Theory of *s-d* Scattering in Dilute Magnetic Alloys. II. Derivation and Solution of Linear Vertex Equations

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A set of linear equations for the vertex function associated with conduction-electron scattering from a spin-dependent impurity potential is derived from perturbation theory and is shown to lead to the Abrikosov vertex function in the limit of a contact interaction and logarithmic accuracy. The general reduction to quadratures of the linear equations is given for *s*-wave, factorizable interactions. A detailed evaluation of the (off-energy-shell) vertex function is performed for that interaction which corresponds to the pole-approximation solution to the *s*-wave Low equation in the limit of spin-independent potentials. The Fermi factors introduced by a spin-dependent *s-d* interaction lead to a (linear) vertex function with analytical properties differing from those of the pole-approximation solution to the coupled Low equations. The influence of ground-state correlations in the linear equations causes the failure of inelastic, single-channel unitarity in both spin channels ($J=S\pm\frac{1}{2}$) for factorizable, *s*-wave antiferromagnetic *s-d* interactions with any strength or form factor.

I. INTRODUCTION

IN the previous paper,[1] hereafter denoted by I, we discussed the perturbation-theory[2] calculation of the electronic self-energy and its associated vertex part for a free-electron gas interacting with a dilute, random distribution of magnetic impurities via the Hamiltonian

$$H_{\rm int}=-N^{-1}\sum_{\mathbf{p},\mathbf{p}',n}\{J_1(\mathbf{p},\mathbf{p}')+J_2(\mathbf{p},\mathbf{p}')\,\boldsymbol{\sigma}_{\alpha'\alpha}\cdot\mathbf{S}_n\}$$
$$C^+_{\mathbf{p}',\alpha'}C_{\mathbf{p},\alpha}\exp[i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{R}_n],\quad(1.1)$$

in which $\boldsymbol{\sigma}_{\alpha',\alpha}$ are the matrix elements of the Pauli spin matrices, $\mathbf{R}_n$ is the coordinate of the nth impurity, $\mathbf{S}_n$ is the impurity spin operator, $J_1(\mathbf{p}',\mathbf{p})$ is the spin-independent impurity potential, $J_2(\mathbf{p}',\mathbf{p})$ is the effective Coulomb exchange integral incorporating hybridization effects,[3,4] and $C_{\mathbf{p}\alpha}$ is the free-electron annihilation operator. In I we reviewed Abrikosov's demonstration that in the pseudofermion representation, the one-electron propogator can be written in terms of a proper self-energy part and that, to logarithmic accuracy,[1,2] those terms in the self-energy which are linear in the impurity concentration can be simply expressed in terms of an electron-impurity vertex function

$$\langle\alpha\beta\mid\Gamma(\mathbf{p},\mathbf{p}';\epsilon)\mid\alpha'\beta'\rangle$$
$$\equiv\langle\alpha\beta\mid\Gamma(\mathbf{p},i\epsilon;0\mid\mathbf{p}',i\epsilon;0)\mid\alpha'\beta'\rangle.$$

The full electron-pseudofermion ("four-tail") vertex function $\langle\alpha\beta\mid\Gamma[\mathbf{p},i\epsilon;i\omega_1\mid\mathbf{p}',i(\epsilon+\omega_1-\omega_2);i\omega_2]\mid\alpha'\beta'\rangle$ is shown in Fig. 1. Its calculation is described in detail in I, where we demonstrated that to logarithmic accuracy the contribution to the self-energy linear in the impurity concentration $N_{\rm i}$ could be written as [Eq. (2.33) in I]

$$\mathrm{Im}\sum_{\alpha'\alpha}(\mathbf{p},\epsilon)=-\delta_{\alpha'\alpha}\tfrac{1}{4}[N_{\rm i}\rho(\epsilon)]\int d\xi_{\mathbf{q}}d\Omega_{\mathbf{p}\cdot\mathbf{q}}\delta(\xi_{\mathbf{q}}-\epsilon)$$
$$\times\{|\,\Gamma^{(S)}(\mathbf{p},\mathbf{q};\epsilon)\,|^2+S(S+1)\,|\,\Gamma^{(V)}(\mathbf{p},\mathbf{q};\epsilon)\,|^2\},\quad(1.2a)$$
$$\Gamma\equiv\Gamma^{(S)}+\boldsymbol{\sigma}\cdot\mathbf{S}_n\Gamma^{(V)}.\quad(1.2b)$$

The primary result obtained in I was the derivation that if on the energy shell $\Gamma(\mathbf{p},\mathbf{p}';\epsilon)$ satisfies the coupled nonlinear Low equations of S-matrix theory[5] then Eq. (1.2a) gives the perturbation-theory self-energy to logarithmic accuracy. In this paper we more precisely delineate the consequences of the restriction to logarithmic accuracy by deriving and solving a set of linear equations for the general (off-energy-shell) $\Gamma(\mathbf{p},\mathbf{p}';\epsilon)$. We find that the analytic properties of the "linear" vertex function are identical to those of the "nonlinear" vertex function only in the absence of the exclusion-principle restrictions imposed by the many-body nature of the Hamiltonian (1.1) for the spin-dependent term $J_2(\mathbf{p}',\mathbf{p})\,\boldsymbol{\sigma}\cdot\mathbf{S}_n$. Therefore, although the vertex-function approximation, Eq. (1.2a), to the self energy is only valid in Abrikosov's limit of logarithmic accuracy, the imposition of this limit does not permit the unique specification of the analytic properties of "effective" electron-impurity vertex function $\Gamma(\mathbf{p},\mathbf{p}';\epsilon)$. An integral equation for the vertex $\Gamma(\mathbf{p},\mathbf{p}';\epsilon)$, which does not depend on the full four-tail vertex

$$\Gamma[\mathbf{p},i\epsilon,i\omega_1\mid\mathbf{p}',i(\epsilon+\omega_1-\omega_2),i\omega_2],$$

can be written only in the limit of logarithmic accuracy, so that the apparent lack of unique specification of the analytic properties of $\Gamma(\mathbf{p},\mathbf{p}';\epsilon)$ in this limit is unsatisfactory. The ill-defined character of these properties emphasizes the more general perturbation theory result that no direct physical significance can be attached to them because (1.2a) is valid only to logarithmic accuracy. As we demonstrate in Sec. II that the linear equations yield Abrikosov's vertex function for a

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contact interaction, we must conclude either that the analytic properties of the effective vertex are ill-defined in the limit of logarithmic accuracy or that Abrikosov's application of this limit is incorrect (see Ref. 31 of I).

In this paper we do not inquire further into Abrikosov's passage to the limit of logarithmic accuracy than to demonstrate in Sec. II that the linear equation yields the correct third-order vertex functions (fourth-order self-energy). In Sec. III we perform the reduction to quadratures of the linear equation for the case of a factorizable s-wave interaction,[6] and give a detailed evaluation of the solution to the linear equation for an s-d potential ($J_1=0$) of the form which directly gives the determinental-method, single-pole-approximation solution to the (nonlinear) Low equation in the absence of exclusion-principle restrictions.[7] We also show in Sec. III that the ground-state correlations included in the linear equation lead to the failure of inelastic, single, channel unitarity for any factorizable, s-wave, antiferromagnetic s-d interaction and relate this result to the variational calculations of Yosida[8] and Okiji.[9]

A linear equation for the vertex function has been proposed independently by Solyom.[10] Unfortunately, his proposed equation undercounts all the perturbation-theory diagrams but the bare vertex and, therefore, is inadequate even in the limit of logarithmic accuracy.

II. DERIVATION OF THE LINEAR INTEGRAL EQUATION

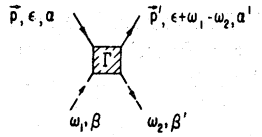
In this section we demonstrate that a linear equation for the vertex function reproduces the full perturbation-theory self-energy via (1.2) to logarithmic accuracy through fourth order for a restricted s-d interaction ($J_1=0$). (The Kondo effect first occurs in the third-order expression for the self-energy so that the fourth-order diagrams provide the first nontrivial test of a diagram summation procedure.)

We proceed as in I by constructing the equation for the vertex function directly. The four-tailed vertex function is defined by

$$\Gamma=\Gamma_0+\Lambda_1+\Lambda_2, \tag{2.1}$$

as in Sec. 3 of I, but for the linear equation Λ_1 and Λ_2 are given in Fig. 2. Similar to the form of the nonlinear equation shown in Fig. 9 of I, the linear equation in its present form suffers both from the omission of classes of

FIG. 1. The four-tail electron-pseudofermion vertex function $\langle\alpha\beta \mid \Gamma[(\mathbf{p}, i\epsilon, i\omega \mid \mathbf{p}', i(\epsilon+\omega_1-\omega_2)] \times \mid \alpha'\beta'\rangle$.



diagrams and from multiple counting of some of the topological forms represented. There are two forms of diagrams omitted. First, the set of nonparquet graphs which are discussed in length in I, and second, the subset of parquet graphs which have both entering and exiting internal pseudofermion lines. Examples of this form are illustrated by the fourth-order graphs $\Gamma^{(2143)}$ and $\Gamma^{(3412)}$ defined in I. This latter subset of graphs is the same order in the logarithmic divergence as the parquet graphs which are summed. However, the variable transformation used to correct the inconsistent counting agparently compensates for the missing terms to logarithmic accuracy (see Ref. 31 in I).

The symmetrized form of the equations for Λ_1 and Λ_2, indicated in Fig. 2 are introduced for the convenience of identifying graphically the topological forms of the vertex functions generated by the integral equation. Alternatively, we can use the unsymmetrized form illustrated in Fig. 3. Although the unsymmetrized version sacrifices the graphical identification of the topological forms, it generates precisely the same terms as the symmetrized equation because the presence therein of overlapping cuts causes several perturbation theory diagrams to be associated with each iterative term in the equation. The results obtained after the transformation of variables used to correct the counting and the analytic continuation are identical for the symmetrized and unsymmetrized equations. The linear equation used by Sólyom[10] is just the unsymmetrized form, without the variable substitution designed to compensate for the incorrect counting of perturbation-theory terms.

We proceed as in Sec. III of I by (1) writing the contributions shown in Fig. 1; (2) performing the sums over the intermediate ω variables using the analytic properties of the vertex function given in I; and (3) setting the external variables equal to λ and analytically continuing ϵ to the real axis from the upper-half complex ϵ plane in both terms of Λ_1 and Λ_2 in Fig. 1. The linear analog of Eqs. (3.4) in I is given by

$$\langle\alpha\beta \mid \Lambda_1(p, \epsilon \mid p', \epsilon) \mid \alpha'\beta'\rangle = -\frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\frac{n(-\xi_q)}{\epsilon-\xi_q} \times \{\langle\alpha\beta \mid \Gamma_0(\mathbf{p}, \mathbf{q}) \mid \alpha''\beta''\rangle\langle\alpha''\beta'' \mid \Gamma(\mathbf{q}, \xi_q; \epsilon-\xi_q \mid \mathbf{p}', \epsilon; 0) \mid \alpha'\beta'\rangle$$
$$+\langle\alpha\beta \mid \Gamma(\mathbf{p}, \epsilon; 0 \mid \mathbf{q}, \xi_q; \epsilon-\xi_q) \mid \alpha''\beta''\rangle\langle\alpha''\beta'' \mid \Gamma_0(\mathbf{q}, \mathbf{p}') \mid \alpha'\beta'\rangle\}; \tag{2.2a}$$

$$\langle\alpha\beta \mid \Lambda_2(\mathbf{p}, \epsilon \mid \mathbf{p}', \epsilon) \mid \alpha'\beta'\rangle = -\frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\frac{n(\xi_q)}{(\epsilon-\xi_q)} \times \{\langle\alpha\beta'' \mid \Gamma_0(\mathbf{p}, \mathbf{q}) \mid \alpha''\beta'\rangle\langle\alpha''\beta \mid \Gamma(\mathbf{q}, \xi_q; 0 \mid \mathbf{p}', \epsilon; \epsilon-\xi_q) \mid \alpha'\beta''\rangle$$
$$+\langle\alpha\beta'' \mid \Gamma(\mathbf{p}, \epsilon; \epsilon-\xi_q \mid \mathbf{q}, \xi_q; 0) \mid \alpha''\beta'\rangle\langle\alpha''\beta \mid \Gamma_0(\mathbf{q}, \mathbf{p}') \mid \alpha'\beta''\rangle\}. \tag{2.2b}$$

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In I we restored paper counting to the nonlinear integral equation by making the variable substitution $\epsilon \to \xi_q$ in the vertex functions appearing in the integrand [I, Eqs. (3.4), (3.5)]. Here we restore the proper counting by making the substitution in the opposite direction, viz., $\xi_q \to \epsilon$ in the vertex function appearing in the kernel of the integral equation. In I the multiple counting was removed by a variable transform which, in effect, decomposed the partial fraction in $\Gamma^{(123)}$. In the linear case we have a fractional counting and we remove the fractional counting using this transform to combine the terms of a similar partial fraction. The linear forms of Λ_1 and Λ_2, which correctly count the perturbation-theory terms, are of the form

$$\langle\alpha\beta \mid \Lambda_1(\mathbf{p},\epsilon \mid \mathbf{p}',\epsilon) \mid \alpha'\beta'\rangle = -\frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\frac{n(-\xi_q)}{\epsilon-\xi_q} \times \{\langle\alpha\beta \mid \Gamma_0(\mathbf{p},\mathbf{q}) \mid \alpha''\beta''\rangle\langle\alpha''\beta'' \mid \Gamma(\mathbf{q},\epsilon \mid \mathbf{p}',\epsilon) \mid \alpha'\beta'\rangle + \langle\alpha\beta \mid \Gamma(\mathbf{p},\epsilon \mid \mathbf{q},\epsilon) \mid \alpha''\beta''\rangle\langle\alpha''\beta'' \mid \Gamma_0(\mathbf{q},\mathbf{p}) \mid \alpha'\beta'\rangle\}; \quad (2.3a)$$

$$\langle\alpha\beta \mid \Lambda_2(\mathbf{p},\epsilon \mid \mathbf{p}',\epsilon) \mid \alpha'\beta'\rangle = -\frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\frac{n(\xi_q)}{\epsilon-\xi_q} \times \{\langle\alpha\beta'' \mid \Gamma_0(\mathbf{p},\mathbf{q}) \mid \alpha''\beta'\rangle\langle\alpha''\beta \mid \Gamma(\mathbf{q},\epsilon \mid \mathbf{p}',\epsilon) \mid \alpha'\beta''\rangle + \langle\alpha\beta'' \mid \Gamma(\mathbf{p},\epsilon \mid \mathbf{q},\epsilon) \mid \alpha''\beta'\rangle\langle\alpha''\beta \mid \Gamma_0(\mathbf{q},\mathbf{p}') \mid \alpha'\beta''\rangle\}. \quad (2.3b)$$

By use of Eq. (3.6) of I the linear coupled integral equations for the scalar and vector vertex functions can now be written as

$$\Gamma^{(S)}(\mathbf{p},\mathbf{p}';\epsilon) = \frac{J_1(\mathbf{p},\mathbf{p}')}{N} - (8\pi N)^{-1}\int_{-\epsilon_F\le\xi_q} \frac{\rho(\xi_q)\,d\Omega_{\mathbf{p},\mathbf{q}}\,d\xi_q}{\epsilon-\xi_q}\{J_1(\mathbf{p},\mathbf{q})\Gamma^{(S)}(\mathbf{p},\mathbf{q}';\epsilon)+\Gamma^{(S)}(\mathbf{p},\mathbf{q};\epsilon)J_1(\mathbf{q},\mathbf{p}') + S(S+1)[J_2(\mathbf{p},\mathbf{q})\Gamma^{(V)}(\mathbf{q},\mathbf{p}';\epsilon)+\Gamma^{(V)}(\mathbf{p},\mathbf{q};\epsilon)J_2(\mathbf{q},\mathbf{p}')]\}; \quad (2.4a)$$

$$\Gamma^{(V)}(\mathbf{p},\mathbf{p}';\epsilon) = \frac{J_2(\mathbf{p},\mathbf{p}')}{N} - (8\pi N)^{-1}\int_{-\epsilon_F<\xi_q} \frac{\rho(\xi_q)\,d\Omega_{\mathbf{p},\mathbf{q}}\,d\xi_q}{\epsilon-\xi_q}\{J_1(\mathbf{p},\mathbf{q})\Gamma^{(V)}(\mathbf{q},\mathbf{p}';\epsilon)+\Gamma^{(V)}(\mathbf{p},\mathbf{q};\epsilon)J_1(\mathbf{q},\mathbf{p}') + J_2(\mathbf{p},\mathbf{q})\Gamma^{(S)}(\mathbf{q},\mathbf{p}';\epsilon)+\Gamma^{(S)}(\mathbf{p},\mathbf{q};\epsilon)J_2(\mathbf{q},\mathbf{p}') - \tanh(\xi_q/2T)[J_2(\mathbf{p},\mathbf{q})\Gamma^{(V)}(\mathbf{q},\mathbf{p}';\epsilon)+\Gamma^{(V)}(\mathbf{p},\mathbf{q};\epsilon)J_2(\mathbf{q},\mathbf{p}')]\}, \quad (2.4b)$$

in which $\rho(\xi) = m[2m(\epsilon_F+\xi)]^{1/2}/2\pi^2$ is the free-electron density of states.

The linear integral equations do not manifestly preclude singularities in the upper half of the complex plane. Their structure is similar to that of the T-matrix form of the Schrödinger equation for potential scattering:

$$T(\epsilon) = V + V(\epsilon - H_0)^{-1}T(\epsilon). \quad (2.5)$$

As discussed in Sec. III, the solutions to the linear integral equation for s-wave separable potentials may exhibit singularities for certain ranges of coupling constants in the upper half of the complex ϵ plane.

Equations (2.4) can be written in a form directly comparable to the Schrödinger equation (2.5) for real symmetric J_1 and J_2 given by

$$J_i(\mathbf{p},\mathbf{p}') = \sum_{l=0}^{\infty} j_i^{(l)}(p,p')\,P_l[\cos\theta_{\mathbf{p},\mathbf{p}'}], \quad (2.6a)$$

$$j_i^{(l)}(p,p') = j_i^{(l)}(p',p). \quad (2.6b)$$

By partial-wave expansion of the second Born term of Eqs. (2.4) and subsequent application of induction, one can demonstrate

$$\Gamma^{(i)}(\mathbf{p},\mathbf{p}';\epsilon) = \Gamma^{(i)}(\mathbf{p}',\mathbf{p};\epsilon). \quad (2.6c)$$

Therefore, Eqs. (2.4) can be written in their final form:

$$\Gamma^{(S)}(\mathbf{p},\mathbf{p}';\epsilon) = \frac{J_1(\mathbf{p},\mathbf{p}')}{N} - (4\pi N)^{-1}\int_{-\epsilon_F}^{\infty} \frac{\rho(\xi_q)\,d\Omega_{\mathbf{p},\mathbf{q}}\,d\xi_q}{\epsilon-\xi_q}\{J_1(\mathbf{p},\mathbf{q})\Gamma^{(S)}(\mathbf{q},\mathbf{p}';\epsilon)+S(S+1)\,J_2(\mathbf{p},\mathbf{q})\Gamma^{(V)}(\mathbf{q},\mathbf{p}';\epsilon)\}, \quad (2.7a)$$

$$\Gamma^{(V)}(\mathbf{p},\mathbf{p}';\epsilon) = \frac{J_2(\mathbf{p},\mathbf{p}')}{N} - (4\pi N)^{-1}\int_{-\epsilon_F}^{\infty} \frac{\rho(\xi_q)\,d\Omega_{\mathbf{p},\mathbf{q}}\,d\xi_q}{\epsilon-\xi_q}\{J_1(\mathbf{p},\mathbf{q})\Gamma^{(V)}(\mathbf{q},\mathbf{p}';\epsilon)+J_2(\mathbf{p},\mathbf{q})\Gamma^{(S)}(\mathbf{q},\mathbf{p}';\epsilon) - \tanh(\xi_q/2T)J_2(\mathbf{p},\mathbf{q})\Gamma^{(V)}(\mathbf{q},\mathbf{p}';\epsilon)\}. \quad (2.7b)$$

We next illustrate how the variable substitution restores the proper counting of the distinct typological forms of the vertex function to logarithmic accuracy. The second- and third-order iterations of the incorrect counting form, Eqs. (2.2), are

$$\Gamma_2 = \Gamma^{(12)} + \Gamma^{(21)}, \quad (2.8a)$$

$$\Gamma_3 = \Gamma^{(123)} + \Gamma^{(321)} + \tfrac{1}{2}(\Gamma^{(132)} + \Gamma^{(231)} + \Gamma^{(213)} + \Gamma^{(312)}). \quad (2.8b)$$

Let us reconsider some of the examples illustrated in Eqs. (3.10)–(3.13) of I. $\Gamma^{(132)}$ for a contact s-d interaction is given by

$$\Gamma^{(132)} = c_{st}^{(132)}(J/N)^2 \int\int_{-\epsilon_F}^{+\epsilon_F} d\xi_1 d\xi_2 \times \frac{\rho(\xi_1)\rho(\xi_2)n(-\xi_1)n(\xi_2)}{(\epsilon-\xi_1+i\delta)(\xi_1-\xi_2)}. \quad (2.9)$$

The form generated from Eqs. (2.2) is just $\frac{1}{2}$ of the exact form given above. The correct counting integral equation, Eqs. (2.3), generates the form

$$\Gamma^{(132)}=\tfrac{1}{2}C_{\mathrm{st}}{}^{(132)}\left(\frac{J}{N}\right)^2\int\int_{-\epsilon_F}^{+\epsilon_F}\frac{d\xi_1 d\xi_2\rho(\xi_1)\rho(\xi_2)}{(\xi_1-\xi_2)(\epsilon-\xi_1+i\delta)}\times[n(-\xi_1)n(\xi_2)+n(\xi_1)n(-\xi_2)]. \quad (2.10)$$

To leading order in the logarithmic divergence, these two expressions (2.9) and (2.10) are identical. The forms $\Gamma^{(123)}$ and $\Gamma^{(321)}$ generated by Eqs. (2.3) are equal to the exact values. We now solve these equations for a cutoff s-d interaction. $J_1=0$; $J_2(\mathbf{p},\ \mathbf{p}')=J$, $-\epsilon_F\leq\xi_p,\ \xi_{p'}\leq\epsilon_F$. Equations (2.4) reduce to

$$\Gamma^{(S)}(\epsilon)=\frac{-S(S+1)}{N}J\Gamma^{(V)}(\epsilon)\int_{-\epsilon_F}^{+\epsilon_F}\frac{\rho(\xi_q)d\xi_q}{\epsilon-\xi_q+i\delta}=i\pi[S(S+1)/N]J\Gamma^{(V)}(\epsilon)\rho(\epsilon), \quad (2.11a)$$

$$\Gamma^{(V)}(\epsilon)=\frac{J}{N}\left[1-\Gamma^{(S)}(\epsilon)J\int_{-\epsilon_F}^{+\epsilon_F}\frac{\rho(\xi_q)d\xi_q}{\epsilon-\xi_q+i\delta}+J\Gamma^{(V)}(\epsilon)\int_{-\epsilon_F}^{+\epsilon_F}\tanh\left(\frac{\xi_q}{2T}\right)\frac{\rho(\xi_q)}{\epsilon-\xi_q+i\delta}d\xi_q\right]. \quad (2.11b)$$

These are now simple algebraic relations. At $T=0$, to logarithmic accuracy, we have

$$\Gamma^{(V)}(\epsilon)=\frac{J/N}{1+(2J/N)\rho(0)\ \ln(\epsilon_F/|\epsilon|)+i\pi(J/N)\rho(\epsilon)[S(S+1)+\mathrm{sgn}(\epsilon)]}. \quad (2.12)$$

This is identical to Abrikosov's solution except for the phase which is beyond logarithmic accuracy.

III. SOLUTIONS TO THE LINEAR EQUATIONS

A. The Factorizable Potential Model

In this section we discuss the solutions to Eqs. (2.7) for the case in which both the scalar and vector bare vertices are linear transformations of rank one. This "factorizable-potential" model[6] is of interest because it leads to exact solutions to Eqs. (2.7) and because for a particular choice of model potential, we recover the linear analog[7] of the pole-approximation to be the left-hand (energy-plane) cut used in Suhl's analysis[5] of the nonlinear dispersion-theory equations. The model bare vertex functions are specified by

$$J_i(p,p')=\frac{4\pi}{N}\sum_{l=0,\ -l\leq m\leq l}^{\infty}\frac{J_i{}^{(l)}}{2l+1}\times\{g_i{}^{(l)}(p)g_i{}^{(l)}(p')Y_{lm}(\Omega_p)Y_{lm}(\Omega_{p'})\}, \quad (3.1)$$

for which we subsequently consider only the $l=0$ component (which causes s-wave scattering) as in the analyses of both Suhl[5] and Abrikosov.[2] All ($l=0$) superscripts will be suppressed for convenience. The extension to a d-wave potential, more relevant for transition-metal impurities in the electron gas, leads to similar results for the $l=2$ partial-wave component of the vertex functions $\Gamma^{(S)}$ and $\Gamma^{(V)}$ after the angular integrals have been performed in Eqs. (2.7).

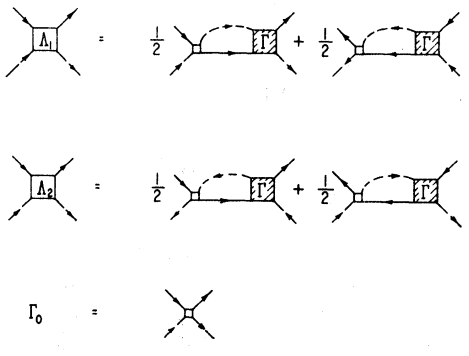

FIG. 2. Diagrammatic expansion of the terms in Eq. (2.1) in the text for the linear, four-tail vertex function. This expansion generates all topological forms through fourth order for the self-energy (third-order for the vertex function itself).

In the case of only the s-wave component in Eq. (3.1), Eqs. (2.7) reduce to a set of four coupled algebraic equations for the quantities

$$M_{i1}(\epsilon;\mathbf{p})=\int\frac{d^3q}{(2\pi)^3}\frac{g_i(q)\Gamma^{(S)}(\epsilon;\mathbf{q},\mathbf{p})}{\xi_q-\epsilon}; \quad (3.2a)$$

$$M_{i2}(\epsilon;\mathbf{p})=\int\frac{d^3q}{(2\pi)^3}\frac{g_i(q)\Gamma^{(V)}(\epsilon;\mathbf{q},\mathbf{p})}{\xi_q-\epsilon}. \quad (3.2b)$$

In terms of the integrals

$$F_{ij}(\epsilon)=\int\frac{d^3q}{(2\pi)^3}\frac{g_i(q)g_j(q)}{\xi_q-\epsilon}, \quad (3.3)$$

$$G_{ij}(\epsilon;T)=\int\frac{d^3q}{(2\pi)^3}\frac{g_i(q)g_j(q)}{\xi_q-\epsilon}\tanh(\xi_q/2T). \quad (3.4)$$

A discussion of the equations, their solution, and the final expressions for $\Gamma^{(S)}$ and $\Gamma^{(V)}$ are given in Appendix A.

Both for the sake of simplicity, and because Suhl's pole approximation restricts his analysis to this limit, we consider in the main text only the case in which the form factors of the scalar and vector vertices are equal:

$$g_1(p)=g_2(p)=g(p); \quad (3.5a)$$

$$F(\epsilon)=\int\frac{d^3q}{(2\pi)^3}\frac{g^2(q)}{\xi_q-\epsilon}; \quad (3.5b)$$

$$G(\epsilon;T)=\int\frac{d^3q}{(2\pi)^3}\frac{g^2(q)}{\xi_q-\epsilon}\tanh(\xi_q/2T). \quad (3.5c)$$

For convenience in discussing unitarity, we write the solutions for the vertex functions characteristic of the

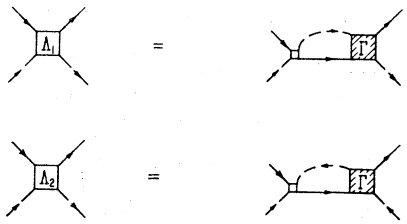

FIG. 3. Diagrammatic expansion of the terms in Eq. (2.1). This expansion is equivalent to that of Fig. 2 after the variable substitutions to correct multiple counting of diagrams has been made as outlined above Eqs. (2.2) in the text.

two angular-momentum eigenstates $J=S\pm\frac{1}{2}$:

$$\Gamma^{(+)}(\mathbf{p}',\mathbf{p};\epsilon,T)=\Gamma^{(S)}(\mathbf{p}',\mathbf{p};\epsilon,T)+S\Gamma^{(V)}(\mathbf{p}',\mathbf{p};\epsilon,T), \tag{3.6a}$$

$$\Gamma^{(-)}(\mathbf{p}',\mathbf{p};\epsilon,T)=\Gamma^{(S)}(\mathbf{p}',\mathbf{p};\epsilon,T) -(S+1)\Gamma^{(V)}(\mathbf{p}',\mathbf{p};\epsilon,T), \tag{3.6b}$$

$$J_{\pm}(\mathbf{p}',\mathbf{p})=J_{\pm}g(p')g(p), \tag{3.7a}$$

$$J_{+}=J_1+SJ_2, \tag{3.7b}$$

$$J_{-}=J_1-(S+1)J_2. \tag{3.7c}$$

From Eqs. (3.6) and the results of Appendix A, we obtain

$$\Gamma^{(\pm)}(p',p;\epsilon,T)=g(p')g(p)N_{\pm}(\epsilon;T)/D(\epsilon;T), \tag{3.8a}$$

$$D(\epsilon;T)=[1-J_{-}F(\epsilon)][1-J_{+}F(\epsilon)] +J_2[G(\epsilon;T)-F(\epsilon)], \tag{3.8b}$$

$$N_{\pm}(\epsilon;T)=J_{\pm}[1-J_{\mp}F(\epsilon)]+J_1J_2[G(\epsilon;T) -F(\epsilon)]. \tag{3.8c}$$

From Eqs. (3.5) and (3.8) we see that even when $T=0$, $G(\epsilon;0)\neq F(\epsilon)$, so that the vertex functions do not reduce to the "potential-scattering" vertices

$$\Gamma_{\rm pot}^{(\pm)}(p',p;\epsilon)=J_{\pm}g(p')g(p)[1-J_{\pm}F(\epsilon)]^{-1} \tag{3.9}$$

which would have been obtained if Eqs. (2.7) were uncoupled by the transformation (3.6) to the representation in which the angular momentum of the coupled impurity-spin-plus-electron system is diagonal. The physical origin of this result lies in the fact that the s-d interaction polarizes the ground state of the Fermi gas in addition to scattering an "extra" electron above the Fermi energy. This fact is reflected in the linear equation by the consequence that already in the second-order term for Γ there is a contribution resulting from an intermediate hole state due to the excitation of a particle-hole state in the Fermi gas by the s-d interaction. The presence of both single and multiple particle-hole correlations in a trial ground-state wave function has been shown by Yosida[8] and by Okiji[9] to stabilize a singlet electron bound state for $S=\frac{1}{2}$ and more complex states for larger values of S, when $J_1=0$, $J_2<0$.

The presence of these correlations in the effective ground-state wave function of the Fermi gas also signals the failure of single-channel, elastic, energy-shell unitarity between the "extra" electron and the impurity potential. The correlations give rise to additional channels of specified singular momentum J, in which the extra electron, particle-hole excitations, and impurity spin are all coupled to the desired value of J. Algebraically, the sign change of the $\tanh(\xi_q/2T)$ term in Eq. (2.7b) for $\xi_q<0$ causes Eqs. (2.7) not to separate into uncoupled equations in the $J=S\pm\frac{1}{2}$ channels and thereby the failure of single-channel elastic unitarity. If we considered only intermediate states $\xi_q>0$, single-channel elastic unitarity would be restored. The failure of single-channel elastic unitarity is therefore basically a consequence of the use, in a scattering problem, of the Fermi-gas model to calculate the ground-state wave function of the "initial" impurity-electron-gas system. The utilization of a more accurate wave function which incorporates the s-d induced correlations might restore an appropriate elastic unitarity condition.

The Kondo-Suhl instability,[5] thought to be related to the occurrence of "singlet" bound states as discussed by Yosida[8,11] and Okiji,[9] is associated in the linear theory with the failure of single-channel inelastic unitarity (in the optical-model sense) as well as single-channel elastic unitarity for antiferromagnetic s-d interactions ($J_2<0$). The restriction of single-channel energy-shell unitarity on the vertex functions (3.8) can be derived by identifying their energy-shell values with the conventional s-wave scattering amplitude, $f_0(p)$, via

$$\Gamma^{(\pm)}(p,p;\xi_p,T)\leftrightarrow 2\pi\hbar^2m^{-1}f_0^{(\pm)}(p). \tag{3.10}$$

Then inelastic, single-channel unitarity for these energy-shell vertex functions is given by[12]

$$\mathrm{Im}[1/\Gamma^{(\pm)}(p,p;\xi_p,T]\leq-\pi\rho(\xi_p), \tag{3.11a}$$

$$\rho(\xi_p)=mp/2\pi^2\hbar^2. \tag{3.11b}$$

From Eqs. (3.8) and (3.11), any given combination of parameters and form-factors can be examined to see if elastic [associated with the equality in (3.11a)] or inelastic single-channel unitarity is satisfied. We consider in detail the case of s-d scattering alone;

$$J_1=0, \qquad J_{+}=SJ_2, \qquad J_{-}=-(S+1)J_2, \tag{3.12}$$

for which Eqs. (3.8) give

$$\mathrm{Im}[1/\Gamma^{(\pm)}(p,p;\xi_p,T)]=-\pi\rho(\xi_p)[1-A_{\pm}(\xi_p,T)], \tag{3.13a}$$

$$A_{\pm}(\xi_p,T)=\frac{J_2}{g^2(p)J_{\pm}}\,\mathrm{Im}\,\frac{G(\xi_p,T)-F(\xi_p)}{1-J_{\mp}F(\xi_p)}. \tag{3.13b}$$

Elastic unitarity is satisfied only in the potential-scattering limit that $G\equiv F$. The inelastic unitarity restric-

[11] K. Yosida I.S.S.P. Report No. A208, Tokyo, 1966 (unpublished); Progr. Theoret. Phys. (Kyoto) 36, 875 (1966).

[12] See, e.g., L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1966), 2nd ed., p. 542.

tion is given by

$$A_{\pm}(\xi_p, T) \leq 0. \tag{3.13c}$$

After some algebra we find

$$A_{+}(\xi_p, T) = \gamma_{+}(\xi_p, T)/S \mid 1+(S+1)J_2F(\xi_p) \mid^2; \tag{3.14a}$$

$$A_{-}(\xi_p, T) = \gamma_{-}(\xi_p, T)/(S+1) \mid 1-SJ_2F(\xi_p) \mid^2; \tag{3.14b}$$

$$\gamma_{+}(\xi_p, T) = \tanh(\xi_p/2T) - 1 - (S+1)J_2[\mathrm{Re}G(\xi_p, T) - \tanh(\xi_p/2T)\,\mathrm{Re}F(\xi_p)]; \tag{3.14c}$$

$$\gamma_{-}(\xi_p, \mathrm{T}) = 1 - \tanh(\xi_p/2T) - SJ_2[\mathrm{Re}G(\xi_p, T) - \tanh(\xi_p/2T)\,\mathrm{Re}F(\xi_p)]. \tag{3.14d}$$

Therefore, at $T=0$, the unitarity condition (3.13c) becomes

$$\gamma_{\pm}(\xi_p, 0) \leq 0. \tag{3.15}$$

For $\xi_p > 0$ both requirements (3.15) reduce to

$$J_2\mathrm{Re}[\mathrm{G}(\xi_p, 0) - F(\xi_p)] \geq 0; \ \xi_p > 0. \tag{3.16}$$

From Eqs. (3.5b) and (3.5c), we find that at $T=0$, $\xi_p > 0$,

$$\mathrm{Re}[G(\xi_p, 0) - F(\xi_p)] = 2P\int_{-\epsilon_F}^{\infty} \frac{\rho(x)g^2(x)\,dx}{\xi_p - x} \geq 0. \tag{3.17}$$

Thus for ferromagnetic interactions ($J_2 > 0$) inelastic single-channel unitarity is identically satisfied, whereas for antiferromagnetic interactions ($J_2 < 0$) it can never be satisfied for incident particles above the Fermi energy ($\xi_p > 0$). These results are consistent with those of Yosida[8,11] and Okiji[9] that the inclusion of particle-hole correlations in the ground-state wave function of the impurity plus Fermi gas stabilizes "singlet" bound states for antiferromagnetic s-d interactions but destabilizes "triplet" bound states for ferromagnetic interactions. The results provide a counterexample to hypothesis[13] that for antiferromagnetic s-d interactions a scattering-theory analysis of the calculation of the self-energy automatically leads to well-defined cross sections which at low temperatures attain the maximum value permitted by elastic unitarity.

B. Results for the Effective-Range Potential

In order to construct a linear-analog to Suhl's pole-approximation solution to the nonlinear ("Low") equations, we utilize the "effective-range" potential[14] in the equal form-factor limit

$$g(p) = (\lambda E_R)^{1/2}[\xi_p + \epsilon_F + E_R]^{-1/2}. \tag{3.18}$$

In the limit that $J_2 = 0$, Eqs. (2.7) reduce to the Schrödinger T-matrix equation with the well-known solution[15]

$$\Gamma^{(\pm)}(p, p'; \epsilon) = J_1 g(p) g(p')[1 - J_1F(\epsilon)]^{-1}. \tag{3.19}$$

In the absence of bound states the potential-scattering s-wave Low equation for the energy-shell vertex is given by[5]

$$\Gamma^{(\pm)}(\mathbf{p}', \mathbf{p}; \epsilon) = \tilde{J}_1(\mathbf{p}, \mathbf{p}; \epsilon) + \int_{-\epsilon_F}^{\infty} \rho(\xi_p)\,d\xi_p \times \frac{\mid \Gamma^{(\pm)}(\mathbf{p}', \mathbf{p}; \xi_p) \mid^2}{\xi_p - \epsilon}, \tag{3.20}$$

where the Γ inside the integral is defined approaching the cut on the real ϵ axis from above, and ξ_p is the energy associated with the (equal-magnitude) vectors $\mathbf{p}$ and $\mathbf{p}'$. $\tilde{J}_1$ is the effective inhomogeneous term which includes not only the s-wave component of the Born term, but also contributions from additional cuts in the full s-wave amplitude Γ.[5,7,16] Suhl approximates Eq. (3.20) and its full s-d analog by the standard replacement of $\tilde{J}$ with a single pole on the negative $(\epsilon + \epsilon_F)$ axis:

$$\tilde{J}_1(p', p; \epsilon) = J_1 \lambda E_R(\epsilon + \epsilon_F + E_R)^{-1}. \tag{3.21}$$

A solution to the resulting equation is well-known to be[5,6,15]

$$\Gamma(\epsilon) = [J_1 \lambda E_R/(\epsilon + \epsilon_F + E_R)][1 - J_1F_L(\epsilon)]^{-1}, \tag{3.22a}$$

$$F_L(\epsilon) = (\epsilon + \epsilon_F + E_R)\lambda \int_{-\epsilon_F}^{\infty} \frac{\rho(x)\,dx}{(x + \epsilon_F + E_R)^2(x - \epsilon)}. \tag{3.22b}$$

As on the energy shell $g(p)g(p') = \lambda E_R(\epsilon + \epsilon_F + E_R)^{-1}$, we see that (3.19) and (3.22) differ (on the energy shell) only in that $F_L(\epsilon) \neq F(\epsilon)$. This difference arises from the fact that the solution (3.19) to the linear equation would lead to a renormalized residue at the "Born-pole" specified by (3.21). Therefore, although Eq. (3.20) is not rigorously valid for a nonlocal interaction[16] (of which the separable-potential model is a particularly simple limiting case), the effective-range potential (3.18) does provide a vertex-function solution to the linear equations (2.7) which, in the absence of s-d scattering, has an energy-shell analytic structure identical to that of (3.22) provided J_1 is sufficiently small that there exist no negative real values ϵ_0 such that

$$1 - J_1F(\epsilon_0) = 0 \tag{3.23}$$

is satisfied. A more detailed comparison between the linear (Schrödinger) and unitarized nonlinear formulations of energy-shell scattering theory has been given by Kantor.[7] Among his results is a demonstration that for a local exponential potential for which (3.20) is

[13] J. R. Schrieffer, J. Appl. Phys. **38**, 1143 (1967).
[14] L. Verlet and J. Gavoret, Nuovo Cimento **10**, 505 (1958).
[15] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).
[16] See, e.g., M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964), p. 598.

known to be valid, the solutions to (3.20) differ substantially from those of the Schrödinger equation.

We can now utilize the form factor (3.18) in the full solutions (3.8), which incorporate the *s-d* scattering, and compare our results to Suhl's solutions to the pole-approximation coupled nonlinear equations. As discussed in Sec. 2, the linear vertex function may exhibit complex poles in the upper-half ϵ plane which are taken as indications of a breakdown of perturbation theory. Unfortunately, even for the simple potential Eq. (3.18) the quantity $G(\epsilon; T)$ cannot be evaluated analytically. One may use one of three tractable methods for studying the roots of

$$D(\epsilon, T)=0 \tag{3.24}$$

for D given by (3.8b): (1) numerical evaluation of G; (2) expansion of $\tanh(\xi/2T)$ about $T=0$; or (3) use of the approximate form[17] of the Fermi factors

$$\begin{aligned}\tanh(x/2T)&=-1; \qquad x<-2T\\ &=x/2T; \qquad -2T<x<2T\\ &=1; \qquad x>2T.\end{aligned} \tag{3.25}$$

Evidently (3.25) will reproduce the exact results in the $T\to 0$ limit. We have utilized both methods (2) and (3) but, for convenience, we henceforth discuss only (3). Using (3.25), taking $2T\ll\epsilon_F$ (which must be the case for (3.25) to be useful), and extracting $\rho(x)\cong\rho\equiv mp_F/2\pi^2\hbar^2$ from inside the integrals (3.5), we find

$$F(\epsilon)=\frac{\lambda E_R\rho}{\epsilon+\epsilon_F+E_R}\left\{\ln\left[\frac{E_R}{\epsilon_F+\epsilon}\right]+i\pi\right\}, \tag{3.26a}$$

$$\begin{aligned}G(\epsilon; T)=&\frac{\lambda E_R\rho}{\epsilon+\epsilon_F+E_R}\left\{\ln\left[\frac{(\epsilon_F+E_R)^2}{(\epsilon+2T)(\epsilon-2T)}\right]\right.\\ &\left.+\frac{\epsilon}{2T}\ln\left[\frac{\epsilon-2T}{\epsilon+2T}\right]+\ln\left[\frac{\epsilon+\epsilon_F}{E_R}\right]+2+i\pi\right\},\end{aligned} \tag{3.26b}$$

in which all cuts of $\ln(z)$ are taken along the positive z axis and $\ln(x+i\delta)$ is real in the $\delta\to 0^+$ limit.

Selecting $J_1=0$ reduces (3.8b) to

$$D_{J1=0}(\epsilon; T)=1+J_2G(\epsilon; T)-S(S+1)J_2^2F^2(\epsilon). \tag{3.27}$$

If we neglect the $J_2^2F^2$ term as small relative to J_2G and take the $T\to 0$ limit in (3.26) we find

$$\begin{aligned}\lim_{T\to 0}G(\epsilon; T)=&\frac{E_R\lambda\rho}{\epsilon+\epsilon_F+E_R}\left\{\ln\left[\frac{\epsilon_F+E_R}{\epsilon}\right]^2\right.\\ &\left.+\ln\left[\frac{\epsilon+\epsilon_F}{E_R}\right]+i\pi\right\},\end{aligned} \tag{3.28a}$$

$$\lim_{\substack{\epsilon\to 0\\ E_R\gg\epsilon_F}}\lim_{T\to 0}G(\epsilon; T)=\rho\lambda[\ln(E_R/\epsilon)^2+i\pi+\ln(\epsilon_F/E_R)], \tag{3.28b}$$

$$D_{J_1=0}(\epsilon, T)=0\Rightarrow\epsilon=\epsilon_0e^{i\pi/2}; \qquad J_2<0, \tag{3.29a}$$

$$\epsilon_0=E_R\exp[-1/(2\rho\lambda\,|\,J_2\,|)]. \tag{3.29b}$$

Therefore, neglecting the constant term of Eq. (3.28b) in the limit of zero temperature and a long-range potential, we recover the Kondo-Suhl instability provided that

$$S(S+1)(J_2\lambda\rho)^2\ln^2(E_R/\epsilon_F)\ll 1, \tag{3.30}$$

which for a given E_R will be satisfied for sufficiently small $|J_2|$. Finite temperature corrections lead to $\epsilon_0(T)$ being the solution to

$$\ln[E_R/\epsilon_0(T)]=(2\rho\lambda\,|\,J_2\,|)^{-1}+2T^2/3\epsilon_0^2(T)$$

if (3.30) is satisfied. Thus, $\epsilon_0(T)<\epsilon_0(0)$ and the zero of $D(\epsilon; T)$ moves toward the origin, eventually disappearing at a (Kondo-Suhl) temperature $T_K<\epsilon_0(0)/2$. The important aspect of these results is the persistence of the pole in the (off-energy-shell or energy-shell) linear vertex functions $\Gamma^{(S)}$ and $\Gamma^{(V)}$, despite the use of the coupled equations (2.7), for sufficiently small $|J_2|$ when $J_2<0$.

The introduction of a nonzero J_1 and smaller values of E_R serve to shift the $T=0$ pole in the complex ϵ plane and for some parameter combinations eliminate its occurrence in the upper half ϵ-plane altogether. However, Eqs. (2.7), as well as those used by Suhl and derived in I, are only valid within logarithmic accuracy. The different analytic properties of the various energy-shell vertex functions [the solutions to the non linear equations manifestly exhibit no poles in the upper-half ϵ-plane] are due to the less-divergent terms in perturbation theory which the two vertex-functions *approximate* differently. As the self-energy calculated with either vertex function differs from the exact results for these less divergent terms, we conclude that the use of either the linear or nonlinear diagram summation technique for temperatures near or below the appropriate Kondo-Suhl temperature T_K cannot be justified if the technique is restricted to logarithmic accuracy.

Note added in proof: Kondo[18] has recently applied a self-consistent adaptation of Brillouin-Wigner perturbation theory to calculate a shift in the ground-state energy of a dilute alloy by $a(T)$. Using method (2) discussed in the text, we find that $\epsilon_0(T)$ is identical with Kondo's $a(T)$ for a contact *s-d* interaction. This result suggests that his hypotheses of logarithmic accuracy and the geometrical character of certain series correspond directly to the diagram summation procedure of the linear theory. It is evidently much easier to find $\epsilon_0(T)$ from the linear theory using a general electron-impurity interaction than it is to calculate $a(T)$ using Kondo's method.

[17] J. Karlovsky, Phys. Rev. **127**, 419 (1962).

[18] J. Kondo, Phys. Rev. **154**, 644 (1967); (to be published).

APPENDIX A: SOLUTION TO THE LINEAR EQUATIONS FOR A GENERAL S-WAVE FACTORIZABLE INTERACTION

In terms of the M_{ij}, F_{ij}, and G_{ij} given in Eqs. (3.2)–(3.4) in the text, the linear Eqs. (2.7) become

$$\Gamma^{(S)}(\epsilon;\, p,\, p') = J_1 g_1(p)\, g_1(p') + J_1 g_1(p) M_{11}(\epsilon;\, p') + S(S+1) J_2 g_2(p) M_{22}(\epsilon;\, p'), \tag{A1a}$$

$$\Gamma^{(V)}(\epsilon;\, p,\, p') = J_2 g_2(p)\, g_2(p') + J_1 g_1(p) M_{12}(\epsilon;\, p') + J_2 g_2(p) M_{21}(\epsilon;\, p') - J_2 g_2(p) \int \frac{d^3q}{(2\pi)^3}\, g_2(q)\, \frac{\tanh(\xi_q/2T)\, \Gamma^{(V)}(\epsilon;\, q,\, p')}{\xi_q - \epsilon}. \tag{A1b}$$

The integral in the last equation is eliminated by multiplying Eq. (A1b) by $g_2(p)\, \tanh(\xi_p/2T) \times (\xi_p - \epsilon)^{-1}$, integrating over d^3p and solving for the integral. We get

$$\Gamma^{(V)}(\epsilon;\, p,\, p') = [1 + J_2 G_{22}(\epsilon;\, T)]^{-1} \{ J_2 g_2(p)\, g_2(p') + J_2 g_2(p) M_{21}(\epsilon;\, p') + M_{12}(\epsilon;\, p') [J_1 g_1(p) + J_1 J_2 (g_1(p) G_{22}(\epsilon;\, T) - g_2(p) G_{21}(\epsilon;\, T))] \}. \tag{A1c}$$

The last term in (A1c) is an example of a cross term which disappears in the case of equal form factors for the scalar and vector bare vertices.

Multiplying Eqs. (A1a) and (A1c) by $g_i(p)\, (\xi_p - \epsilon)^{-1}$ and integrating over d^3p we obtain a set of four coupled algebraic equations for the $M_{ij}(\epsilon;\, p')$ which can be solved by standard methods. The solutions are given by (the T dependence is now explicitly indicated)

$$M_{ij}(\epsilon;\, T;\, p) = N_{ij}(\epsilon;\, T;\, p)/D(\epsilon;\, T), \tag{A2}$$

$$D(\epsilon;\, T) = [1 - J_1 F_{11}(\epsilon)][1 - J_1 F_{11}(\epsilon) + J_2 G_{22}(\epsilon;\, T) - J_1 J_2 \Delta_{FG}] - S(S+1) J_2^2 (F_{22}(\epsilon) - J_1 \Delta_{FF})^2, \tag{A3}$$

$$\Delta_{FG} = F_{11}(\epsilon) G_{22}(\epsilon;\, T) - F_{12}(\epsilon) G_{21}(\epsilon;\, T), \tag{A4a}$$

$$\Delta_{FF} = F_{11}(\epsilon) F_{22}(\epsilon) - F_{12}(\epsilon) F_{21}(\epsilon), \tag{A4b}$$

$$N_{11}(\epsilon;\, T;\, p) = J_1 g_1(p) F_{11}(\epsilon)[1 + J_2 G_{22}(\epsilon;\, T) - J_1 F_{11}(\epsilon) - J_1 J_2 \Delta_{FG}] - S(S+1)\, (F_{22}(\epsilon) - J_1 \Delta_{FF}) \times [J_1 J_2^2 g_1(p) \Delta_{FF} - J_2^2 g_2(p) F_{12}(\epsilon)], \tag{A5a}$$

$$N_{22}(\epsilon;\, T;\, p) = J_2 (F_{22}(\epsilon) - J_1 \Delta_{FF}) \times \{ g_2(p) + J_1 [F_{21}(\epsilon)\, g_1(p) - F_{11}(\epsilon)\, g_2(p)] \}, \tag{A5b}$$

$$N_{12}(\epsilon;\, T;\, p) = J_2 F_{12}(\epsilon) \{ g_2(p) + J_1 [g_1(p) F_{21}(\epsilon) - F_{11}(\epsilon)\, g_2(p)] \}, \tag{A5c}$$

$$N_{21}(\epsilon;\, T;\, p) = S(S+1) J_2^2 g_2(p)\, (F_{22}(\epsilon) - J_1 \Delta_{FF})^2 + J_1 g_1(p) F_{21}(\epsilon)[1 + J_2 G_{22}(\epsilon;\, T) - J_1 F_{11}(\epsilon) - J_1 J_2 \Delta_{FG}]. \tag{A5d}$$

The vertex functions are obtained by inserting the results (A2) into Eqs. (A1a) and (A1c):

$$\Gamma^{(S)}(\epsilon;\, T;\, p,\, p') = g_1(p)\, g_1(p')\, A_1(S)(\epsilon;\, T) + [g_1(p) g_2(p') + g_2(p) g_1(p')] A_2(S)(\epsilon);\, T) + g_2(p) g_2(p') A_3(S)(\epsilon;\, T), \tag{A6a}$$

$$A_1^{(S)}(\epsilon;\, T) = D^{-1}(\epsilon;\, T) J_1 \{ 1 - J_1 F_{11}(\epsilon) + J_2 G_{22}(\epsilon;\, T) - J_1 J_2 \Delta_{FG} - S(S+1) J_2^2 (F_{22}(\epsilon) - J_1 \Delta_{FF}) F_{22}(\epsilon) \}, \tag{A6b}$$

$$A_2^{(S)}(\epsilon;\, T) = D^{-1}(\epsilon;\, T) J_1 J_2^2 S(S+1) F_{12}(\epsilon)[F_{22}(\epsilon) - J_1 \Delta_{FF}], \tag{A6c}$$

$$A_3^{(S)}(\epsilon;\, T) = D^{-1}(\epsilon;\, T)\, S(S+1) J_2^2 [1 - J_1 F_{11}(\epsilon)][F_{22}(\epsilon) - J_1 \Delta_{FF}], \tag{A6d}$$

$$\Gamma^{(V)}(\epsilon;\, T;\, p,\, p') = g_2(p)\, g_2(p')\, A_1(V)(\epsilon;\, T) + [g_1(p) g_2(p') + g_2(p) g_1(p')] A_2(V)(\epsilon;\, T) + g_1(p)\, g_1(p')\, A_3(V)(\epsilon;\, T), \tag{A7a}$$

$$A_1^{(V)}(\epsilon;\, T) = D^{-1}(\epsilon;\, T) J_2 [1 - J_1 F_{11}(\epsilon)]^2, \tag{A7b}$$

$$A_2^{(V)}(\epsilon;\, T) = D^{-1}(\epsilon;\, T) J_1 J_2 [1 - J_1 F_{11}(\epsilon)] F_{12}(\epsilon), \tag{A7c}$$

$$A_3^{(V)}(\epsilon;\, T) = J_1^2 J_2 F_{21}(\epsilon) F_{12}(\epsilon) D^{-1}(\epsilon;\, T). \tag{A7d}$$

The equal-form-factor limits discussed in the text are given by taking the $g_1(k) = g_2(k) \equiv g(k)$ limit to obtain

$$\Gamma^{(S)}(\epsilon;\, T;\, p,\, p') = D^{-1}(\epsilon;\, T)\, g(p)\, g(p')\, N^{(S)}(\epsilon;\, T), \tag{A8a}$$

$$\Gamma^{(V)}(\epsilon;\, T;\, p,\, p') = D^{-1}(\epsilon;\, T) J_2 g(p)\, g(p'), \tag{A8b}$$

$$D(\epsilon;\, T) = [1 - J_1 F(\epsilon)][1 - J_1 F(\epsilon) + J_2 G(\epsilon;\, T)] - S(S+1) J_2^2 F^2(\epsilon), \tag{A8c}$$

$$N^{(S)}(\epsilon;\, T) = J_1 [1 + J_2 G(\epsilon)] + F(\epsilon)[S(S+1) J_2^2 - J_1^2], \tag{A8d}$$

$$F(\epsilon) = \int \frac{d^3q}{(2\pi)^3} \frac{g^2(q)}{\xi_q - \epsilon}, \tag{A8e}$$

$$G(\epsilon;\, T) = \int \frac{d^3q}{(2\pi)^3} \frac{g^2(q)\, \tanh(\xi_q/2T)}{\xi_q - \epsilon}. \tag{A8f}$$