\langle\chi_{0'}|U|\phi_0^{+}\rangle+\langle\phi_{0'}^{-}|X(E+i\eta)|\phi_0^{+}\rangle. \tag{12}$$

The first term on the right-hand side of (12) may be immediately identified with the amplitude for pure (screened) Coulomb scattering. All the interesting physics therefore resides in the second term. We now note that there will be only a finite number of terms, in the formal expansion of $X(E+i\eta)$, which are *not* spatially bounded. These are the "disconnected terms," $\tilde{t}_\alpha(E+i\eta)$, and the "rescattering" terms, such as $\tilde{t}_\alpha g\tilde{t}_\beta$.[6] Obviously, the spatially bounded terms restrict the integrations in evaluating $\langle\phi_{0'}^{-}|X_{\text{bounded}}(E+i\eta)\times|\phi_0^{+}\rangle$ to volumes small compared to $\mathfrak{R}^3$, so the screening radius now may be allowed to become infinite in this part of $\langle\phi_{0'}^{-}|X(E+i\eta)|\phi_0^{+}\rangle$. What of the disconnected terms and the terms with rescattering singularities? If the Coulomb potential U is positive, the mathematical properties of $|\phi_0^{\pm}\rangle$ will be exactly the same as those of $|\chi_0\rangle$. That is, $\langle\mathbf{r},\mathbf{R}|\phi_0^{\pm}\rangle$ is a bounded, continuous function of six variables, independent of whether the screening $\mathfrak{R}$ is finite or infinite. Thus, the matrix elements

$$\lim_{\eta\rightarrow 0}\langle\phi_{0'}^{-}|\tilde{t}_\alpha(E+i\eta)|\phi_0^{+}\rangle,$$

$$\lim_{\eta\rightarrow 0}\langle\phi_{0'}^{-}|\tilde{t}_\alpha(E+i\eta)g(E+i\eta)\tilde{t}_\beta(E+i\eta)|\phi_0^{+}\rangle,$$

as $R\rightarrow\infty$, are no more pathological than those from the non-Coulomb problem:

$$\lim_{\eta\rightarrow 0}\langle\chi_{0'}|t_\alpha(E+i\eta)|\chi_0\rangle,$$

$$\lim_{\eta\rightarrow 0}\langle\chi_{0'}|t_\alpha(E+i\eta)G_0(E+i\eta)t_\beta(E+i\eta)|\chi_0\rangle,$$

where $G_0(W)=(W-H_0)^{-1}$ is the free-particle Green's function, and the t_α satisfy Eq. (8) with $g(W)$ replaced by $G_0(W)$. In general, the disconnected terms will still have δ-function singularities, and the rescattering terms will have rescattering singularities. It must be emphasized that this is true whether U is long- or short-ranged, as long as $\lim_{U\rightarrow 0}|\phi_0^{\pm}\rangle=|\chi_0\rangle$. In fact, the presence or absence of screening in U is irrelevant to a discussion of the difficulties involved in evaluating matrix elements of the disconnected and rescattering terms. The major problem is that there is not yet known an explicit representation for $|\phi_0^{\pm}\rangle$ or equivalently, $g(W)$. Thus, to proceed further, one must employ either approximate forms of $g(W)$, or else use the explicit form in the special cases where it is known. These matters will be explored further in the next section.

Let us conclude this section by pointing out that for rearrangement collisions one has the option of using either expression (5) for the Green's function, or of solving the particular Lippmann-Schwinger equation for the appropriate scattering operator by methods akin to those introduced above.[7] In either case, the limit $\mathfrak{R}\rightarrow\infty$ causes no difficulty.

III. THE COULOMB GREEN'S FUNCTION

Hunziker[2] has recently proved several properties of the Hamiltonian H_0+U, which hold even if U is the sum of two-body Coulomb interactions. First, H_0+U is bounded below and is essentially self-adjoint (in the sense of von Neumann) on the domain of H_0. Therefore, its eigenfunction spectrum is "as complete as" that of H_0. Secondly, the continuous part of the eigenvalue spectrum of H_0+U is bounded below and includes that of H_0. Finally, the discrete eigenvalues of H_0+U can accumulate only at the lower end of its continuous spectrum. (Obviously, if U is repulsive, there is no discrete spectrum.) Thus, $g(W)$ is, for positive energies, certainly a well-defined bounded operator (on the Banach space of bounded functions satisfying a Hölder condition[5]), even on the boundaries of the "kinematic" branch cuts given by the spectrum of H_0.

[5] L. D. Faddeev, *Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory* (Israel Program for Scientific Translations, Jerusalem, 1966).

[6] M. Rubin, R. Sugar, and G. Tikopoulos, Phys. Rev. **146**, 1130 (1966).

[7] J. V. Noble, Ph.D. thesis, Princeton University, 1966 (unpublished).

Keeping this in mind, we see that Faddeev's method[5] may be taken over without alteration, to show that the kernel of Eq. (7) is completely continuous.

At present there does not seem to be any rigorous *and* useful general method for constructing $g(W)$. Even if U is positive, so that $G(W)$ has no poles, and only one cut, the difficulties are formidable. If the screening distance is kept large but finite, any of the approaches given by Faddeev,[5] Weinberg,[8] or Blankenbecler and Sugar[9] will be formally suitable. The problems arise from the fact that a large number of partial waves will add coherently in the forward direction to produce a nearly-divergent sum, just as in ordinary two-body screened-Coulomb scattering. Furthermore, the iterated kernels such as $U_1G_0(W)U_2G_0(W)$ (in the usual notation) are not bounded on the previously mentioned Banach space when the screening is turned off *after* the energy is made real. It should be noted, however, that for negative, real W the kernels $U_\alpha G_0(W)U_\beta G_0(W)$ represent completely continuous operators on the Banach space of bounded, continuous functions of six variables, as is shown in the Appendix. Thus, for example, the bound states of the three-alpha particle model of C^{12} [10] may be investigated in a formally straightforward way.

There are two important special cases for which the Green's function $g(W)$ may be explicitly constructed. Because of the awesome difficulties introduced by *attractive* long-range forces, the remainder of the discussions will be restricted to purely repulsive potentials. Suppose one of the particles, say 3, is neutral. Then defining

$$\mathbf{q}_3=(m_2\mathbf{p}_1-m_1\mathbf{p}_2)/(m_1+m_2)\,, \tag{13}$$

$$\mu_3=m_1m_2/(m_1+m_2)\,, \tag{14}$$

$$M_3=m_3(m_1+m_2)/(m_1+m_2+m_3)\,, \tag{15}$$

we have, in the center-of-mass system

$$\langle\mathbf{p}_3'\mathbf{q}_3'|\,g(W)\,|\mathbf{p}_3\mathbf{q}_3\rangle=\delta(\mathbf{p}_3'-\mathbf{p}_3)\times\int d\mathbf{k}''\langle\mathbf{q}_3'|\,\phi_{\mathbf{k}''}{}^c\rangle$$
$$\times[W-(2M_3)^{-1}p_3{}^2-(2\mu_3)k''{}^2]^{-1}\langle\phi_{\mathbf{k}''}{}^c|\,\mathbf{q}_3\rangle\,. \tag{16}$$

In (16) above, $\langle\mathbf{q}_3|\,\phi_{\mathbf{k}}{}^c\rangle$ is the Fourier transform of the Coulomb-distorted plane wave satisfying

$$[\nabla^2+k^2-2Z_1Z_2e^2\mu_3r^{-1}]\langle\mathbf{r}|\,\phi_{\mathbf{k}}{}^c\rangle\,, \tag{17}$$

and such that

$$\lim_{e^2\to0}\langle\mathbf{r}|\,\phi_{\mathbf{k}}{}^c\rangle=(2\pi)^{-3/2}\exp(i\mathbf{k}\cdot\mathbf{r})\,.$$

The other special case is that in which particle No. 3 is infinitely massive, and is strongly charged. If the Coulomb interaction, U_{12}, between 1 and 2 is small compared to U_{13} and U_{23} and therefore is neglected, we may write $g(W)$ as (note that $\mathbf{p}_1$, $\mathbf{p}_2$ is a useful coordinate system with m_3 infinite)

$$\langle\mathbf{p}_1'\mathbf{p}_2'|\,g(W)\,|\mathbf{p}_1\mathbf{p}_2\rangle=\int d\mathbf{x}\int d\mathbf{y}\langle\mathbf{p}_1'|\,\phi_{\mathbf{x}}{}^c(1,3)\rangle$$
$$\times\langle\mathbf{p}_2'|\,\phi_{\mathbf{y}}{}^c(2,3)\rangle[W-(2m_1)^{-1}x^2-(2m_2)^{-1}y^2]^{-1}$$
$$\times\langle\phi_{\mathbf{y}}{}^c(2,3)\,|\,\mathbf{p}_2\rangle\langle\phi_{\mathbf{x}}{}^c(1,3)\,|\,\mathbf{p}_1\rangle\,. \tag{18}$$

Let us now examine what happens when we evaluate the $\tilde{t}_\alpha$ using the Green's function $g_3(W)$ defined in (16). First consider $\tilde{t}_3(W)$: since both V_3 and g_3 conserve $\mathbf{p}_3$, there will be an over-all factor of $\delta(\mathbf{p}_3'-\mathbf{p}_3)$ to factor out. Then, $[Z=W-(2M_3)^{-1}p_3{}^2]$

$$\langle\mathbf{q}_3'|\,\tilde{t}_3(Z)\,|\mathbf{q}_3\rangle=\langle\mathbf{q}_3'|\,V_3\,|\mathbf{q}_3\rangle+\int\langle\mathbf{q}_3'|\,V_3\,|\phi_{\mathbf{k}''}{}^c\rangle$$
$$\times(Z-(2\mu_3)^{-1}k''{}^2)^{-1}\langle\phi_{\mathbf{k}''}{}^c|\,\tilde{t}_3(Z)\,|\mathbf{q}_3\rangle d\mathbf{k}''\,, \tag{19}$$

where

$$\langle\mathbf{q}_3'|\,V_3\,|\phi_{\mathbf{k}}{}^c\rangle\equiv\int d\mathbf{q}_3\langle\mathbf{q}_3'|\,V_3\,|\mathbf{q}_3\rangle\langle\mathbf{q}_3|\,\phi_{\mathbf{k}}{}^c\rangle\,,$$

and so forth. Defining $\langle\mathbf{k}'|\,V_3{}^c\,|\mathbf{k}\rangle\equiv\langle\phi_{\mathbf{k}'}{}^c|\,V_3\,|\phi_{\mathbf{k}}{}^c\rangle$, we have

$$\langle\mathbf{q}_3'|\,\tilde{t}_3(Z)\,|\mathbf{q}_3\rangle=\int d\mathbf{k}'\int d\mathbf{k}\langle\mathbf{q}_3'|\,\phi_{\mathbf{k}'}{}^c\rangle$$
$$\times\langle\mathbf{k}'|\,t_3{}^c(Z)\,|\mathbf{k}\rangle\langle\phi_{\mathbf{k}}{}^c|\,\mathbf{q}_3\rangle\,, \tag{20}$$

where $\langle\mathbf{k}'|\,t_3{}^c(Z)\,|\mathbf{k}\rangle$ is the solution of the ordinary two-body Lippmann-Schwinger equation, using $\langle\mathbf{k}'|\,V_3{}^c\,|\mathbf{k}\rangle$ as the potential. Since the screened functions $|\phi_{\mathbf{k}}{}^c\rangle$, are used only in such matrix elements as $\langle\mathbf{k}'|\,V_3{}^c\,|\mathbf{k}\rangle$, they may obviously be replaced by the Coulomb wave functions without altering any of our results.

Next, consider, say, $\tilde{t}_1(W)$: it represents the scattering matrix of two particles (2 and 3), one of which is charged, interacting via a short-range force in the presence of a third, charged, particle (1). We have [the coordinates are defined by cyclic permutation of the indices in Eqs. (13)–(15)]

$$\langle\mathbf{p}_1'\mathbf{q}_1'|\,\tilde{t}_1(W)\,|\mathbf{p}_1\mathbf{q}_1\rangle=\delta(\mathbf{p}_1'-\mathbf{p}_1)\langle\mathbf{q}_1'|\,V_1\,|\mathbf{q}_1\rangle$$
$$+\int d\mathbf{q}_1''\int d\mathbf{p}_1'''\int d\mathbf{q}_1'''\langle\mathbf{q}_1'|\,V_1\,|\mathbf{q}_1''\rangle$$
$$\times\langle\mathbf{p}_1'\mathbf{q}_1''|\,g_3(W)\,|\mathbf{p}_1'''\mathbf{q}_1'''\rangle\langle\mathbf{p}_1'''\mathbf{q}_1'''|\,\tilde{t}_1(W)\,|\mathbf{p}_1\mathbf{q}_1\rangle\,. \tag{21}$$

In the notation of Ref. 6, letting $\alpha=m_3/(m_2+m_3)$, $\beta=m_1/(m_1+m_2)$, and $\gamma=m_2(m_1+m_2+m_3)/[(m_1+m_2)\times(m_2+m_3)]$, we have $\mathbf{p}_3=-\mathbf{q}_1-\alpha\mathbf{p}_1$, $\mathbf{q}_3=-\beta\mathbf{q}_1+\gamma\mathbf{p}_1$, and $\alpha\beta+\gamma=1$. Then

$$\langle\mathbf{p}_1'\mathbf{q}_1'|\,g_3(W)\,|\mathbf{p}_1\mathbf{q}_1\rangle$$
$$=\delta(\mathbf{q}_1'+\alpha\mathbf{p}_1'-\mathbf{q}_1-\alpha\mathbf{p}_1)\langle-\beta\mathbf{q}_1'+\gamma\mathbf{p}_1'|$$
$$\times g_3(W-(2M_3)^{-1}(\mathbf{q}_1+\alpha\mathbf{p}_1)^2)|-\beta\mathbf{q}_1+\gamma\mathbf{p}_1\rangle\,. \tag{22}$$

[8] S. Weinberg, Phys. Rev. **133**, B232 (1964).
[9] R. Sugar and R. Blankenbecler, Phys. Rev. **136**, B472 (1964).
[10] D. R. Harrington, Phys. Rev. **147**, 685 (1966).

Without some sort of approximation, it is difficult to proceed beyond Eq. (21). Schulman[11] has studied the local plane-wave approximation which is based on the idea that in momentum space, the Coulomb-modified plane wave is nearly a δ function. That is,

$$\int d\mathbf{p}\ f(\mathbf{p})\langle \mathbf{p}|\phi_k{}^c\rangle \simeq f(\mathbf{k})C_0(\mathbf{k}), \tag{23}$$

where $|C_0(k)|^2 \equiv 2\pi\eta[\exp(2\pi\eta)-1]^{-1}$ is the usual Coulomb penetration factor.[12] This approximation seems to be reasonable for functions $f(\mathbf{p})$ which are square-integrable, such as $(p^2+a^2)^{-1}$. It is clearly not correct for functions which are not $\mathcal{L}_2$, for example, $f(\mathbf{p})=\exp(i\mathbf{p}\cdot\mathbf{r})$. Thus one must be somewhat cautious about applying (23) in order to avoid incorrect results. A more physically motivated approximation comes from noting that in (21), particles 2 and 3 can interact only at short spatial separations, in which case they look like one charged particle as far as particle No. 1 is concerned. Since it is the long range of the Coulomb force which causes the difficulties, we need not worry particularly about what happens when all three particles are close, since the nuclear interaction between 2 and 3 then dominates the dynamics, relative to the Coulomb interaction 1 and 2. Let us define $\bar{U}_1$ to be the Coulomb potential which depends on the distance between particle 1 and the center of mass of 2 and 3. Then we may write

$$g_3(W)=(W-H_0-U_3)^{-1} \equiv [W-H_0-\bar{U}_1-(U_3-\bar{U}_1)]^{-1}. \tag{24}$$

Defining $\bar{g}_1(W)=(W-H_0-\bar{U}_1)^{-1}$, we have

$$g_3(W)=\bar{g}_1(W)+\bar{g}_1(W)(U_3-\bar{U}_1)g_3(W), \tag{25}$$

and so

$$\langle \mathbf{p}_1'\mathbf{q}_1'|\tilde{t}_1(W)|\mathbf{p}_1\mathbf{q}_1\rangle=\delta(\mathbf{p}_1'-\mathbf{p}_1)\langle \mathbf{q}_1'|V_1|\mathbf{q}_1\rangle+\int\langle \mathbf{q}_1'|V_1|\mathbf{q}_1''\rangle\langle \mathbf{p}_1'|\bar{g}_1(W-(2\mu_1)^{-1}q_1''^2)|\mathbf{p}_1''\rangle$$
$$\times\langle \mathbf{p}_1''\mathbf{q}_1''|\tilde{t}_1(W)|\mathbf{p}_1\mathbf{q}_1\rangle d\mathbf{p}_1''d\mathbf{q}_1''+\frac{Z_1Z_2e^2}{2\pi^2}\int\Big\{\int\{\langle \mathbf{q}_1'|V_1|\mathbf{q}_1''\rangle\langle \mathbf{p}_1'|\bar{g}_1(W-(2\mu_1)^{-1}q_1''^2)|\mathbf{p}_1''\rangle(\mathbf{p}_1''-\mathbf{p}_1''')^{-2}$$
$$\times[\delta(\mathbf{q}_1''+\alpha\mathbf{p}_1''-\mathbf{q}_1'''-d\mathbf{p}_1''')-\delta(\mathbf{q}_1''-\mathbf{q}_1''')]\}d\mathbf{p}_1''d\mathbf{q}_1''\Big\}\langle \mathbf{p}_1'''\mathbf{q}_1'''|g_3(W)\tilde{t}_1(W)|\mathbf{p}_1\mathbf{q}_1\rangle d\mathbf{p}_1'''d\mathbf{q}_1'''. \tag{26}$$

Let us now consider

$$I=\int\langle \mathbf{q}_1'|V_1|\mathbf{q}_1''\rangle\langle \mathbf{p}_1'|\bar{g}_1(W-(2\mu_1)^{-1}q_1''^2)|\mathbf{p}_1''\rangle(\mathbf{p}_1''-\mathbf{p}_1)^{-2}[\delta(\mathbf{q}_1''+\alpha\mathbf{p}_1''-\mathbf{q}_1-\alpha\mathbf{p}_1)-\delta(\mathbf{q}_1''-\mathbf{q}_1)]d\mathbf{q}_1''d\mathbf{p}_1''$$
$$=\int d\mathbf{k}_1\langle \mathbf{p}_1'|\phi_{k_1}{}^c\rangle\int d\mathbf{p}_1''\langle \phi_{k_1}{}^c|\mathbf{p}_1''\rangle(\mathbf{p}_1''-\mathbf{p}_1)^{-2}\{\langle \mathbf{q}_1'|V_1|\mathbf{q}_1+\alpha(\mathbf{p}_1''-\mathbf{p}_1)\rangle$$
$$\times[W-(2M_1)^{-1}k_1^2-(2\mu_1)^{-1}(\mathbf{q}_1+\alpha(\mathbf{p}_1''-\mathbf{p}_1))^2]^{-1}-\langle \mathbf{q}_1'|V_1|\mathbf{q}_1\rangle[W-(2M_1)^{-1}k_1^2-(2\mu_1)^{-1}q_1^2]^{-1}\}. \tag{27}$$

[Note that the Coulomb wave functions which appear in (27) are slightly different from those of Eq. (17).] The integrand of the p_1'' integral in (27) is precisely the type for which approximation (23) works well, and so

$$I\simeq|C_0(p_1';\bar{U}_1)|^2(\mathbf{p}_1'-\mathbf{p}_1)^{-2}[\langle \mathbf{q}_1'|V_1|\mathbf{q}_1+\alpha(\mathbf{p}_1'-\mathbf{p}_1)\rangle$$
$$\times\{W-(2M_1)^{-1}p_1'^2-(2\mu_1)^{-1}(\mathbf{q}_1+\alpha(\mathbf{p}_1'-\mathbf{p}_1))^2\}^{-1}$$
$$-\langle \mathbf{q}_1'|V_1|\mathbf{q}_1\rangle\{W-(2M_1)^{-1}p_1'^2-(2\mu_1)^{-1}q_1^2\}^{-1}]. \tag{28}$$

Because (28) is not a singular or strongly peaked function of $|\mathbf{p}_1-\mathbf{p}_1'|$, the correlations it contains are relatively negligible. It is thus reasonable to treat the order e^2 terms in (26) as a small perturbation if the potentials V_α have a strength parameter much greater than $Z_1Z_2e^2$; this is almost always the case in nuclear physics. In other words, defining $\tilde{t}_1$ by

$$\langle \mathbf{p}_1'\mathbf{q}_1'|\tilde{t}_1(W)|\mathbf{p}_1\mathbf{q}_1\rangle=\delta(\mathbf{p}_1'-\mathbf{p}_1)\langle \mathbf{q}_1'|V_1|\mathbf{q}_1\rangle+\int d\mathbf{p}_1''d\mathbf{q}_1''$$
$$\times\langle \mathbf{q}_1'|V_1|\mathbf{q}_1''\rangle\langle \mathbf{p}_1'|\bar{g}_1(W-(2\mu_1)^{-1}\mathbf{q}_1''^2)|\mathbf{p}_1''\rangle$$
$$\times\langle \mathbf{p}_1''\mathbf{q}_1''|\tilde{t}_1(W)|\mathbf{p}_1\mathbf{q}_1\rangle, \tag{29}$$

we have $\tilde{t}_1(W)=\tilde{t}_1(W)+O(e^2)$. We note that (29) is exactly solvable in terms of the eigenfunctions $\bar{\phi}_k{}^c$ of $H_0+\bar{U}_1$:

$$\langle \mathbf{p}_1'\mathbf{q}_1'|\tilde{t}_1(W)|\mathbf{p}_1\mathbf{q}_1\rangle=\int d\mathbf{k}_1\langle \mathbf{p}_1'|\bar{\phi}_{k_1}{}^c\rangle$$
$$\times\langle \mathbf{q}_1'|t_1(W-(2M_1)^{-1}k_1^2)|\mathbf{q}_1\rangle\langle\bar{\phi}_{k_1}{}^c|\mathbf{p}_1\rangle, \tag{30}$$

where $\langle \mathbf{q}_1'|t_1(Z)|\mathbf{q}_1\rangle$ is the ordinary two-body scattering

[11] L. Schulman, Phys. Rev. **156**, 1129 (1967). This paper includes the more important references to the literature on Coulomb wave functions.

[12] A. Messiah, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1961), Vol. I, p. 486.

matrix generated by V_1 in the usual Lippmann-Schwinger equation. A similar approximation for $\tilde{t}_2$ also is easily constructed. Equation (29) has the advantage of possessing the correct high-energy limit [obtained from (21) by asymptotic approximation for large $|\mathbf{p}_1|$ and $|\mathbf{p}_1'|$] as well as the physical low-energy behavior. [From its structure, we see (26) is the adiabatic approximation.]

It is not yet possible to explore the problem of three charged particles in detail, since a general expression for $g(W)$ is not available. This is not a handicap in the bound-state problem, since for W real and negative, $g(W)$ may be constructed by standard techniques (see the Appendix); in fact, for this case it may not even be advantageous to split up the two-body potentials into their short- and long-range parts. In the spirit of the local plane-wave approximation Eq. (23), an approximation for the special case of three identical charged particles may be developed. Since its applicability is at present restricted to one specific physical situation, this approximation will be introduced in the next section.

IV. APPLICATIONS TO NUCLEAR THREE-BODY PROBLEMS

A. Three-Alpha Model C^{12}

Harrington[10] has numerically solved the bound-state Faddeev equations for three identical uncharged spinless particles, interacting via 1-S separable potentials and has applied the results to the alpha-cluster model of C^{12}, with fair success. Because of its astrophysical significance, the three-alpha problem needs to be studied at positive energies as well as at negative energies. The constructive technique for $g(W)$ outlined in the Appendix fails for $W>0$, so to proceed further, drastic approximations are necessary.

We consider the Coulomb wave function for three identical charged bosons, and conjecture that since each two-body scattering is essentially forward, it will be reasonable to write

$$\langle \mathbf{q}_i\mathbf{p}_i|\phi_{\mathbf{k}_i\mathbf{K}_i}{}^c\rangle \simeq C(\mathbf{k}_i,\mathbf{K}_i)\delta(\mathbf{q}_i-\mathbf{k}_i)\delta(\mathbf{p}_i-\mathbf{K}_i). \quad (31)$$

We now require the three-body analog of the barrier penetration factor. We have no exact representation of the three-body Coulomb wave function, so we first write

$$C=(2\pi)^3\langle \mathbf{r}_i,\mathbf{R}_i|\phi_{\mathbf{k}_i,\mathbf{K}_i}{}^c\rangle|_{r_i=R_i=0}. \quad (32)$$

The total symmetry of Eq. (32) with regard to interchange of the particles implies that C can depend only on the energy $k_i{}^2+\frac{3}{4}K_i{}^2$, or must consist of symmetric products like $c(k_1)c(k_2)c(k_3)$. For low (total) energies, rather than depend on an approximation of doubtful validity such as the product form just mentioned, we prefer to assume that

$$C(\mathbf{k}_i,\mathbf{K}_i)\equiv C(k_i{}^2+\tfrac{3}{4}K_i{}^2), \quad (33)$$

which can be derived from a full three-particle wave equation (with some approximations which are clearly correct in the low-energy limit). We consider the coordinate system suggested by Bayman[13] in which the three particles are assumed to define the $x-y$ plane. Then the radius coordinate is

$$\rho^2=\tfrac{1}{3}\sum_{\alpha>\beta=1}^{3}[(x_\alpha-x_\beta)^2+(y_\alpha-y_\beta)^2] \quad (34)$$

and the Schrödinger equation becomes (only the components with lowest angular momentum and with greatest symmetry survive in $\langle \mathbf{r},\mathbf{R}|\phi_{\mathbf{k},\mathbf{K}}{}^c\rangle$ at $r=R=0$)

$$\rho^{-5}\frac{d}{d\rho}\left(\rho^5\frac{du_0(\rho)}{d\rho}\right)+[\bar{k}^2-2\eta\bar{k}\rho^{-1}]u_0(\rho)=0, \quad (35)$$

where $\eta=3Me^2\bar{k}^{-1}$, $\bar{k}^2=2(k_i{}^2+\frac{3}{4}K_i{}^2)$, and we have assumed that in this most symmetric state the particles are at the vertices of an equilateral triangle of side ρ [note that ρ defined by (34) is the average interparticle distance]. Making the usual transformation to eliminate the first derivative term, i.e., $u_0(\rho)=\rho^{-5/2}v(\rho)$, we get

$$\left[\frac{d^2}{d\rho^2}+\bar{k}^2-\frac{\frac{3}{2}(\frac{3}{2}+1)}{\rho^2}-\frac{2\eta\bar{k}}{\rho}\right]v(\rho)=0. \quad (36)$$

The solution of (36) which is regular at the origin is proportional to $\exp(i\bar{k}\rho)\rho^{5/2}\,{}_1F_1(\frac{5}{2}+i\eta|5|-2i\bar{k}\rho)$. Normalizing to the asymptotic behavior $v(\rho)\sim\cos[\bar{k}\rho+\sigma_0(\bar{k})-\eta\ln(2\bar{k}\rho)-\frac{1}{4}\pi]$, we find the appropriate penetration factor to be[14] [note $C(e^2=0)=1$]

$$C(\tfrac{1}{2}\bar{k}^2)\equiv C(\eta)=\exp(-\tfrac{1}{2}\pi\eta)\,|\Gamma(\tfrac{5}{2}+i\eta)|/\Gamma(\tfrac{5}{2})$$
$$=\tfrac{3}{4}\exp(-\tfrac{1}{2}\pi\eta)[(\eta^2+\tfrac{1}{4})(\eta^2+9/4)/\cosh(\pi\eta)]^{-1/2}. \quad (37)$$

With approximation (31), the Green's function becomes

$$\langle \mathbf{p}_i'\mathbf{q}_i'|\,g(W)\,|\mathbf{p}_i\mathbf{q}_i\rangle \simeq \delta(\mathbf{p}_i'-\mathbf{p}_i)\delta(\mathbf{q}_i'-\mathbf{q}_i)C^2(q_i{}^2+\tfrac{3}{4}p_i{}^2)$$
$$\times(W-M^{-1}q_i{}^2-\tfrac{3}{4}M^{-1}p_i{}^2)^{-1},\quad i=1,2,3. \quad (38)$$

Using (38), we obtain the following equations defining the bound-state wave function ($J=0$), in the case of separable two-body interactions[15]:

$$\psi(q,p)=-2C^2(q^2+\tfrac{3}{4}p^2)v(q)\tau(E,p)$$
$$\times[E-E(q,p)]^{-1}f(p), \quad (39)$$

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[13] B. Bayman, in *Proceedings of the Eastern Theoretical Physics Conference—1962* (Gordon and Breach Science Publishers, New York, 1963).

[14] *Handbook of Mathematical Functions*, edited by M. Abramowitz and L. A. Stegun (U. S. Department of Commerce, National Bureau of Standards, Washington, D. C., 1964), Appl. Math. Ser. 55, p. 256.

[15] These equations generalize Harrington's Eq. (11) (Ref. 10) for the case when the Coulomb Green's function is given by Eq. (37).

where

$$f(p)=-2\int d\mathbf{p}'\frac{v^*(\mathbf{p}'+\frac{1}{2}\mathbf{p})v(\mathbf{p}+\frac{1}{2}\mathbf{p}')\tau(E,p')C^2(p^2+p'^2+\mathbf{p}\cdot\mathbf{p}')f(p')}{E-E(|\mathbf{p}+\frac{1}{2}\mathbf{p}'|,\,p')}, \tag{40}$$

$$\tau(E,p)=\left[\lambda^{-1}+\int d\mathbf{q}''\frac{|v(q'')|^2C^2(q''^2+\frac{3}{4}p^2)}{E-E(q'',p)}\right]^{-1}, \tag{41}$$

$$E(p,q)=M^{-1}(q^2+\tfrac{3}{4}p^2), \tag{42}$$

and the two-body scattering matrix in the presence of the over-all Coulomb field [using approximation (31)] is

$$\langle\mathbf{q}'|\tilde{t}(E-\tfrac{3}{4}M^{-1}p^2;\,p)|\mathbf{q}\rangle=-v(\mathbf{q}')\tau(E,p)v^*(\mathbf{q}). \tag{43}$$

[Note that the function $v(q)$ in (43) is not the same as that in (36).]

For the 3α scattering problem the analog of Eq. (40) is simply

$$h(\mathbf{p},\mathbf{p}';W)=v(\mathbf{p}+\tfrac{1}{2}\mathbf{p}')v^*(\mathbf{p}'+\tfrac{1}{2}\mathbf{p})C^2(p^2+p'^2+\mathbf{p}\cdot\mathbf{p}')[W-E(|\mathbf{p}'+\tfrac{1}{2}\mathbf{p}|,\,p)]^{-1}$$
$$-2\int d\mathbf{p}''\frac{v^*(\mathbf{p}''+\frac{1}{2}\mathbf{p})v(\mathbf{p}+\frac{1}{2}\mathbf{p}'')C^2(p^2+p''^2+\mathbf{p}\cdot\mathbf{p}'')\tau(W,p'')h(\mathbf{p}'',\mathbf{p}';W)}{W-E(|\mathbf{p}''+\frac{1}{2}\mathbf{p}|,\,p)}, \tag{44}$$

where we may define the $3\rightarrow 3$ scattering matrix by $T(W)=T^{(1)}(W)+T^{(2)}(W)+T^{(3)}(W)$[5] and where

$$\langle\mathbf{p}_1\mathbf{q}_1|\,T^{(1)}(W)\,|\mathbf{p}_1'\mathbf{q}_1'\rangle=-v(\mathbf{q}_1)\tau(E,p_1)v^*(\mathbf{q}_1')$$
$$\times[\delta(\mathbf{p}_1-\mathbf{p}_1')-2h(\mathbf{p}_1,\mathbf{p}_1';W)\tau(W,p_1')]. \tag{45}$$

It is clear that for three identical particles, the functions $T^{(2)}$ and $T^{(3)}$ are obtained simply by substituting $p_{2,3}$ and $q_{2,3}$ everywhere in place of p_1, q_1 in Eq. (45)

It should be most interesting to see whether calculations using the approximate theory outlined above will give a reasonable value for the Coulomb energy. At any rate, this theory should enable the present crude estimates of the reaction rates for[16] $3\,He^4\rightarrow C^{12}+2\gamma$ to be considerably improved, with consequent implications for the theory of stellar structure and element formation.

B. Deuteron Reactions with Heavy Targets

We now use what we have learned in Sec. III to write down the equations for Coulomb-modified deuteron scattering in approximate form suitable for calculations. The theory of Ref. 7 will be taken as the starting point, rather than the Faddeev equations. We assume the Hamiltonian is given by

$$H=H_0+V_n+V_p+V_{np}+U_p, \tag{46}$$

where V_n and V_p are the neutron-target and proton-target nuclear interactions, respectively, V_{np} is the neutron-proton interaction, and U_p is the proton-target Coulomb potential, which will be assumed to be that for point charges. The Coulomb Green's function is defined by

$$g_p(W)=(W-H_0-U_p)^{-1}. \tag{47}$$

[16] D. Frank-Kamenetskii, *Physical Processes in Stellar Interiors* (translated from Russian) (Israel Program for Scientific Translations, Jerusalem, 1962), p. 200. The estimated reaction rate is probably uncertain by a factor of 10.

It will also be convenient to define the Coulomb potential acting on the deuteron center-of-mass, system which in coordinate space is given by

$$U_d=Ze^2[\tfrac{1}{2}(\mathbf{r}_n+\mathbf{r}_p)]^{-1}(1+2m/M_c)^{-1}. \tag{48}$$

The momentum-space coordinates are

$$\mathbf{K}=\mathbf{k}_n+\mathbf{k}_p \tag{49a}$$

and

$$\mathbf{q}=\tfrac{1}{2}(\mathbf{k}_n-\mathbf{k}_p). \tag{49b}$$

We shall also need the reduced mass appropriate to K: $M=2mM_c/(2m+M_c)$, where M_c is the target mass and m the nucleon mass. In terms of M and m, the kinetic energy is $(2M)^{-1}K^2+m^{-1}q^2$. The elastic deuteron scattering amplitude is defined, for example, by

$$T_{el}^+(E)=\langle\phi_{d'}^-|[V_n+V_p+(U_p-U_d)]|\psi_d^+\rangle$$
$$+\langle\phi_{d'}^-|\,U_d\,|\chi_d\rangle, \tag{50}$$

where

$$|\psi_d^+\rangle=|\chi_d\rangle+(E+i\eta-H_0-V_{np})^{-1}$$
$$\times(V_n+V_p+U_p)|\psi_d^+\rangle \tag{51}$$

is the appropriate solution of the Schrödinger equation asymptotic to the direct product $|\chi_d\rangle$ of a deuteron internal wave function and a plane wave for the deuteron motion; and where the Coulomb-distorted deuteron wave is formally given by

$$|\phi_d^\pm\rangle=|\chi_d\rangle+(E+i\eta-H_0-V_{np})^{-1}U_d|\phi_d^\pm\rangle. \tag{52}$$

Analogously to Eq. (12), the scattering amplitude (50) is the sum of a nuclear part, plus the amplitude for scattering by the effective Coulomb potential acting on the deuteron, plus a small, short-ranged correction, U_p-U_d, for the fact that the $|\phi_d^\pm\rangle$ are eigenfunctions of $H_0+V_{np}+U_d$.

The next problem is to construct $|\psi_d^+\rangle$; we have

$$|\psi_d^+\rangle=|\phi_p^+\rangle+(E+i\eta-H_0-V_{np}-U_p)^{-1}\times(V_n+V_p)|\psi_d^+\rangle, \quad (53)$$

where

$$|\phi_p^+\rangle=|\chi_d\rangle+(E+i\eta-H_0-V_{np})^{-1}U_p|\phi_p^+\rangle. \quad (54)$$

Now $|\phi_p^+\rangle$ may be written

$$|\phi_p^+\rangle=|\phi_d^+\rangle+(E+i\eta-H_0-V_{np}-U_d)^{-1}\times(U_p-U_d)|\phi_p^+\rangle. \quad (55)$$

Since U_p and U_d are both repulsive and so support no bound states, the iterative solution of Eq. (55) at least has the possibility of existing.[7] In fact, it is possible to show that on the Banach space of bounded Hölder functions,[5] $(E+i\eta-H_0-U_d)^{-1}(U_p-U_d)$ is a bounded operator, whose norm decreases with energy, so that for sufficiently large energies,

$$|\phi_p^\pm\rangle\simeq|\phi_d^\pm\rangle. \quad (56)$$

Furthermore, in view of our previous remarks on the applicability of the local plane-wave approximation, Eq. (56) is probably quite good even at low energies, when $\langle\mathbf{Kq}|\phi_p^+\rangle$ is multiplied by a square-integrable function of $\mathbf{q}$ and integrated with respect to $\mathbf{q}$. The fact that Eq. (56) is frequently a good approximation does not imply that we can with impunity replace U_p everywhere by U_d, however. Instead, we formally keep the Green's function $g_p(W)$ defined by Eq. (47), except in circumstances where the use of

$$g_d(W)=(W-H_0-U_d)^{-1} \quad (57)$$

can be shown not to be misleading.

We now continue calculating $|\psi_d^+\rangle$: as has been discussed in many previous works,[17] the kernel of Eq. (53) is noncompact[18] as it stands, and further manipulations are needed to make it suitable for numerical solution. We define the two-body scattering matrices as in Sec. III:

$$\tilde{t}_n(W)=V_n+V_ng_p(W)\tilde{t}_n(W), \quad (58)$$

$$\tilde{t}_p(W)=V_p+V_pg_p(W)\tilde{t}_p(W), \quad (59)$$

$$\tilde{t}_{np}(W)=V_{np}+V_{np}g_p(W)\tilde{t}_{np}(W). \quad (60)$$

We also define operators $X_n(W)$ and $X_p(W)$ by the Faddeev-type equations[5]

$$X_n(W)=\tilde{t}_n(W)+\tilde{t}_n(W)g_p(W)X_p(W), \quad (61a)$$

$$X_p(W)=\tilde{t}_p(W)+\tilde{t}_p(W)g_p(W)X_n(W). \quad (61b)$$

Note that $\tilde{t}_n$ is easily calculated in the recoilless case, $\tilde{t}_p$ is obtained exactly as in Eq. (19), and $g_p(W)$ may be replaced by $g_d(W)$ in (60) without causing grief, so that $\tilde{t}_{np}$ may be expressed in the form (30). We define the deuteron elastic-scattering operator by $(W=E+i\eta)$

$$\mathbf{T}(W)|\phi_p^+\rangle=(V_n+V_p)|\psi_d^+\rangle, \quad (62)$$

and therefore,

$$\mathbf{T}(W)|\phi_p^+\rangle=(V_n+V_p)|\phi_p^+\rangle+(V_n+V_p)g_p(W)\mathbf{T}(W)|\phi_p^+\rangle+(V_n+V_p)g_p(W)\tilde{t}_{np}(W)g_p(W)\mathbf{T}(W)|\phi_p^+\rangle. \quad (63)$$

Using (61), Eq. (63) may be rewritten

$$\mathbf{T}(W)|\phi_p^+\rangle=[X_n(W)+X_p(W)]|\phi_p^+\rangle+[X_n(W)+X_p(W)]g_p(W)\tilde{t}_{np}(W)\times g_p(W)\mathbf{T}(W)|\phi_p^+\rangle. \quad (64)$$

[Recall that X_n and X_p were chosen so that

$$[1-(V_n+V_p)g_p(W)]^{-1}(V_n+V_p)=X_n(W)+X_p(W).]$$

Finally, we replace $g_p(W)$ by $g_d(W)$ in Eq. (60), and use a separable form for the ordinary two-body matrix t_{np} [defined by $t_{np}=V_{np}+V_{np}(W-H_0)^{-1}t_{np}$ in the n-p barycentric subspace]:

$$\langle\mathbf{q}'|t_{np}(Z)|\mathbf{q}\rangle=-v_1(\mathbf{q}')\Lambda^T\tau_1(Z)v_1^*(\mathbf{q})-v_0(\mathbf{q}')\Lambda^S\tau_0(Z)v_0^*(\mathbf{q}), \quad (65)$$

where Λ^T and Λ^S are the triplet and singlet spin projection operators, respectively. Putting these approximations together, we get

$$\mathbf{T}(W)|\phi_p^+\rangle=[X_n(W)+X_p(W)]|\phi_p^+\rangle-\sum_{s''=0}^{1}\sum_{\nu''=-s''}^{s''}\int d\mathbf{K}''\times\left\{\left[\int d\mathbf{q}''(X_n(W)+X_p(W))g_p(W)|\mathbf{q}''s''\nu'';\phi_{\mathbf{K}''}{}^c\rangle v_{S''}(\mathbf{q}'')\right]\times\tau_{S''}(W-(2M)^{-1}K''^2)\left[\int d\mathbf{q}'''v_{S''}{}^*(\mathbf{q}''')\langle\mathbf{q}'''s''\nu'';\phi_{\mathbf{K}''}{}^c|g_p(W)\mathbf{T}(W)|\phi_p^+\rangle\right]\right\}. \quad (66)$$

(S is the total spin of the n-p system, and ν is its z projection.) Left-multiplying Eq. (66) by $\phi_{\mathbf{K}'}{}^{c*}v_{S'}(\mathbf{q}')g_p(W)$ and

[17] See Ref. 7 for a more complete bibliography.

[18] In this connection, "noncompact" means "not a Fredholm kernel." An excellent summary of the mathematics of integral equations, in a form accessible to physicists, is given by S. Weinberg, in Appendix A of Ref. 8.

integrating appropriately, we have, in an obvious notation,

$$\langle v_{S'\nu'};\phi_{\mathbf{K}'}{}^{c}|g_p(W)\mathbf{T}(W)|\phi_p{}^{+}\rangle$$
$$=\langle v_{S'\nu'};\phi_{\mathbf{K}'}{}^{c}|g_p(W)[X_n(W)+X_p(W)]g_p(W)|\phi_p{}^{+}\rangle-\sum_{s''\nu''}\int d\mathbf{K}''\langle v_{S'\nu'};\phi_{\mathbf{K}'}{}^{c}|g_p(W)[X_n+X_p]g_p(W)|v_{S''\nu''};\phi_{\mathbf{K}''}{}^{c}\rangle$$
$$\times\tau_{S''}(W-(2M)^{-1}K''^{2})\langle v_{S''\nu''};\phi_{\mathbf{K}''}{}^{c}|g_p(W)|\phi_p{}^{+}\rangle. \quad (67)$$

We shall now assume that $g_p(W)|v_{1\nu};\phi_{\mathbf{K}}{}^{c}\rangle$ is the off-shell continuation of $|\phi_p{}^{+}\rangle$; this is consistent with approximation (56) and the respresentation, Eq. (30), we have been tacitly using for $\tilde{t}_{np}(W)$; it is also most convenient. With this assumption implemented in Eq. (67), we can now calculate $\mathbf{T}(W)|\phi_p{}^{+}\rangle\simeq\mathbf{T}(W)g_p(W)|v_{1\nu};\phi_{\mathbf{K}}{}^{c}\rangle$ by solving the integral equation (67) and substituting the solution in (66); then using the definition (62) and Eq. (53) [with approximation (56) if convenient], we see that

$$|\psi_d{}^{+}\rangle\simeq|\phi_p{}^{+}\rangle+g_p(W)\mathbf{T}(W)g_p(W)|v_{1\nu};\phi_{\mathbf{K}}{}^{c}\rangle$$
$$-\sum_{s''\nu''}\int d\mathbf{K}''\{g_p(W)|v_{S''\nu''};\phi_{\mathbf{K}''}{}^{c}\rangle\tau_{S''}(W-(2M)^{-1}K''^{2})$$
$$\times\langle v_{S''\nu''};\phi_{\mathbf{K}''}{}^{c}|g_p(W)\mathbf{T}(W)|v_{1\nu};\phi_{\mathbf{K}}{}^{c}\rangle\}. \quad (68)$$

Since all the amplitudes describing transitions with a deuteron in the initial state may be expressed as matrix elements involving $|\psi_d{}^{+}\rangle$, our task of describing deuteron reactions with Coulomb modification is formally complete. The final expression for the deuteron scattering amplitude is reminiscent of the distorted-wave Born approximation. This is no accident.[7,19] The object which plays the role of the deuteron optical potential in Eq. (67),[20]

$$\langle v_{S'\nu'};\phi_{\mathbf{K}'}{}^{c}|g_p(W)[X_n(W)+X_p(W)]g_p(W)|v_{S\nu};\phi_{\mathbf{K}}{}^{c}\rangle,$$

is, however, much more complicated than the optical potential used in DWBA calculations. It is derived from a detailed theory of three-particle scattering, and is not at all phenomenological in nature. It is exactly known, given the ability to solve Eqs. (61), and to perform the quadratures implied by the matrix element. Finally, in contrast with the phenomenological optical potential, the "optical potential" derived here contains the possibility of $S=1\rightarrow S=0$ transitions, even when the nuclear interactions V_n and V_p are identical. Such transitions lead to isospin violating components of the deuteron-scattering wave function $|\psi_d{}^{+}\rangle$, so that transitions, from initial states with deuterons and $T=0$ targets to $T=1$ final states, can take place via this direct mechanism.[21]

Let us suppose $V_n\equiv V_p$, and briefly examine how the Coulomb effects bring about $S=1\rightarrow S=0$ transitions. Equation (61) implies that X_n+X_p has the form

$$X_n(W)+X_p(W)=\tilde{t}_n(W)+\tilde{t}_p(W)$$
$$+\text{connected terms.} \quad (69)$$

The connected terms are in fact square-integrable, and so do not propagate two-body momentum correlations nearly as strongly as $\tilde{t}_p(W)$, which contains the factor $\delta(\mathbf{k}_n'-\mathbf{k}_n)$, and $\tilde{t}_n(W)$, which may be written (leaving the spin-dependence implicit, and assuming a recoilless target)

$$\langle\mathbf{k}_n'\mathbf{k}_p'|\tilde{t}_n(W)|\mathbf{k}_n\mathbf{k}_p\rangle=\int d\mathbf{k}''\langle\mathbf{k}_p'|\phi_{\mathbf{k}''}{}^{c}\rangle$$
$$\times\langle\mathbf{k}_n'|t_n(W-(2m)^{-1}k''^{2})|\mathbf{k}_n\rangle\langle\phi_{\mathbf{k}''}{}^{c}|\mathbf{k}_p\rangle, \quad (70)$$

where $t_n(Z)$ is the usual two-body n-C scattering amplitude, and where $\langle\phi_{\mathbf{k}''}{}^{c}|\mathbf{k}_p\rangle$ is the momentum-space wave function appropriate to proton-target Coulomb scattering. In the recoilless approximation, and using the approximate orthogonality implied by (56), we obtain the explicit result ($|\phi_{\mathbf{K}}{}^{c}\rangle$ is the *deuteron*-target Coulomb wave function)

$$\langle v_{S'\nu'}\phi_{\mathbf{K}'}{}^{c}|g_p(W)[\tilde{t}_n(W)+\tilde{t}_p(W)]g_p(W)|v_{S\nu};\phi_{\mathbf{K}}{}^{c}\rangle\simeq\sum_{\sigma_n'\sigma_p'\sigma_n\sigma_p}\langle s'\nu'|\tfrac{1}{2}\tfrac{1}{2};\sigma_n'\sigma_p'\rangle$$
$$\times\Big\{\int d\mathbf{x}\,\frac{v_{S'}{}^{*}(\mathbf{x}-\frac{1}{2}\mathbf{K}')}{W-(4m)^{-1}K'^{2}-m^{-1}(\mathbf{x}-\frac{1}{2}\mathbf{K}')^{2}}[\delta_{\sigma_n'\sigma_n}\langle\mathbf{K}'-\mathbf{x},\sigma_p'|t_p{}^{c}(W-(2m)^{-1}x^{2})|\mathbf{K}-\mathbf{x},\sigma_p\rangle$$
$$+\delta_{\sigma_p'\sigma_p}\langle\mathbf{K}'-\mathbf{x},\sigma_n'|t_n(W-(2m)^{-1}x^{2})|\mathbf{K}-\mathbf{x},\sigma_n\rangle]\frac{v_S(\mathbf{x}-\frac{1}{2}\mathbf{K})}{W-(4m)^{-1}K^{2}-m^{-1}(\mathbf{x}-\frac{1}{2}\mathbf{K})^{2}}\Big\}\langle\tfrac{1}{2}\tfrac{1}{2};\sigma_n\sigma_p|s\nu\rangle. \quad (71)$$

In Eq. (71), we mean $t_p{}^{c}$ to be the ordinary off-shell proton-target scattering matrix obtained from the usual Lippmann-Schwinger equation, except that the momentum-space matrix element of V_p is taken in the basis of Coulomb wave functions rather than in the basis of plane waves. This effectively reduces the proton-target nuclear potential by approximately the

[19] J. V. Noble, Phys. Rev. **157**, 939 (1967).

[20] From now on, where necessary, we shall denote this object by $\langle\mathbf{K}s'\nu'|B(W)|\mathbf{K}s\nu\rangle$.

[21] J. V. Noble (to be published).

factor $C_0(k_p')C_0(k_p)$; it also shifts the positions of any poles (bound states) in $t_p{}^c$ by an amount approximately equal to the Coulomb energy in that nucleus. Now, if the two-body scattering matrices, t_n and $t_p{}^c$, are independent of spin, examination of Eq. (71) reveals that the amplitude for $S=1\rightarrow S=0$ vanishes identically, independent of the presence of the Coulomb modifications. The only way the spin of the neutron can flip relative to that of the proton is if, say, the nucleon-target interactions have spin-orbit parts; the proton wave function is excluded from the target region by the Coulomb force preferentially relative to the neutron wave function, in the intermediate states, and the spin-orbit parts of V_n and V_p can thus flip one nucleon's spin more often than the other's.

The nature of the "connected terms" appearing in Eq. (69) to a large extent precludes their contributing much to the isospin-violating matrix elements $\langle \mathbf{K}'00|B(W)|\mathbf{K}1\nu\rangle$. The major asymmetry in Eq. (71) between the neutron and proton coordinates arises from the presence of the function $t_p{}^c$ instead of t_p, inside the brackets. This came about through the fact that $\tilde{t}_p$ was diagonal in neutron coordinates, and that $\tilde{t}_n$ had the form (70) which, because of the completeness relation for Coulomb wave functions, led to the appearance of two fewer Coulomb wave functions in the t_n term than in the $t_p{}^c$ terms. This reduction and consequent symmetry occurs to a much smaller extent in such higher iterations as $\tilde{t}_n g_p \tilde{t}_p + \tilde{t}_p g_p \tilde{t}_n$.[22] From the computational point of view, therefore, it is probably sufficient to calculate the T-violating part of the "optical potential" using expression (71) alone. This is the procedure followed in an application of this theory to isospin-violating (d,α) reactions, which will be published separately.[21]

C. The Three-Nucleon Problem

The modified Faddeev equations introduced in Sec. II are obviously directly applicable to the calculation of the properties of the ground state of He^3, and to the discussion of p-d scattering. These calculations are being carried out by Adya,[23] so we shall not discuss this application further here.

V. SUMMARY AND CONCLUSION

The object of this paper has been to show that the formal treatment of repulsive Coulomb interactions in the framework of an otherwise well-behaved three-body problem is not particularly hard. On the other hand, as we have seen, drastic approximations of various kinds are required to render the Coulomb-modified nuclear-scattering problem numerically tractable. This is partly because, lacking a general solution, we do not understand the detailed properties of the Coulomb wave function of three charged particles; and partly because the simplest three-body problems already tax the capacity of the largest and fastest currently available computers. The approximations made in Secs. III and IV were suggested with this limitation in mind, with the aim if including at least some of the physics arising from Coulomb forces in three-body scattering into the canon of numerically accessible problems.

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APPENDIX: CALCULATION OF THE COULOMB GREEN'S FUNCTION AT NEGATIVE ENERGIES

It is straightforward to show that for Yukawa potentials, the kernel $V_1G_0V_2G_0$ is a completely continuous operator on the Banach space of bounded, continuous functions of six variables, when the energy is negative and real. To see this, one merely notes that

$$\int d\mathbf{p}_1' \int d\mathbf{p}_2' |\langle \mathbf{p}_1\mathbf{p}_2| \times V_1G_0(-|E|)V_2G_0(-|E|)|\mathbf{p}_1'\mathbf{p}_2'\rangle| < \infty \quad \text{(A1)}$$

and is continuous in p_1 and p_2.[24] In fact, Faddeev has shown[5] that this kernel is completely continuous on a slightly more restricted Banach space at all energies. Now consider the kernel K_n in which V_1 and V_2 are Yukawa potentials of range n (in appropriate units), $n=1, 2, \cdots$. Clearly the case of repulsive Coulomb potentials is obtained by taking the limit as $n\rightarrow\infty$. We now ask at what energies the limiting kernel, $K(W)=\lim_{n\rightarrow\infty} K_n(W)$, is bounded on the appropriate Banach space. For these energies, we can apply the well-known theorem[25] that the limit in norm of a sequence of completely continuous transformations is itself completely continuous. We anticipate the result of this query by stating that $K(W)$ is only bounded for negative, real W, or when $\mathrm{Im}(W)\neq 0$.

In appropriate units, the kinetic energy in the center-of-mass system can be written

$$\omega = p_1{}^2 + p_2{}^2 + 2\alpha \mathbf{p}_1\cdot\mathbf{p}_2, \quad \text{(A2)}$$

where $\alpha\leqslant 1$. We shall assume that all the masses are finite, so that $\alpha<1$.

We first show that $K(E+i\eta)$ maps a typical member of either of the previously mentioned Banach spaces into

[22] To be completely honest, we should note that the asymmetry occurs again in the next terms, $\tilde{t}_n g_p \tilde{t}_p g_p \tilde{t}_n + \tilde{t}_p g_p \tilde{t}_n g_p \tilde{t}_p$, in the series expansion of Eq. (61), and in all odd-order terms. However, these terms contribute substantially only in the lowest partial waves, and so their spin dependence is not nearly so strong as that of the first-order terms.

[23] S. Adya and R. D. Amado (private communication).

[24] F. Riesz and B. Sz.-Nagy, *Functional Analysis* (F. Ungar Publishing Company, New York, 1955), p. 224.

[25] Reference 24, p. 178.

a function which becomes unbounded as $\eta \to 0$, if $E \geq 0$. The typical element is just the constant function, whose norm we choose for convenience to be 1. Stripped of superfluous coupling constants, the kernel $K(W)$ maps the function 1 into the function

$$I(\mathbf{p}_1,\mathbf{p}_2; W)=\int d\mathbf{p}_1' \int d\mathbf{p}_2' (\mathbf{p}_1-\mathbf{p}_1')^{-2}(\mathbf{p}_2-\mathbf{p}_2')^{-2} \times (W-p_1^2-p_2'^2-2\alpha \mathbf{p}_1\cdot\mathbf{p}_2')^{-1}(W-w')^{-1}. \quad \text{(A3)}$$

We now show that the function $I(0, \mathbf{p}_2; E+i\eta)$ becomes infinite as $\eta \to 0$, regardless of the value of E or $\mathbf{p}_2$. With $p_1=0$, we may do the integration over $\mathbf{p}_1'$; the result is

$$2\pi^2(\alpha p_2')^{-1}\tan^{-1}[\alpha p_2'(p_2'^2(1-\alpha^2)-W)^{-1/2}].$$

Substituting this expression into (A3), and doing the $\hat{p}_2'$ integration, we obtain

$$|I(0,\mathbf{p}_2; W)|=\frac{8\pi^3}{\alpha}\left|\int_0^\infty dp_2'[p_2'(W-p_2'^2)]^{-1} \times \tan^{-1}[\alpha p_2'(p_2'^2(1-\alpha^2)-W)^{-1/2}] \times Q_0((p_2^2+p_2'^2)/2p_2p_2')\right|, \quad \text{(A4)}$$

where $Q_0(z)$ is the Legendre function of the second kind, of order 0. Adding and subtracting $Q_0((p_2^2+W)/2p_2\sqrt{W})$ in the integrand of (A4), we obtain two integrals:

$$I_1=\int_0^\infty dx[x(W-x^2)]^{-1}\left[Q_0\left(\frac{p_2^2+x^2}{2p_2x}\right)-Q_0\left(\frac{W+x^2}{2xW^{1/2}}\right)\right] \times \tan^{-1}[\alpha x(x^2(1-\alpha^2)-W)^{-1/2}]. \quad \text{(A5)}$$

$$I_2=Q_0\left(\frac{W+p_2^2}{2p_2\sqrt{W}}\right)\int_0^\infty dx \tan^{-1}[\alpha x(x^2(1-\alpha^2)-W)^{-1/2}] \times [x(W-x^2)]^{-1}. \quad \text{(A6)}$$

The integral (A5) is finite since its integrand has only logarithmic singularities. The integral in (A6) is infinite, however, since its integrand has a simple pole coinciding with a logarithmic singularity. This shows that $K(E+i\eta)$ is not a bounded operator for $E \geq 0$, in the limit as $\eta \to 0$.

What of the case $E<0$? Consider the integral

$$J(\mathbf{p}_1, \mathbf{p}_2; -|E|)=\int d\mathbf{p}_1' \int d\mathbf{p}_2' (\mathbf{p}_1-\mathbf{p}_1')^{-2} \times (\mathbf{p}_2-\mathbf{p}_2')^{-2}[|E|+p_1^2+p_2'^2+2\alpha \mathbf{p}_1\cdot\mathbf{p}_2']^{-1} \times (|E|+\omega')^{-1} f(\mathbf{p}_1',\mathbf{p}_2'), \quad \text{(A7)}$$

where $f(\mathbf{p}_1,\mathbf{p}_2)$ is any bounded, continuous function; if we define $\|f\|=\max[\mathbf{p}_1,\mathbf{p}_2] f(\mathbf{p}_1,\mathbf{p}_2)$, $|J|$ is majorized by

$$|J| \leq \|f\|\left[\int d\mathbf{p}_1'(\mathbf{p}_1-\mathbf{p}_1')^{-2}[|E|+p_1'^2(1-\alpha^2)]^{-1}\right] \times \left[\int d\mathbf{p}_2'(\mathbf{p}_2-\mathbf{p}_2')^{-2}[|E|+p_2'^2(1-\alpha^2)]^{-1}\right]. \quad \text{(A8)}$$

In obtaining the bound (A8), we have used the inequalities

$$p_1^2+p_2'^2+2\alpha \mathbf{p}_1\cdot\mathbf{p}_2'=(\mathbf{p}_1+\alpha\mathbf{p}_2')^2 +(1-\alpha^2)p_2'^2 \geq (1-\alpha^2)p_2'^2, \quad \text{(A9)}$$
$$\omega'=(\mathbf{p}_2'+\alpha\mathbf{p}_1')^2+(1-\alpha^2)p_1'^2 \geq (1-\alpha^2)p_1'^2.$$

Both integrals in (A8) are finite, and the end result is that

$$|J| \leq C\|f\|\left\{p_1^{-1}\tan^{-1}\left(p_1\left[\frac{1-\alpha^2}{|E|}\right]^{1/2}\right)\right\} \times \left\{p_2^{-1}\tan^{-1}\left(p_2\left[\frac{1-\alpha^2}{|E|}\right]^{1/2}\right)\right\}, \quad \text{(A10)}$$

where C is a constant involving α's and π's, and whose exact form is irrelevant here. We have thus shown that the kernel $U_iG_0U_jG_0$ is completely continuous at negative real energies, for $i \neq j$.

Finally, it remains to be seen how to construct the Green's function $g(W)$, defined in Eq. (3). One way to do this is as follows. We first define the formal operators

$$k_i(W)=[1-U_iG_0(W)]^{-1}, \quad i=1, 2, 3. \quad \text{(A11)}$$

We next write $g(W)$ formally as

$$g(W)=G_0k_1k_2k_3[(1-UG_0)k_1k_2k_3]^{-1}, \quad \text{(A12)}$$

which satisfies the Lippmann-Schwinger equation

$$g(W)=G_0(W)+g(W)UG_0(W), \quad \text{(A13)}$$

if the inverse of the object in the square brackets in (A12) is well-defined.[9] (By U we just mean $U_1+U_2+U_3$.) Expanding (A11), and performing some manipulations, we find that

$$[(1-UG_0)k_1k_2k_3]=1-[U_2G_0U_1G_0+U_3G_0U_1G_0]k_1k_2k_3 -U_3G_0U_2G_0k_2k_3. \quad \text{(A14)}$$

But at negative energies, with purely repulsive Coulomb potentials, the k_i are bounded operators (in fact, their norms are less than 1 for all negative energies), and so the right side of (A14) is just 1 minus a completely continuous kernel. Therefore, the inverse of the operator in the square brackets in (A12) may be constructed by the usual methods. Finally, if we write the right-hand side of (A14) as $1-K$, and write K as

$$K=[(1-U_3G_0)(1-U_2G_0)(1-U_1G_0) -(1-(U_1+U_2+U_3)G_0)]k_1k_2k_3, \quad \text{(A15)}$$

that is, as the difference between a positive operator of unit norm, and a positive-definite operator of norm less than 1, we see that the norm of K itself is less than 1, and so the Neumann series for $(1-K)^{-1}$ converges.

We briefly recapitulate the results of this Appendix: we have shown that no reformulation, of the Faddeev,[5] Weinberg,[8] or Blankenbecler and Sugar[9] type, of the scattering equation for $g(W)$, yields a tractable integral equation at positive, real energies. They all evidently suffer from the same disease as the two-body Lippmann-Schwinger equation for the Coulomb Green's function, namely the scattering amplitude diverges in the forward direction. This strongly implies that an entirely new approach is required to construct the Coulomb Green's function for three charged particles at positive energies. Secondly, we have shown how a method suggested by Sugar and Blankenbecler[9] may be applied to the case of repulsive potentials, at negative energies. This particular form of the solution [Eq. (A12)] has the advantage of being nearly in the product form, which Amado has found more closely resembles the true solution than the partial-sum form.[26] Evidently there is some merit to Amado's contention, since the kernel K has norm less than 1, indicating that the product $G_0k_1k_2k_3$ is the first term in a convergent series expansion for $g(-|E|)$.

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Migdal's Quasiparticle Model and Partial Muon Capture in O^{16}

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By means of the process involving the capture of muons in O^{16} (g.s.) leading to definite final states in N^{16}, we examine simultaneously (a) the *quasiparticle* model of nuclear structure developed by Migdal and (b) the pseudoscalar coupling generated by the axial-vector coupling in the effective weak-interaction Hamiltonian. In (a) we clarify the basic assumptions essential for the model and the connection between this model and other better-known (nuclear) models. In (b), it is shown that the Migdal model successfully eliminates the well-known discrepancy between theory and experiment in $\mu^-+O^{16}(0^+) \rightarrow \nu_\mu+N^{16}(2^-)$ and also in $e^-+O^{16}(0^+) \rightarrow e^-+O^{16*}(2^-)$. This in turn enables us to make use of the nuclear model to obtain a reasonable estimate of $C_P=m_\mu F_P/F_A$. The conclusion is that the one-pion-pole dominance hypothesis is compatible with all available data in O^{16} and that there seems to be no urgent need to introduce the tensor coupling as some people have suggested.

I. INTRODUCTION

THE major difficulty with the use of complex nuclei as a means of studying the muon capture process is the inherent uncertainty associated with the nuclear structure. One process which does not involve nuclear physics is capture in hydrogen. But since the capture takes place mainly in muon-hydrogen molecules, there is some uncertainty associated with the molecular structure. Moreover, not all the necessary information on the weak-coupling constants can be deduced from this muon-hydrogen experiment.

The μ capture process involves a large momentum transfer $q \sim m_\mu \sim 100$ MeV/c, and for this reason it can provide valuable information about effects which are not found in processes like β decay; e.g., the induced pseudoscalar (P) coupling generated by the axial-vector coupling.[1,2] Goldberger and Trieman (GT) have obtained a theoretical estimate of the P coupling constant by relating the constant to the pion lifetime and pion-nucleon coupling constant in the one-pion-pole dominance hypothesis.[2] At this moment, there is no clear experimental verification of the GT result. There is evidence, however, which suggests that the actual P constant might be much larger than the GT estimate. One set of experiments which seems to indicate this is the measurement of the asymmetry of the neutrons emitted after the capture of partially polarized muons.[3] The other is the radiative μ capture in complex nuclei.[4] These two seem to require a larger P coupling constant than the theoretical estimate.

Here we are concerned with another type of experiment, which seems to have been proposed originally by Shapiro and Blokhintsev.[5] This is to look at the partial-

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