Solution of and Standard Approximations for an Exactly Soluble Three-Particle Model of Stripping

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An exact numerical solution is given of a model for stripping and elastic scattering based on three pairwise interacting spinless particles. Results are given (a) in the absence of a proton-nucleus Coulomb interaction and (b) when its presence is included approximately. The distorted-wave Born approximation as defined for this model has been calculated exactly and, together with the absorption model, is compared with the exact results.

1. INTRODUCTION

CONSIDERABLE progress has been recently made in the study of potential scattering of three-particle systems.[1,2] The same problem has been independently discussed by Amado[3] on the basis of a field-theoretical model with Lee-model-type vertices between fields n, p, A, d, and B, viz.

$$n+p \rightleftharpoons d$$
$$n+A \rightleftharpoons B. \tag{1.1}$$

It had been shown before by Amado *et al.*[4] that a simple Lee model in the limit of the normalization constant $Z \to 0$, and a two-particle system interacting through a separable potential, yield the same S matrix in the n-p (θ-N) sector. The same has been proved by Rosenberg[5] for Amado's extended Lee model[3] with vertices (1.1). Finally, it can be shown[6] that a further generalization of the Lee model including a vertex

$$p+A \rightleftharpoons B' \tag{1.2}$$

is again equivalent to a model with three pairwise separable potentials.

It was first observed by Amado that a theory with vertices (1.1) may serve as a model for deuteron-stripping and elastic-scattering reactions on a nucleus A which has no internal degrees of freedom and which can bind a neutron but not a proton. Calculations have subsequently been made by Aaron and Shanley.[7] Mitra[8] extended Amado's model in a potential formulation by assuming proton and neutron to be identical and Shanley[9] actually performed the calculations suggested by Mitra.

The following is an account of an extended study of deuteron stripping and elastic scattering which we undertook for two reasons.

(a) The models of Amado and Mitra either disregard the p-A interaction completely or neglect the Coulomb interaction, which breaks charge independence. The effect of the latter interaction must be pronounced in a three-particle model, where it is solely responsible for the difference between (d,p) and (d,n) reactions.

(b) Various approximations have been applied in the past to the description of stripping reactions. Some of these approximations may be precisely defined within the framework of what will be termed in the following the exact three-particle model. Consequently, we may compare the outcome of the approximate calculation with the exact one. We shall illustrate our point by a formulation and *calculation* of the distorted-wave Born approximation (DWBA) without recourse to an optical potential, the parameters of which have to be determined by fitting elastic scattering data.

In Sec. 2 we treat the deuteron and target nucleus as a three-particle system, interacting through three separable pair potentials. Our interest is focused on the pair interaction v_{pA} between the proton and the nucleus. We consider in particular interactions v_{pA} which are quantitatively the same as v_{nA} but lead to a bound proton. The calculated cross sections are compared with those where an additional Coulomb interaction reduces the binding of the proton.

In Sec. 3 we calculate on one hand the DWBA for stripping, and on the other hand some versions of the absorption model. The results are again compared with the exact ones. The last section contains a discussion of our results and a comparison with the calculations by Aaron and Shanley[7] and Shanley.[9]

2. THREE-PARTICLE MODEL WITH THREE PAIR INTERACTIONS

From here on we shall base our discussion of a three-particle system from the point of view of a potential theory. Consider a nucleus A together with a proton and neutron, all three spinless and interacting through relative s waves only. A, p, n are in this order numbered by 1, 2, 3, while pairs are denoted by the index of the particle absent in the pair.

We denote by $\mathbf{k}_i$ the particle momenta in the center-

[1] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1961) [English transl.: Soviet Physics—JETP **12**, 1014 (1961)].
[2] C. A. Lovelace, Phys. Rev. **135**, B1225 (1964).
[3] R. D. Amado, Phys. Rev. **132**, 485 (1963).
[4] M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961).
[5] L. Rosenberg, Phys. Rev. **140**, B217 (1965).
[6] A. I. Jaffe and A. S. Reiner, Phys. Rev. **152**, 1304 (1966).
[7] R. Aaron and P. E. Shanley, Phys. Rev. **142**, 608 (1966).
[8] A. N. Mitra, Phys. Rev. **139**, B1472 (1965).
[9] P. E. Shanley, thesis, Northeastern University (unpublished).

of-mass system and define[2]

$$\mathbf{p}_1=[2m_2m_3(m_2+m_3)]^{-1/2}(m_3\mathbf{k}_2-m_2\mathbf{k}_3)$$
$$\mathbf{q}_1=[2m_1(m_2+m_3)(m_1+m_2+m_3)]^{-1/2}$$
$$\times(m_1(\mathbf{k}_2+\mathbf{k}_3)-(m_2+m_3)\mathbf{k}_1). \quad (2.1)$$

Four other momenta, which are pairwise linearly dependent on $\mathbf{p}_1$ and $\mathbf{q}_1$, are defined by cylic permutation of the indices in Eq. (2.1).

We are interested in reactions initiated by a deuteron, but out of the four possible reactions

$$d+A\rightarrow d+A$$
$$d+A\rightarrow p+B$$
$$d+A\rightarrow n+B'$$
$$d+A\rightarrow n+p+A \quad (2.2)$$

only elastic scattering and stripping reactions will be considered here.

We recall the definition of bound-state scattering amplitudes $T_{\beta\alpha}$:

$$\langle\mathbf{q}_\beta'|T_{\beta\alpha}^+(s)|\mathbf{q}_\alpha\rangle=\int\varphi_\beta^*(\mathbf{p}_\beta')$$
$$\times\langle\mathbf{p}_\beta\mathbf{q}_\beta'|U_{\beta\alpha}^+(s)|\mathbf{p}_\alpha\mathbf{q}_\alpha\rangle\varphi_\alpha(\mathbf{p}_\alpha)d\mathbf{p}_\alpha d\mathbf{p}_\beta', \quad (2.3)$$

with φ_α, φ_β bound-state wave functions of pairs α and β belonging to energies $-\epsilon_\alpha$, $-\epsilon_\beta$. The transition operators $U_{\beta\alpha}^+$ in (2.3) are defined as

$$U_{\beta\alpha}^+(s)=\sum_{\gamma\neq\beta}v_\gamma-\sum_{\substack{\gamma\neq\beta\\ \delta\neq\alpha}}v_\gamma G(s)v_\delta \quad (2.4)$$

in which $G(s)$, and $G^0(s)$ to be used later, are, respectively, the resolvents $(H-s)^{-1}\equiv(H_0+\sum_\alpha v_\alpha-s)^{-1}$ and $(H_0-s)^{-1}$ of the total and unperturbed three-particle Hamilton operators. Finally, physical amplitudes are obtained from Eq. (2.3) by relating the energy parameter s to the (in general off-shell) momenta as follows

$$s=q_\alpha^2-\epsilon_\alpha=q_\beta'^2-\epsilon_\beta. \quad (2.5)$$

We now assume for each pair α ($\alpha=1,2,3$) the existence of a separable s-wave potential[2]

$$\langle\mathbf{p}_\alpha'|v_\alpha|\mathbf{p}_\alpha\rangle=-\lambda_\alpha g_\alpha(p_\alpha')g_\alpha(p_\alpha). \quad (2.6)$$

The coupling constants λ_α are chosen to be sufficiently large to bind the pair α with energy $-\epsilon_\alpha$. The quantity g_α may in that case be shown to be the form factor of the bound-state wave function[2]

$$g_\alpha(p_\alpha)=-(p_\alpha^2+\epsilon_\alpha)\varphi_\alpha(p_\alpha) \quad (2.7)$$

and is normalized as follows

$$\int\frac{g_\alpha^2(p_\alpha)d\mathbf{p}_\alpha}{(p_\alpha^2+\epsilon_\alpha)^2}=1. \quad (2.8)$$

A tool of paramount interest in the Faddeev theory to be applied here is the off-energy-shell scattering matrices $\hat{t}_\alpha$ belonging to the different pair interactions v_α. If those are of the separable form (2.6), $\hat{t}_\alpha$ reads simply

$$\langle\mathbf{p}_\alpha'|\hat{t}_\alpha(s)|\mathbf{p}_\alpha\rangle=g_\alpha(p_\alpha')\tau_\alpha(s)g_\alpha(p_\alpha) \quad (2.9)$$

with

$$\tau_\alpha(s)=-\left[\lambda_\alpha^{-1}+\int\frac{g_\alpha^2(p_\alpha)d\mathbf{p}_\alpha}{s-p_\alpha^2}\right]^{-1}. \quad (2.10)$$

Finally, one may eliminate the coupling constant λ_α from (2.10) in favor of the bound-state energy. The condition that each pair interaction supports a bound state is

$$\lambda_\alpha^{-1}=\int\frac{g_\alpha^2(p_\alpha)d\mathbf{p}_\alpha}{\epsilon_\alpha+p_\alpha^2}. \quad (2.11)$$

It is useful to introduce state vectors $|g_\alpha\rangle$ whose projections $\langle\mathbf{p}_\alpha|g_\alpha\rangle$ on momentum eigenstates equal the (s-state) form factors $g_\alpha(p_\alpha)$. We further need potentials $Z_{\beta\alpha}$,

$$Z_{\beta\alpha}(s)\equiv\langle g_\beta|G_0(s)|g_\alpha\rangle(1-\delta_{\alpha\beta}), \quad (2.12)$$

as well as amplitudes $X_{\beta\alpha}$ both defined by Lovelace,[2]

$$X_{\beta\alpha}(s)\equiv\langle g_\beta|G_0(s)U_{\beta\alpha}^\dagger(s)G_0^+(s)|g_\alpha\rangle$$
$$-Z_{\beta\alpha}(s)(1+\lambda_\alpha\langle g_\alpha|G_0(s)|g_\alpha\rangle). \quad (2.13)$$

For later reference, we explicitly write down an off-energy-shell element of $X_{\beta\alpha}$ which reads

$$\langle\mathbf{q}_\beta'|X_{\beta\alpha}(s)|\mathbf{q}_\alpha\rangle=\int\varphi_\beta^*(\mathbf{p}_\beta')\langle\mathbf{p}_\beta'\mathbf{q}_\beta'|U_{\beta\alpha}^+(s)|\mathbf{p}_\alpha\mathbf{q}_\alpha\rangle$$
$$\times\varphi_\alpha(\mathbf{p}_\alpha)\frac{(\epsilon_\beta+p_\beta'^2)(\epsilon_\alpha+p_\alpha^2)}{(p_\beta'^2+q_\beta'^2-s)(p_\alpha^2+q_\alpha^2-s)}d\mathbf{p}_\alpha d\mathbf{p}_\beta'$$
$$-\langle\mathbf{q}_\beta'|Z_{\beta\alpha}(s)|\mathbf{q}_\alpha\rangle\left[1+\lambda_\alpha\int\frac{g_\alpha^2(p_\alpha)d\mathbf{p}_\alpha}{p_\alpha^2+q_\alpha^2-s}\right]. \quad (2.14)$$

Comparison of Eq. (2.14) with (2.3) shows by use of Eq. (2.8) and (2.10) that X and T are only identical on the energy shell (2.5). In anticipation of a relevant discussion in Sec. 3A we wish to stress here that the two amplitudes differ off shell in spite of the term "off-shell amplitude" used for X [cf. Sec. (3A)].

As shown by Lovelace, the amplitudes $X_{\beta\alpha}$ satisfy coupled integral equations.[2] We therefore just cite the result for deuteron-induced reactions in a situation where all channels are governed by a single s-wave boundstate:

$$\langle\mathbf{q}_\beta'|X_{\beta1}(s)|\mathbf{q}_\alpha\rangle=-\langle\mathbf{q}_\beta'|Z_{\beta1}(s)|\mathbf{q}_1\rangle$$
$$-\sum_{\gamma=1}^{3}\int\langle\mathbf{q}_\beta'|Z_{\beta\gamma}(s)|\mathbf{q}_\gamma''\rangle\tau_\gamma(s-q_\gamma''^2)$$
$$\times\langle\mathbf{q}_\gamma''|X_{\gamma1}(s)|\mathbf{q}_1\rangle d\mathbf{q}_\gamma'' \quad (2.15)$$

The model is finally specified by the masses m_i and a choice for the form factors $g_\alpha(p_\alpha)$. It will be assumed that $m_A=\infty$ and $m_p=m_n=m$, although these restrictions could be removed easily. For the form factors, we adopt for the p-n and n-A interaction functions of the Hulthén type

$$g_\alpha(p_\alpha)=\frac{N_\alpha}{p_\alpha^2+\beta_\alpha^2};\quad \alpha=1,2. \tag{2.16}$$

The parameters for the p-n interaction are chosen to fit the deuteron binding energy $\epsilon_d=2.22$ MeV and the p-n triplet scattering length $a_t=5.38$F. The parameters for the n-A system are taken, in agreement with Aaron and Shanley,[7] to fit a fixed binding energy and radius R_n of the neutron orbit.

We finally come to v_{pA}, which is completely neglected in Amado's original work. A first choice for $g_3(p_3)$ would be a Hulthén form factor like (2.16). Mitra[8] suggested, and Shanley[9] actually used, $v_{pA}^{(0)}=v_{nA}$. If, however, one wants to account for a reduced proton binding energy on one hand, and a stripping pattern which is different for (d,p) and (d,n) reactions on the other hand,[10] one is bound to alter the simplest assumption made above for v_{pA}.

As a first attempt one may search for parameters of v_{pA} which account for the observed differences, without ascribing the modifications to a qualitatively different dynamical cause. We shall see later on that reasonable p-A parameters will not produce d-n cross sections which are smaller than those for d-p reactions as is actually observed.

We therefore suggest incorporating a Coulomb interaction v_c without spoiling the separability of the total v_{pA} interaction. Since v_c itself is non-separable, one has to have recourse to approximations. It seems reasonable to express the effect of v_c by a modification of the unperturbed form factor $g_3^{(0)}=g_2$. The ensuing change in the t matrix, Eqs. (2.9) and (2.10), will cause some distortion of the proton wave function due to the Coulomb interaction.

Let H_{pA} be the total Hamiltonian of the proton in the field of a point nucleus

$$H_{pA}=T+v_{pA}^{(0)}+v_c, \tag{2.17}$$

where $v_{pA}^{(0)}=v_{nA}$ is separable. The perturbation v_c changes the energy and wave function of the bound state to lowest order as follows

$$\epsilon_3^{(1)}\sim\epsilon_3^0-\langle\varphi_3^{(0)}|v_c|\varphi_3^{(0)}\rangle$$
$$|\varphi_3^{(1)}\rangle\sim[1-\mathcal{P}\mathcal{G}^{(0)}(-\epsilon_3^0)v_c]|\varphi_3^{(0)}\rangle, \tag{2.18}$$

where $\mathcal{P}$ means the principal value.

$$\mathcal{G}^{(0)}(s)\equiv(T_p+v_{pA}-s)^{-1}$$

is the resolvent of the unperturbed part of the Hamiltonian (2.17) and replaces v_{pA} in Eq. (2.6):

$$\langle\mathbf{p}_3'|\mathcal{G}^{(0)}(s)|\mathbf{p}_3\rangle=\frac{\delta(\mathbf{p}_3'-\mathbf{p}_3)}{s-p_3^2}$$
$$+\frac{g_3^{(0)}(p_3')\tau_3(s)g_3^{(0)}(p_3)}{(s-p_3'^2)(s-p_3^2)}. \tag{2.19}$$

$\tau_3(s)$ is the same bound-state propagator as given in Eq. (2.10). Using the Fourier transform of $v_c=Ze^2/r$, we find for $\varphi_3^{(1)}$ in Eq. (2.18), by means of Eqs. (2.7) and (2.19),

$$\varphi_3^{(1)}(p_3)=N\Big[\frac{g_3^{(0)}(p_3)}{\epsilon_3^0+p_3^2}$$
$$-\mathcal{P}\int\int\Big\{\frac{\delta(\mathbf{p}_3-\mathbf{p}_3')}{\epsilon_3^{(0)}+p_3^2}-\frac{g_3^{(0)}(p_3)\tau_3(-\epsilon_3^{(0)})g_3^{(0)}(p_3')}{(\epsilon_3^{(0)}+p_3^2)(\epsilon_3^{(0)}+p_3'^2)}\Big\}$$
$$\times\frac{Ze^2}{2\pi^2}\frac{1}{|\mathbf{p}_3'-\mathbf{p}_3''|^2}\frac{g_3^{(0)}(p_3'')}{\epsilon_3^{(0)}+p_3''^2}d\mathbf{p}_3'd\mathbf{p}_3''\Big]. \tag{2.20}$$

One obtains from Eqs. (2.7) and (2.18), the perturbed form factor

$$g_3^{(1)}(p_3)=-(\epsilon_3^{(1)}+p_3^2)\varphi_3^{(1)}(p_3), \tag{2.21}$$

which is not of the Hulthén type. It will therefore be impossible to treat expressions like (2.21) analytically which is the main reason for the choice of Hulthén form factors (2.16). In practice we searched therefore for a best fit of Eq. (2.21) with a sum of a few Hulthén form factors (2.16).

Before continuing, we wish to discuss the validity of the procedure followed. The separation energies of the last bound nucleons in, for instance, O^{17} and F^{17} are 4.14 and 0.60 MeV, respectively. The difference is small compared to the total energy of O^{16} and can then be treated as a perturbation. However, the situation is different when the nuclei are considered as three-particle systems: The difference of the separation energies is not small compared to any one of them. We therefore simply assume that first-order perturbation theory will incorporate some dynamical features of the Coulomb force. The normalization constant N in Eq. (2.20) (usually of second order in the perturbation) is then crucial and may differ considerably from 1.[11]

To summarize: We added to two separable s-wave potentials v_{pn} and v_{nA} a third one v_{pA}. Starting from $v_{pA}^{(0)}=v_{nA}$, we perturbed the form factor $g_3^{(0)}=g_{nA}$ by v_c and used the resulting form factor to construct a corrected separable s-wave interaction with a built-in, simulated effect of the Coulomb potential.

[10] For $l=1$ capture, see for instance B. Zeidman and J. M. Fowler, Phys. Rev. **112**, 2020 (1958).

[11] The procedure becomes better as Z becomes smaller. It will be used in particular for a perturbation calculation of the properties of He^3 starting from H^3.

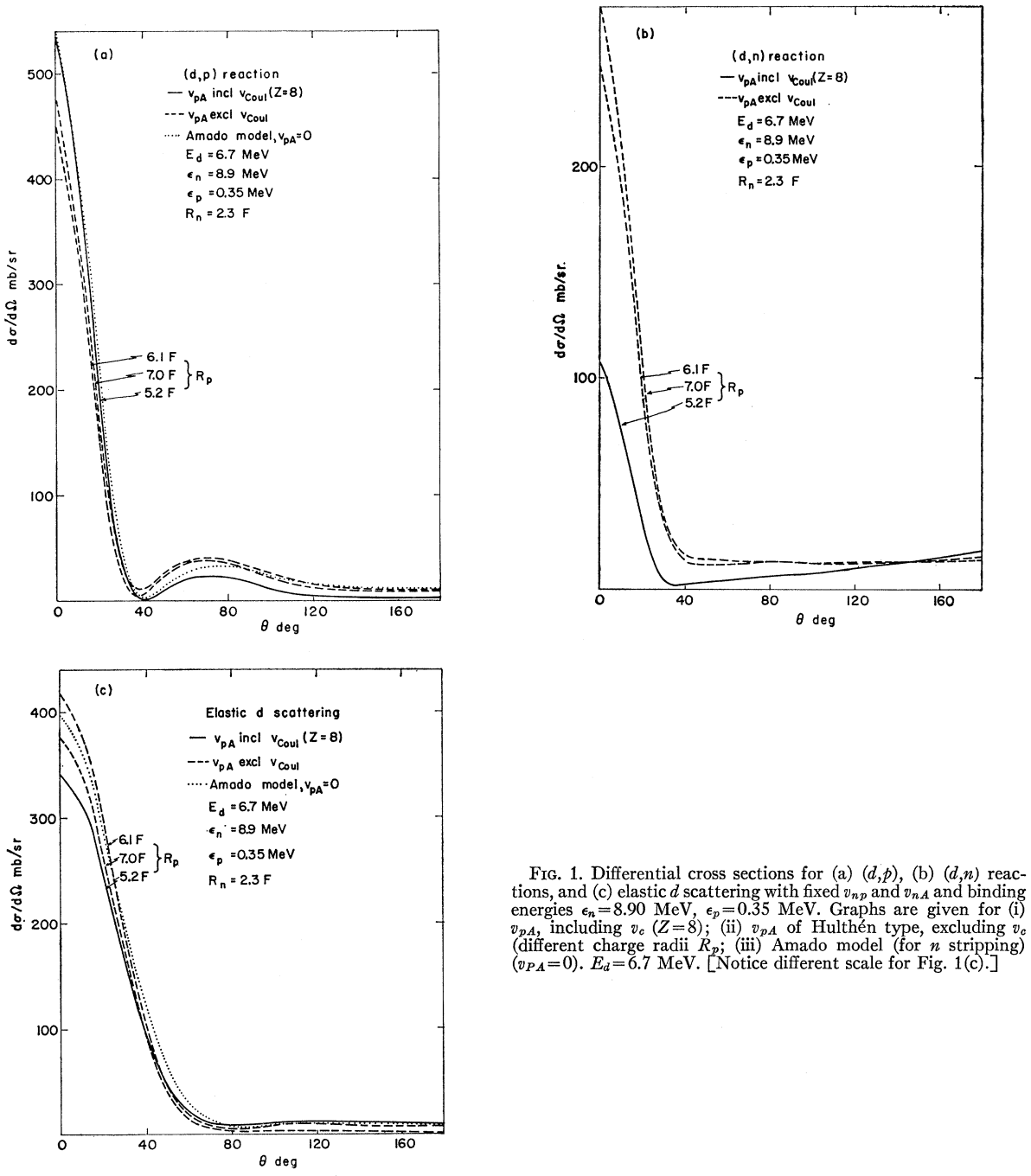

FIG. 1. Differential cross sections for (a) (d,p), (b) (d,n) reactions, and (c) elastic d scattering with fixed v_{np} and v_{nA} and binding energies $\epsilon_n=8.90$ MeV, $\epsilon_p=0.35$ MeV. Graphs are given for (i) v_{pA}, including v_c $(Z=8)$; (ii) v_{pA} of Hulthén type, excluding v_c (different charge radii R_p; (iii) Amado model (for n stripping) $(v_{PA}=0)$. $E_d=6.7$ MeV. [Notice different scale for Fig. 1(c).]

We now turn to the numerical results pertaining to the three models discussed. After a partial-wave analysis of the amplitudes $X_{\beta 1}$ [Eq. (2.15)], the resulting equations for $X_{\beta 1}{}^l$ were numerically solved by performing a contour deformation, which procedure is by now standard.[12,13]

For the parameters, we used throughout the fixed values $\epsilon_n=8.90$ MeV and a neutron radius $R_n=2.3$F. Notice that such a radius refers to the size of the orbit of the added particle, which is not necessarily of the order of the nuclear radius. We then compare the exact results (i.e., for v_{pA} including v_c taken for $Z=8$), with, on one hand, the Amado model (which neglects v_{pA} altogether) and, on the other hand, two choices of Hulthén potentials for v_{pA} not including v_c. Whenever $v_{pA}\neq 0$, we took a fixed proton binding energy $\epsilon_p=0.35$ MeV. The size parameter β_n in the exact theory equals,

[12] J. H. Hetherington and L. H. Schick, Phys. Rev. **137**, B935 (1965).
[13] R. Aaron and R. D. Amado, Phys. Rev. **150**, 857 (1966).

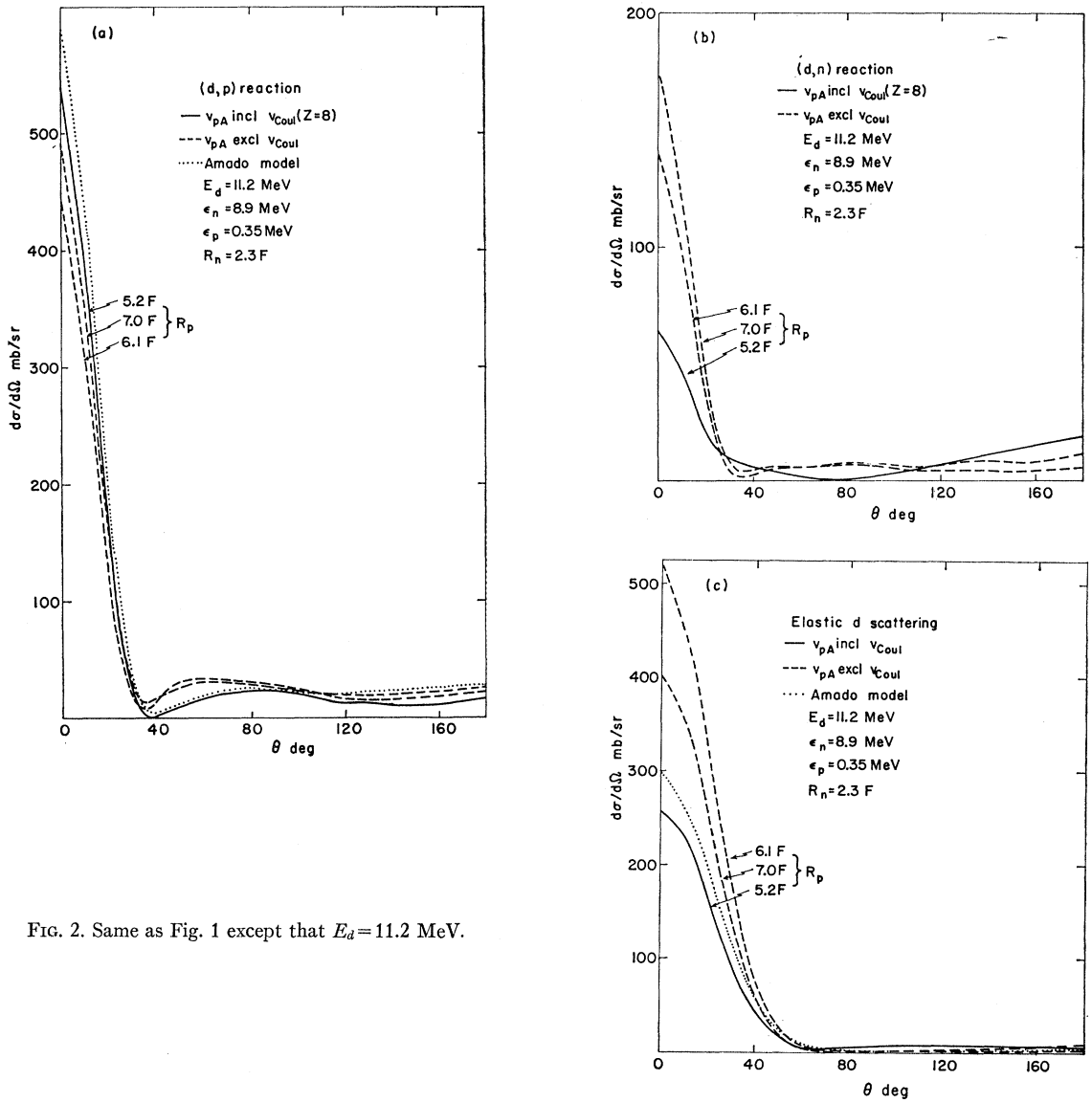

FIG. 2. Same as Fig. 1 except that $E_d = 11.2$ MeV.

as mentioned, the one in v_{nA}. On the other hand, two different values for β_p were tested in the pure Hulthén approximation for v_{pA}.

Instead of the shape parameters β, we give in the figures the neutron radius R_n in terms of β_n and $\alpha_n^2 = \epsilon_n$:

$$R_n^2 = [(\alpha_n^3+\beta_n^3)(\alpha_n+\beta_n)^3 - 16\alpha_n^3\beta_n^3] \times [2\alpha_n^2\beta_n^2(\alpha_n+\beta_n)^2(\alpha_n-\beta_n)^2]^{-1}. \quad (2.22)$$

Figures 1(a), 1(b), and 1(c) display the resulting differential cross sections for (d,p) and (d,n) reactions and elastic d scattering. The incident deuteron energy was chosen to be $E_d = 6.7$ MeV, whereas Fig. 2(a), 2(b), and 2(c) give parallel results for $E_d = 11.2$ MeV.

One notices the apparently negligible influence of the proton-nucleus interaction for (d,p) stripping. This is at least the case for a v_{pA} which gives a low binding energy, i.e., for a peripheral proton. (d,n) stripping is of course impossible if $v_{pA} = 0$. Here one sees that the dynamical cause of a weak proton-nucleus binding is crucial; the inclusion of a Coulomb potential substantially reduces the cross section. In the case of elastic deuteron scattering we find it hard to interpret the cross sections. In particular, the curve for a v_{pA} of the Hulthén type is certainly unexpected. We further tested Hulthén's potentials for v_{pA} by varying ϵ_p, and $R_p(\alpha_p,\beta_p)$. For the same $\epsilon_n = 8.90$ MeV, we now choose $R_n = 1.7$F. Two sets of values were chosen for β_p, both leading to the same R_p [cf. Eq. (2.22)] and $E_d = 1.77$ MeV. The results are shown in Figs. 3(a), 3(b), and 3(c). Figures 4(a), 4(b), and 4(c) refer to the same R_n and two values of R_p, viz, 1.7 and 1.9F. Here the deuteron energy is 11.2 MeV.

The influence of any v_{pA} in (d,p) cross sections remains small, although for very low incident energies

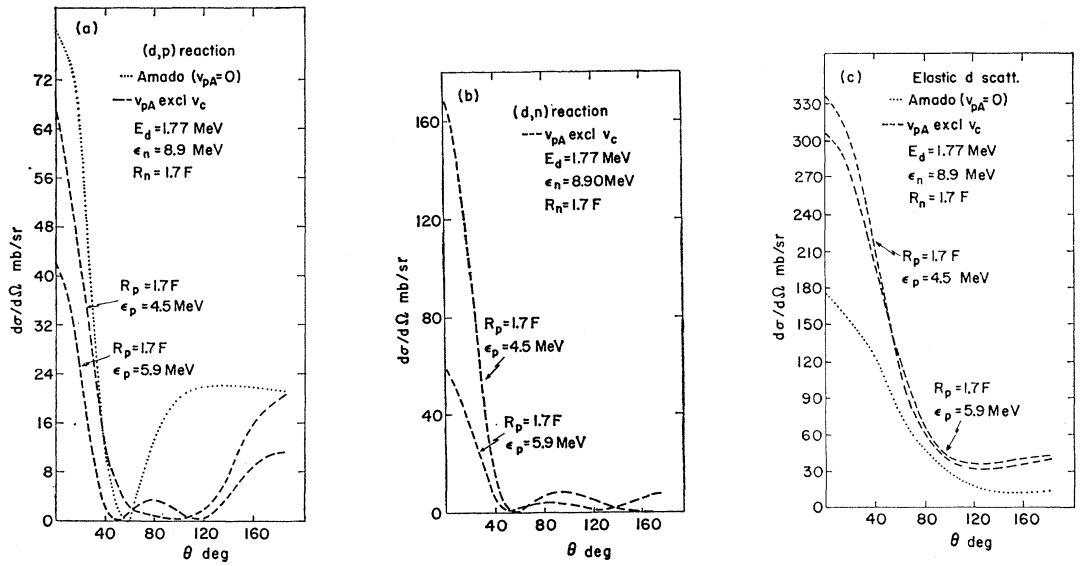

FIG. 3. Differential cross sections for (a) (d,p), (b) (d,n) reactions, and (c) elastic d scattering with fixed v_{np} and v_{nA}. Graphs given for (i) no interaction between proton and nucleus (Amado model); (ii) Hulthén-type interactions between proton and nucleus with varying proton binding energy and range parameters. $E_d=1.77$ MeV.

an effect is seen in backward directions. (d,n) stripping for low E_d is apparently very sensitive to small changes in the parameters of v_{pA}. Deuteron elastic cross sections increase under the influence of an over-all attractive v_{pA} of Hulthén form. From Figs. 1(c) and 2(c) however, we infer that v_c suppresses forward scattering.

3. STANDARD APPROXIMATIONS FOR STRIPPING

(A) Distorted-Wave Born Approximation

The exact amplitude for a transition between channels α and β may, in the distorted-wave representation, be written as[14]

$$T_{\beta\alpha}{}^{+}=\langle\chi_\beta{}^{-}|(\bar{v}_\beta-w_\beta{}^{\dagger})-(\bar{v}_\beta-w_\beta{}^{\dagger})\times G^{+}(\bar{v}_\alpha-w_\alpha)|\chi_\alpha{}^{+}\rangle. \quad (3.1)$$

G^{+} in Eq. (3.1) is again the total Green function as in Eq. (2.4), while $\bar{v}_\gamma=\sum_{\alpha\neq\gamma}v_\alpha$ is the interaction missing in the channel Hamiltonian $H_\gamma=H_0+v_\gamma$. The states $|\chi_\gamma{}^{\pm}\rangle$ are distorted waves which develop from the channel wave function $|\phi_\gamma\rangle=|\varphi_\gamma,\mathbf{q}_\gamma\rangle$ by action of some channel potential w_γ:

$$|\chi_\gamma{}^{\pm}\rangle=\{1+(E-H+\bar{v}_\gamma-w_\gamma\pm i\epsilon)^{-1}w_\gamma\}|\phi_\gamma\rangle. \quad (3.2)$$

For $\gamma=\beta$, w_β is chosen in such a fashion that the distorted wave $|\chi_\gamma{}^{\pm}\rangle$ generated through Eq. (3.2), does not possess a component corresponding to a rearrangement of channel β, or[14]

$$\lim_{\epsilon\to 0} i\epsilon\langle\chi_\beta{}^{-}|\phi_\alpha\rangle=0. \quad (3.3)$$

[14] L. R. Dodd and K. R. Greider, Phys. Rev. 146, 671 (1966).

One now defines the amplitude in the DWBA by neglecting in Eq. (3.1) the second term within the brackets. Thus[15]

$$T_{\beta\alpha}{}^{\mathrm{DWBA}+}=\langle\chi_\beta{}^{-}|\bar{v}_\beta-w_\beta{}^{\dagger}|\chi_\alpha{}^{+}\rangle. \quad (3.4)$$

It is natural to determine the (as yet unspecified) w_α in such a fashion that $|\chi_\alpha{}^{+}\rangle$ and the actual scattering state $|\psi_\alpha{}^{+}\rangle$ will lead to identical elastic scattering amplitudes $T_{\alpha\alpha}$. In other words, w_α will just be the *exact* optical potential for channel α, with infinite energy resolution.

We shall demonstrate below that $|\chi_\alpha{}^{+}\rangle$ may be determined from the elastic scattering amplitudes, without entering an actual calculation of w_α.[16] One may then wonder why $|\chi_\beta{}^{-}\rangle$ has not been chosen to be the exact optical-model wave function for channel β. The answer lies in the explicit presence of the optical potential $w_\beta{}^{\dagger}$ in Eq. (3.4), which necessitates either the computation of $w_\beta{}^{\dagger}$ or manipulations with $|\chi_\beta{}^{-}\rangle$. In order to avoid these, we propose the choice $w_\beta{}^{\dagger}=v_{pA}$ (v_{nA}) for the outgoing proton (neutron) channel. For both (d,p) and (d,n) reactions one sees that the DWBA amplitude (3.4) becomes

$$T_{\beta\alpha}{}^{\mathrm{DWBA}+}=\langle\chi_\beta{}^{-}|v_{pn}|\chi_\alpha{}^{+}\rangle. \quad (3.4')$$

The customary choice of T with $\chi_\beta{}^{-}$ the wave function

[15] Notice the apparent asymmetry in Eq. (3.4) with respect to initial and final channels, which is due to our choice to express the amplitude (2.3) by means of the operator U^{+} Eq. (2.4). Different operators U^{-} lead to amplitudes T^{-}, which in its DWBA read (Ref. 14) $T_{\beta\alpha}{}^{\mathrm{DWBA}-}=\langle\chi_\beta{}^{-}|\bar{v}_\alpha-w_\alpha|\chi_\alpha{}^{+}\rangle$. T^{+} and T^{-} can be shown to be identical on the energy shell.

[16] The possibility of determining the d optical potential in this fashion is discussed in Ref. 18.

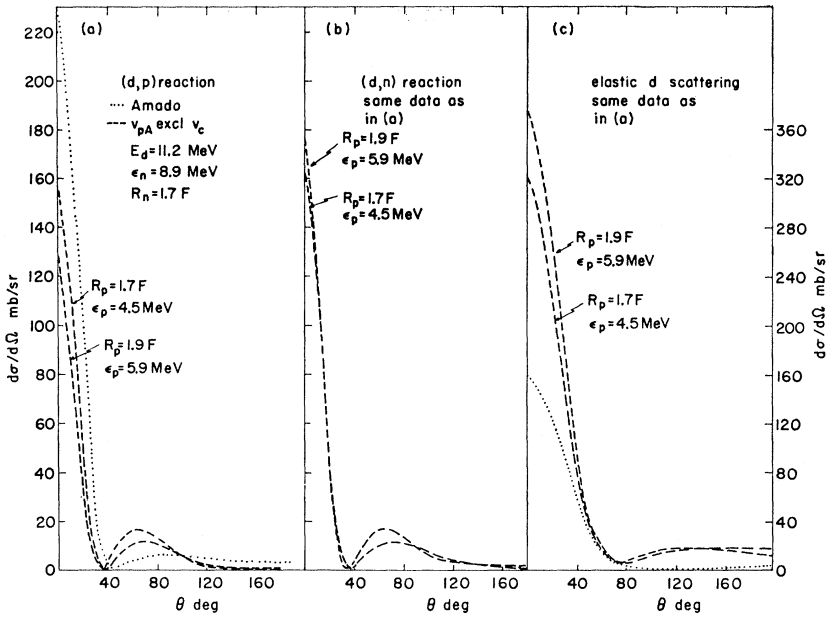

FIG. 4. Same as Fig. 3 except that $E_d=11.2$ MeV.

of an elastically scattered proton (neutron) from a bound state of a neutron (proton) and the nucleus will, when derived from (3.4), lead to $\bar{v}_\beta - w_\alpha^\dagger \neq v_{pn}$. Nevertheless the equality sign is used in the standard DWBA.

We now study Eq. (3.4) for stripping and consider first a nucleon channel β. The state $|\chi_\beta\rangle$ then describes the scattering of a proton (neutron) from a nucleus B (B') in the absence of a p-n interaction. It is therefore the product of the wave function φ_2 (φ_3) of the nucleus B (B') and a scattering state $|\xi_{pA}^-\rangle$ ($|\xi_{nA}^-\rangle$) for the proton (neutron). Clearly no neutron (proton) pickup is possible, and w_β thus satisfies the requirement (3.3) explained above.

$|\xi_{pA}^-\rangle$ satisfies an effective single-particle equation, which for separable v_{pA} (and correspondingly for v_{nA}) can be solved as follows ($\beta=2$):

$$\langle \mathbf{p}_2\mathbf{q}_2|\chi_2\rangle = \varphi_2(p_2)\langle \mathbf{q}_2|\xi_{pA}^-\rangle = \varphi_2(p_2) \times\left[\delta(\mathbf{q}_2-\mathbf{q}_2^0)+\frac{g_3(q_2)g_3(q_2^0)\tau_3(E_2+\epsilon_2-i\epsilon)}{E_2+\epsilon_2-q_2^2-i\epsilon}\right]. \quad (3.5)$$

E_2 is the total energy of the three-particle system and $\mathbf{q}_2^0$ the momentum of the outgoing proton.

Next we turn to the $\alpha=1$ channel. As proposed above, we construct $|\chi_1^+\rangle$ taking into account the requirement that it shall reproduce the exact d elastic scattering. Thus $|\chi_1^+\rangle$ is by definition given by

$$\langle \mathbf{p}_1\mathbf{q}_1|\chi_1^+\rangle = \varphi_1(p_1)[\delta(\mathbf{q}_1-\mathbf{q}_1^0)+(q_1^{0\,2}-q_1^2+i\epsilon)^{-1} \times\langle \mathbf{q}_1\varphi_1|\bar{v}_1|\psi_1^+\rangle], \quad (3.6)$$

where $\mathbf{q}_1^0$ and $\varphi_1(p_1)$ are, respectively, the incident momentum and bound-state wave function of the deuteron. $|\psi_1^+\rangle$ is the total scattering state which develops from $|\phi_1\rangle$, describing a deuteron incident on the core nucleus A. $|\chi_1^+\rangle$ is therefore known, provided the same holds for the partially off-energy-shell elastic-scattering amplitude ($E_1=q_1^{0\,2}-\epsilon_1$)

$$\langle \mathbf{q}_1\varphi_1|\bar{v}_1|\psi_1^+\rangle = \langle \mathbf{q}_1|T_{11}^+(E_1)|\mathbf{q}_1^0\rangle. \quad (3.7)$$

We noticed already in Sec. 2 the difference between the truly off-shell amplitudes and the Lovelace amplitudes X. The Lovelace equations (2.15) are thus unsuited to determine the former. We shall now show how T_{11}(3.7) may be obtained, in particular for the energy variable $E_1=q_1^{0\,2}-\epsilon_1$.

We first decompose the scattering state $|\psi_1^+\rangle$ appearing in Eq. (3.7) as proposed by Faddeev.[1]

$$|\psi_1^+\rangle = \sum_{\gamma=1} |\psi_1^\gamma\rangle. \quad (3.8)$$

The components $|\psi_1^\gamma\rangle$ satisfy

$$|\psi_1^\gamma\rangle = \delta_{1\gamma}|\phi_1\rangle - G_0(s)t_\gamma(s)\sum_{\delta\neq\gamma}|\psi_1^\delta\rangle, \quad (3.9)$$

with t_β the scattering matrix of the channel Hamiltonian H_β. Its matrix elements are related to $\hat{t}_\beta$ [Eq. (2.9)] by

$$\langle \mathbf{p}_\beta'\mathbf{q}_\beta'|t_\beta(s)|\mathbf{p}_\beta\mathbf{q}_\beta\rangle = \delta(\mathbf{q}_\beta'-\mathbf{q}_\beta')\langle \mathbf{p}_\beta'|\hat{t}_\beta(s-q_\beta^2)|\mathbf{p}_\beta\rangle. \quad (2.9')$$

The set of coupled integral equations (3.9) is treated in much the same manner as Phillips[17] treated a corresponding set for three-body bound states. Let us start with the momentum representative of Eq. (3.9) which,

[17] A. C. Phillips, Phys. Rev. **142**, 984 (1966).

by use of Eqs. (3.10) and (2.9), is written as

$$\langle \mathbf{p}_\gamma \mathbf{q}_\gamma | \psi_1{}^\delta \rangle = \delta_{1\delta} \langle \mathbf{p}_\gamma \mathbf{q}_\gamma | \phi_1 \rangle + (s - p_\gamma{}^2 - q_\gamma{}^2)^{-1} \int\int \langle \mathbf{p}_\gamma \mathbf{q}_\gamma | \mathbf{p}_\delta' \mathbf{q}_\delta' \rangle g_\delta(p_\delta') \tau_\delta(s - q_\delta'^2) d\mathbf{p}_\delta' d\mathbf{q}_\delta' \int g_\delta(p_\delta'') \langle \mathbf{p}_\delta'' \mathbf{q}_\delta' | \sum_{\epsilon \neq \delta} \psi_1{}^\epsilon \rangle d\mathbf{p}_\delta''$$

$$\equiv \delta_{1\delta} \langle \mathbf{p}_\delta \mathbf{q}_\delta | \phi_1 \rangle + (s - p_\gamma{}^2 - q_\gamma{}^2)^{-1} \int\int \langle \mathbf{p}_\gamma \mathbf{q}_\gamma | \mathbf{p}_\delta' \mathbf{q}_\delta' \rangle g_\delta(p_\delta') \tau_\delta(s - q_\delta'^2) \langle \mathbf{q}_\delta' | Q_\delta \rangle d\mathbf{p}_\delta' d\mathbf{q}_\delta' . \tag{3.10}$$

Equation (3.10) defines new amplitudes $|Q\rangle$. It is tedious but straightforward to show that for $s = E_1 = q_1{}^{0\,2} - \epsilon_1$ these Q satisfy the equation

$$\langle \mathbf{q}_\gamma | Q_\gamma \rangle = -\langle \mathbf{q}_\gamma | Z_{\gamma 1}(E_1) | \mathbf{q}_0' \rangle - \sum_{\delta \neq \gamma} \int \langle \mathbf{q}_\gamma | Z_{\gamma\delta}(E_1) | \mathbf{q}_\delta' \rangle \tau_\delta(E_1 - q_\delta'^2) \langle \mathbf{q}_\delta' | Q_\delta \rangle d\mathbf{q}_\delta' , \tag{3.11}$$

where the potentials Z are given by Eq. (2.12). Comparison of Eqs. (3.11) and (2.15) shows that

$$\langle \mathbf{q}_\gamma | Q_\gamma \rangle = \langle \mathbf{q}_\gamma | X_{\gamma\alpha}(E_1) | \mathbf{q}_1{}^0 \rangle . \tag{3.12}$$

We now substitute Eq. (3.8) into Eq. (3.7), using Eqs. (3.10) and (2.6), and obtain after some algebra

$$\langle \mathbf{q}_1 | T_{11}{}^+(E_1) | \mathbf{q}_1{}^0 \rangle = \langle \mathbf{q}_1 \varphi_1 | \bar{v}_1 | \psi_1{}^+ \rangle = -\sum_{\delta \neq \gamma} \int \langle \mathbf{q}_1 | Z_{1\delta}(q_1{}^2 - \epsilon_1) | \mathbf{q}_\delta' \rangle \tau_\delta(E_1 - q_\delta'^2) \langle \mathbf{q}_\delta' | Q_\delta \rangle d\mathbf{q}_\delta' . \tag{3.13}$$

One then invokes Eqs. (3.4), (3.5), (3.6), (3.7), and (3.13) in order to derive the final result ($\bar{v}_\beta - w_\beta{}^+ = v_\alpha = v_{np}$):

$$T_{\beta 1}{}^{\mathrm{DWBA}+} = -\langle \mathbf{q}_\beta | Z_{\beta 1}(E_1) | \mathbf{q}_1{}^0 \rangle - \int \frac{\langle \mathbf{q}_\beta | Z_{\beta 1}(E) | \mathbf{q}_1' \rangle \langle \mathbf{q}_1' | T_{11}{}^+(E_1) | \mathbf{q}_1{}^0 \rangle d\mathbf{q}_1'}{q_1{}^{02} - q_1'^2 + i\epsilon} - g_\beta(q_\beta) \tau_\beta(q_\beta{}^2 + i\epsilon) \int \frac{g_\beta(q_\beta')}{q_\beta{}^2 - q_\beta'^2 + i\epsilon}$$

$$\times \left[\langle \mathbf{q}_\beta' | Z_{\beta 1}(q_\beta'^2 - \epsilon_\beta) | \mathbf{q}_1{}^0 \rangle d\mathbf{q}_\beta' + \langle \mathbf{q}_\beta' | Z_{\beta 1}(q_\beta'^2 - \epsilon_\beta) | \mathbf{q}_1'' \rangle \frac{\langle \mathbf{q}_1'' | T_{11}{}^+(E_1) | \mathbf{q}_1{}^0 \rangle}{q_1{}^{02} - q_1''^2 + i\epsilon} d\mathbf{q}_\beta' d\mathbf{q}_1'' \right] . \tag{3.14}$$

After putting $\mathbf{q}_\beta$ on the energy shell

$$q_\beta{}^2 - \epsilon_\beta = (q_1{}^0)^2 - \epsilon_1 = E_1 ,$$

Eq. (3.14) constitutes the *exact* DWBA for stripping. The presence of the partially off-shell amplitude $T_{11}{}^+$ shows [cf. Eq. (3.13)] that amongst others one has to solve the set of coupled integral equations (3.11). This is done by first performing a partial-wave analysis and proceeding as in the solution of the Lovelace equations (2.15). Again, in both Eqs. (3.11) and (3.14), one has to deform the contour to avoid singularities.[12] Notice further the fact that the energy variable in the last term of Eq. (3.14) is not a parameter but a function of the integration variable.

The theory given above holds for the DWBA pertaining to stripping. A DWBA analysis for d elastic scattering will yield exact results once a d optical potential is determined. This is possible in principle, as has already been mentioned above.[18]

(B) Absorption Model

Angular distributions of stripping reactions which display pronounced diffraction patterns have recently been analyzed in several ways. One may, for instance, endow the nucleus with definite optical properties, which amounts to prescribing definite boundary conditions for the wavefunction of ingoing and outgoing particles on a well-defined nuclear surface region.[19]

In an equivalent procedure one assumes absorption of low partial waves in ingoing and outgoing channels.[20,21] In the simplest model, one only studies the Born term or the equivalent one-particle-exchange term. Such a term is described in our model by the potential Z in Eq. (2.15). We therefore ask whether removal of a few low partial waves from the Born term will reproduce the "observed" (in this case the exactly calculated) angular distributions. This, of course, amounts in a semiclassical picture to a nucleus acting as an object with some absorption radius.

If the cross section shows diffraction due to strong absorption, a DWBA fit should also show it through the imaginary parts of optical potentials. It is now to our detriment that the DWBA calculation circumvents a determination of those potentials. A comparison was nevertheless thought to be advantageous.

In Figs. 5(a) and 5(b), we plot differential cross sections for (d,p) and (d,n) reactions calculated with the

[18] A. I. Jaffe, A. S. Reiner, and J. E. Ventura, Nucl. Phys. **A95**, 235 (1967).

[19] A. Dar, Nucl. Phys. **55**, 305 (1964).

[20] N. J. Sopkovich, Nuovo Cimento **26**, 186 (1962).

[21] L. Durand and Y. T. Chiu, Phys. Rev. **139**, B646 (1965).

same parameters leading to Fig. 1 (E_d=6.7 MeV). Those results are compared on the one hand with the DWBA calculation, and on the other hand with the result for the one-particle exchange model with partial waves $l<2$ absorbed. Figures 6(a) and 6(b) give the same cross sections for E_d=11.2 MeV. In Figs. 7(a), 7(b), 8(a), and 8(b) curves are given for the absorption model with a variable number of partial waves removed.

Most prominent is the apparently correct representation of the DWBA for the entire angular range, which, we stress again, is obtained without parameter adjustments. The absorption model, on the other hand, gives a reasonable fit for the (d,p) reaction over the entire angular range. This result is certainly striking in view of the low incident energy. The fit is substantially less good for the (d,n) reaction. It will, however, be noticed from Figs. 7 and 8 that the fit is sharp. It is therefore clear that the use of a rounded-off absorption function[22] instead of a step function in l would considerably improve the fit.

A final remark concerns the interpretation of the angular distributions as diffraction patterns. One may derive from the value of the largest absorbed partial wave an impact parameter of the order of the nuclear radius ($b\sim$2.5 F). However, the interaction radii estimated from diffraction minima are about a factor 2 larger, and are thus of the order of the proton radius and definitely larger than the size of the neutron orbit.

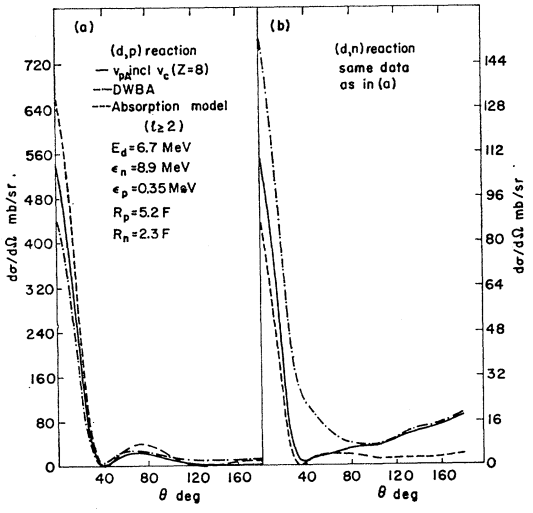

FIG. 5. Differential cross section for (a) (d,p) and (b) (d,n) reactions with fixed v_{np} and v_{nA}. Graphs given for (i) v_{pA} incorporating Coulomb effect (Z=8); (ii) absorption model; (iii) DWBA, E_d=6.7 MeV.

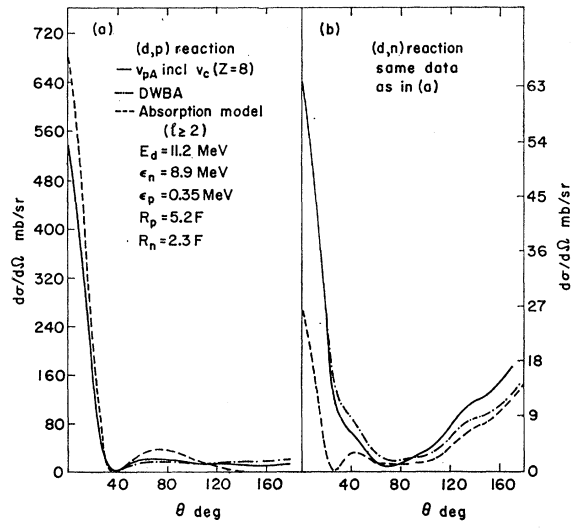

FIG. 6. Same as Fig. 5 except that E_d=11.2 MeV.

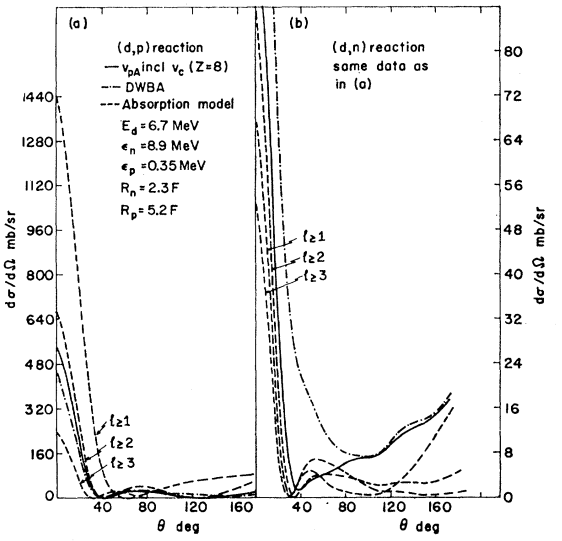

FIG. 7. Differential cross sections for (a) (d,p) and (b) (d,n) reactions in comparison with DWBA and model for absorption of various partial waves. E_d=6.7 MeV.

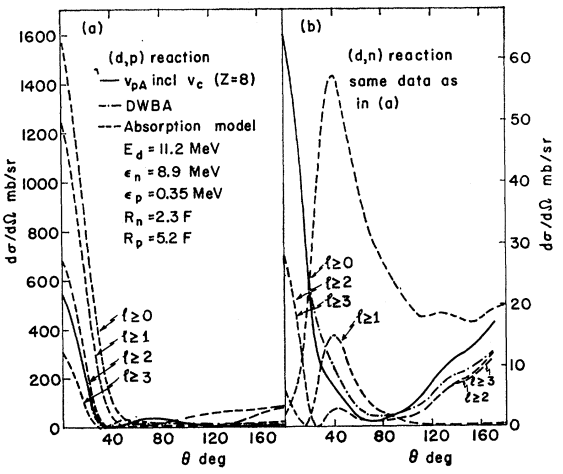

FIG. 8. Same as Fig. 7 except that E_d=11.2 MeV.

[22] W. E. Frahn and R. H. Venter, Ann. Phys. (N.Y.) 24, 243 (1963).

In view of the impossibility of ascribing a radius to our nucleus which is different from the orbit sizes of the two outer nucleons, we find it hard to apply criteria for the validity of the diffraction picture based on the above observations.

4. DISCUSSION AND COMPARISON

We have described above a three-particle model for nuclear stripping. Exact calculations have been performed for a system of spinless particles interacting through separable interactions leading to a s-wave bound state in each channel. One of our two objectives has been the study of the effect of a proton-"nucleus" interaction v_{pA}, which had initially been neglected.[3] Mitra[8] and Aaron and Shanley[7] included a $v_{pA}=v_{nA}$. In a three-particle model this entails identical p and n stripping patterns, contrary to what is observed.

We therefore assumed $v_{pA}\neq v_{nA}$ in two alternative ways. We first tried to account for the reduced binding of the proton by changing the parameters of the neutron-nucleus interaction. For reasonable parameters the thus calculated stripping cross sections for emerging protons are smaller than those for neutrons, contrary to observation. However, inclusion of a Coulomb interaction yields the correct behavior.

There is, of course, no way to compare the outcome of our computations with a real experiment, except for the deuteron-nucleon system. Nevertheless, the angular distributions obtained very much resemble typical stripping patterns and show that stripping by a real nucleus may presumably be accounted for in part by the dynamics of a three-particle system. A three-particle *model* of course does not give any spectroscopic correction factor.

We computed for our model what is commonly termed the DWBA and found very satisfactory agreement with the exact solution. Our result is to be contrasted with the calculations of Shanley[9] who employs optical potentials, with parameters chosen to fit p and d elastic scattering.

We further tested the model where out of the one-particle exchange term the lowest partial waves were absorbed. The fit, in spite of the low energies involved, is reasonable for (d,p) stripping throughout the entire angular range. Shanley[9] reproduces diffraction patterns by applying rescattering corrections in initial and final channels,[20,21] but the resulting patterns fit less well than those of the simple absorption model.

Finally we touch upon a last approximation to stripping recently proposed by Butler.[23] He assumes that for sufficiently high incident-deuteron energies, the constituent proton and neutron are scattered like plane waves, the momenta of which remain correlated as in the deuteron bound-state wave function. Butler's model cannot be compared with ours since his cross section for a three-particle model vanishes exactly.[24–26]

In conclusion we consider deuteron elastic scattering, which in our model, for the parameters chosen, is of the same order of magnitude as stripping. It is known that for medium-weight nuclei, deuteron elastic cross sections exceed stripping cross sections by far. Also the shape shows far more oscillation than in our curves.

Only a few deuteron elastic scattering data exist for light nuclei. For O^{16} the elastic angular cross sections, although still larger than the stripping cross section, no longer differs by orders of magnitude.

We do not aim to compare our results in detail with experiment. In spite of the inclusion of a proton-nucleus interaction we still think the model to be crude. On the other hand, we doubt whether it pays to include spins and in particular more single-particle bound states; such an attempt in any case is hardly possible with present-day computers. It has merely been our goal to see the influence of v_{pA} (till now neglected), and in particular to test standard approximations in stripping calculations. We feel that inferences from those tests are of relevance beyond the scope of our model.

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[23] S. T. Butler, Nature **207**, 1346 (1965).
[24] R. M. May, Nature **207**, 1348 (1965).
[25] Y. Tikochinsky (unpublished).
[26] C. F. Clement, Phys. Rev. Letters **17**, 759 (1966).