Quantum Theory of a Simple Maser Oscillator

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We discuss the quantum theory of a simple model of a maser oscillator, consisting of one radiation-field mode interacting with a large number of stationary three-level atoms. The field and the atoms also interact with separate heat reservoirs which represent dissipation mechanisms and an incoherent pumping mechanism. The model is sufficiently simple that some analytical progress can be made with the nonlinear quantal equations before further approximation is necessary. We start from the quantal equation of motion of the field-atom density operator. We make immediate use of a diagonal coherent-state expansion for the field part of the density operator and a somewhat similar expansion for the atom part. This yields an exact equation of the Fokker-Planck form for a c-number weight distribution, which retains all the significance of the original operator equation, and which has the semiclassical equation for the same model as a first, fluctuation-free approximation. We make use of our basic Fokker-Planck equation in a variety of ways. We discuss the reduction of the equation under conditions that the atomic decay constants are large (large atomic linewidth), arriving finally at an equation of motion for a field-only weight function which serves to demonstrate the basic coherence properties of a maser. We derive and discuss the equation of motion of the generalized Wigner density for the maser model. The generalized Wigner density is a smoothed version (a convolution) of our basic weight distribution, and from it we derive an equivalent classical model including noise sources. Finally, we discuss other useful weight distributions and the number representation for the field. The equations we derive in these discussions make contact with the rate equations of Shimoda, Takahasi, and Townes, as well as with the more recent work of Lax and Louisell, Lax, and of Scully and Lamb.

I. INTRODUCTION

THE theory of a maser oscillator is an interesting meeting ground for quantal and classical physics. A simple maser represents possibly the most elementary quantal problem involving many particles which obey nonlinear equations. In addition, of course, the maser (laser) is a useful device which is the basis of quantum-electronics research. Hence, it is important to gain as much insight as possible into the behavior of this device, for example, to understand its inherent quantal fluctuations.

Since masers operating at levels near or above the threshold of oscillation normally involve many atoms and many photons, one might expect that there should exist a close equivalent classical model. A tuned circuit oscillator involving a saturable negative resistance is such an equivalent, and part of the effort of this work is to demonstrate precisely the nature of the equivalence. Our equivalent classical model contains appropriate sources of fluctuations (noise).

We discuss a simple model of a maser oscillator, consisting of one radiation field mode interacting with a large number of stationary three-level atoms. The field and the atoms also interact with separate heat reservoirs which represent dissipation mechanisms and an incoherent pumping mechanism. The model is sufficiently simple that some analytical progress can be made with the nonlinear quantal equations before further approximation is necessary.

A considerable amount of progress has already been made in other work on fully quantal treatments of the maser, wherein the radiation field is quantized as well as the atoms. Our model is the same as that of Lax,[1] who has approached the problem by deriving and then working from a set of Langevin-type operator equations involving noncommuting random forces. Lax and Louisell[2] and later Lax[3] have extended this treatment to obtain equations of motion of an "associated classical function" (c-number function) which represents a certain (antinormal) ordering of the field density operator. Haken[4] and his co-workers have pursued more or less the same approach, generally treating more complicated models with many field modes, moving atoms, etc. Scully and Lamb[5] have taken a different and independent approach starting from a model nearly equivalent to Lax's. They treat the case of short atomic lifetimes, and work in the photon number (energy) representation for the field. They find the effect on the field density operator of the temporal passage of one atom through the field. They then multiply the result for one atom by the rate of passage of atoms to find the total effect. Two different kinds of atoms represent gain and loss mechanisms.

We have found yet another approach to this problem, which we feel has some advantages over the others. We start (Sec. II) from the quantal equation of motion of the field-atoms density operator. We make immediate use (Sec. III) of a diagonal coherent state expansion for the field part of the density operator and a somewhat similar expansion for the atoms part. This yields an exact equation of the Fokker Planck form for a c-number weight distribution, which retains all the

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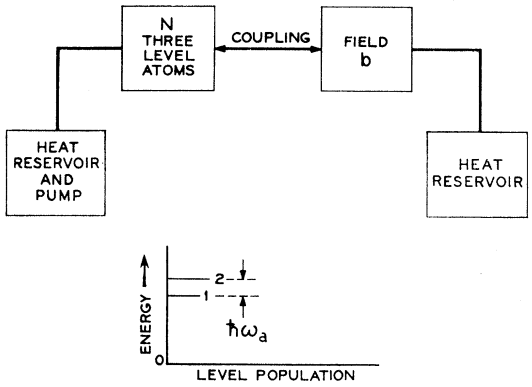

FIG. 1. Block diagram of the maser model.

significance of the original operator equation, and which has the semiclassical equation for the same model as a first, fluctuation free, approximation. The importance of coherent state (minimum uncertainty wave packet) representations has been emphasized by Glauber,[6] who has used them to gain understanding of coherence properties of quantized fields. Further understanding of diagonal coherent state representations has been afforded by the work of Sudarshan[7,8] and Klauder.[8,9]

In the remainder of the paper we make use of our basic Fokker–Planck equation in a variety of ways. In Sec. IV we discuss the reduction of the equation under conditions that the atomic decay constants are large (large atomic linewidth), arriving finally at an equation of motion for a field-only weight distribution which serves to demonstrate the basic coherence properties of a maser. In Sec. V, we derive and discuss the equation of motion of what we call the generalized Wigner density for the maser model. The generalized Wigner density is a smoothed version (a convolution) of our basic weight distribution, and from the diffusion approximation to its equation of motion we derive an equivalent classical model, including noise sources. Finally, in Sec. VI we discuss other useful weight distributions, and the number representation for the field. The equations we derive in these discussions make contact with the rate equations of Shimoda, Takahasi, and Townes[10] as well as with the more recent work of Lax and Louisell,[2] Lax,[3] and Scully and Lamb.[5]

II. THE MODEL AND EQUATION OF MOTION

The model describing the simple maser we consider here is illustrated in Fig. 1. It is identical with one used by Lax.[1] A resonator contains a single important radiation field mode of frequency ω_b described by the uncoupled Hamiltonian

$$\mathbf{H}_b=\hbar\omega_b\mathbf{b}^\dagger\mathbf{b}, \tag{2.1}$$

where $\mathbf{b}$ is the usual annihilation operator and $\mathbf{b}^\dagger$ its adjoint creation operator. The photon number operator $\mathbf{b}^\dagger\mathbf{b}$ has the positive integers and zero as eigenvalues. In contact with the resonator field are a large number N of atoms having three important nondegenerate energy levels labeled 0, 1, 2 in order of increasing energy. The uncoupled Hamiltonian describing the atoms is

$$\mathbf{H}_a=\sum_{j=0}^{2}\epsilon_j\mathbf{N}_j, \tag{2.2}$$

where $\mathbf{N}_j$ is the operator for the number of atoms in the jth level, and ϵ_j is the energy of the jth level. We use the notation of second quantization to describe the atoms, with operators

$$(\mathbf{a}_j)_m \quad \text{and} \quad (\mathbf{a}_j^\dagger)_m$$

being, respectively, annihilation and creation operators for the jth state of the mth atom. The properties of the atomic operators which we shall need are discussed in Appendix A. In terms of these,

$$\mathbf{N}_j=\sum_{m=1}^{N}(\mathbf{a}_j^\dagger\mathbf{a}_j)_m, \tag{2.3}$$

since $(\mathbf{a}_j^\dagger\mathbf{a}_j)_m$ is unity if the mth atom is in the jth state, and zero otherwise.

The coupling between the atoms and the field is described in the rotating wave approximation by the interaction Hamiltonian

$$\mathbf{V}=i\hbar\mu[\mathbf{M}\mathbf{b}^\dagger-\mathbf{M}^\dagger\mathbf{b}], \tag{2.4}$$

where μ is a real coupling constant proportional to the atomic dipole moment associated with the $1\leftrightarrow2$ transition, and

$$\mathbf{M}=\sum_{m=1}^{N}(\mathbf{a}_1^\dagger\mathbf{a}_2)_m,$$

$$\mathbf{M}^\dagger=\sum_{m=1}^{N}(\mathbf{a}_2^\dagger\mathbf{a}_1)_m. \tag{2.5}$$

The macroscopic polarization of the atoms is proportional to $\mu\mathbf{M}$. Application of a single term of the interaction [expanded according to (2.5)] removes one photon from the field and simultaneously raises one atom from level 1 to level 2, or adds a photon to the field and lowers one atom from level 2 to level 1. Without the rotating wave approximation, terms

$$\mathbf{M}\mathbf{b} \quad \text{and} \quad \mathbf{M}^\dagger\mathbf{b}^\dagger$$

would appear in V. These terms, which do not conserve energy in the individual transitions without a lot of

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help from the heat reservoirs, are ignored here from the beginning. This restricts the discussion to the case of sharp resonance, or high Q.

Finally, the model is completed by allowing both the field and the atoms to interact with separate heat reservoirs. The field reservoir simulates the lossy walls of the resonator and the lossy resonator medium. The atom reservior simulates the maser's energy source (the pump) and the various objects which randomly perturb the atoms; i.e., radiation field modes other than the one singled out here, phonon modes, atomic collision, etc. We specifically do not include direct interactions among the N atoms of the maser, as these would have to be considered in more detail than is implied by a reservoir. Hence no maser atom feels the others except by way of the radiation field.

The effects of the reservoirs on the density operator describing the field and the atoms are not describable in terms of a Hamiltonian. We have taken them from Lax's work,[1] and include them directly in the equation of motion of the density operator.

In the Shrödinger picture, the equation of motion of the density operator describing the combined system of field and N atoms can be written

$$\frac{d\rho}{dt}=-\frac{i}{\hbar}[(\mathbf{H}_a+\mathbf{H}_b),\rho]+\mu[(\mathbf{M}\mathbf{b}^\dagger-\mathbf{M}^\dagger\mathbf{b}),\rho]+\sum_{m=1}^{N}\sum_{i,j=0}^{2}w_{ij}\{(\mathbf{a}_i^\dagger\mathbf{a}_j)_m\rho(\mathbf{a}_j^\dagger\mathbf{a}_i)_m-\tfrac{1}{2}(\mathbf{a}_j^\dagger\mathbf{a}_j)_m\rho-\tfrac{1}{2}\rho(\mathbf{a}_j^\dagger\mathbf{a}_j)_m\}$$
$$-\tfrac{1}{2}\gamma\{[\bar{n}\mathbf{b}\mathbf{b}^\dagger+(\bar{n}+1)\mathbf{b}^\dagger\mathbf{b}]\rho+\rho[\bar{n}\mathbf{b}\mathbf{b}^\dagger+(\bar{n}+1)\mathbf{b}^\dagger\mathbf{b}]\}+\gamma\{(\bar{n}+1)\mathbf{b}\rho\mathbf{b}^\dagger+\bar{n}\mathbf{b}^\dagger\rho\mathbf{b}\}. \quad (2.6)$$

In this equation of motion (2.6), the first two terms result from using

$$\mathbf{H}=\mathbf{H}_a+\mathbf{H}_b+\mathbf{V}$$

in the usual equation of motion for the density operator in the Schrödinger picture; i.e.,

$$i\hbar(d\rho/dt)=[\mathbf{H},\rho], \quad (2.7)$$

where the right side is the symbol for the commutator of $\mathbf{H}$ and ρ. The succeeding terms contain the effects of the reservoirs on the atoms and on the field. The positive term in the curly brackets multiplying w_{ij} when $i\neq j$, gives the increase in the population of level i caused by reservoir-induced transitions from level j, with w_{ij} being the probability per unit time and per atom for such a transition. The negative terms, when $i\neq j$, give the decrease in the population of level j caused by reservoir-induced transitions to level i, and also effect the associated decay of the atomic polarization. The terms with $i=j$ do not involve level transitions, and do not affect the level populations. They correspond to reservoir-induced random-phase shifts, and effect additional decay of the polarization.

Finally, γ is the field decay constant caused by its reservoir. In the n representation for the field, one may see that the negative terms involving γ yield a decay of ρ_{nm} proportional to ρ_{nm} while the positive terms yield an increase in ρ_{nm} proportional to $\rho_{n+1,m+1}$ and $\rho_{n-1,m-1}$. The parameter $\bar{n}$ is the mean number of photons in the field when it is allowed to come to thermal equilibrium with its reservoir in the absence of any maser atoms. That is, $\bar{n}=[\exp(\hbar\omega_b/kT)-1]^{-1}$, where T is the resonator temperature.

These reservoir terms constitute a slightly abbreviated version of the more general results of Lax.[1] We have omitted terms whose only effect is to shift the atomic or field frequencies, assuming that any such shifts are incorporated in the definitions of ω_a and ω_b, respectively.

III. THE ANSATZ AND THE FOKKER-PLANCK EQUATION

We will show that (2.6) is consistent with the following expansion of the density operator ρ, namely,

$$\rho(t)=\int^{(6)}P_1(\beta,\beta^*,\mathfrak{X}_j,t)\sigma(\beta,\beta^*,\mathfrak{X}_j)\,d^{(2)}\beta\,d^{(4)}\mathfrak{X}. \quad (3.1)$$

In (3.1), P_1 is a weight distribution and σ is a simple product density operator for the field and N maser atoms, of the form

$$\sigma(\beta,\beta^*,\mathfrak{X}_j)=|\,\beta\rangle\langle\beta\,|\prod_{m=1}^{N}\sigma_m(\mathfrak{X}_j), \quad (3.2)$$

where $|\,\beta\rangle$ represents a pure coherent state of the field,[6] and

$$\sigma_m(\mathfrak{X}_j)=N^{-1}\{(N-\mathfrak{N}_1-\mathfrak{N}_2)(\mathbf{a}_0^\dagger\mathbf{a}_0)_m$$
$$+\mathfrak{N}_1(\mathbf{a}_1^\dagger\mathbf{a}_1)_m+\mathfrak{N}_2(\mathbf{a}_2^\dagger\mathbf{a}_2)_m$$
$$+\mathfrak{M}(\mathbf{a}_2^\dagger\mathbf{a}_1)_m+\mathfrak{M}^*(\mathbf{a}_1^\dagger\mathbf{a}_2)_m\} \quad (3.3)$$

represents a mixed state of the mth atom. We have shortened the writing of (3.1)–(3.3) by using the notation

$$\mathfrak{X}_j(j=1,2,3,4)\equiv(\mathfrak{N}_1,\mathfrak{N}_2,\mathfrak{M},\mathfrak{M}^*),$$
$$d^{(2)}\beta\equiv d(\mathrm{Re}\beta)\,d(\mathrm{Im}\beta),$$
$$d^{(4)}\mathfrak{X}\equiv d\mathfrak{N}_1 d\mathfrak{N}_2 d(\mathrm{Re}\mathfrak{M})\,d(\mathrm{Im}\mathfrak{M}). \quad (3.4)$$

We have also used a superscript (6) on the integral sign to represent a sixfold integral. To further simplify later writing we shall use

$$\sigma_b(\beta,\beta^*)\equiv|\,\beta\rangle\langle\beta\,|$$

and

$$\sigma_a(\mathfrak{X}_j)\equiv\prod_{m=1}^{N}\sigma_m(\mathfrak{X}_j), \tag{3.5}$$

so that we can write

$$\sigma(\beta,\beta^*,\mathfrak{X}_j)=\sigma_b(\beta,\beta^*)\,\sigma_a(\mathfrak{X}_j). \tag{3.6}$$

Thus, the subscript b pertains to the field, the subscript a pertains to all the atoms, while the subscript m pertains to the mth atom only.

The usefulness of the expansion (3.1) lies in the fact that the "elementary" density operator σ is of product (a symptom of statistical independence) form, and that in σ each atom has an identical description. The expansion is possible because the density operator equation (2.6) is symmetrical among all the atoms.

The motivation for making the expansion (3.1) was the desire to display the relation between semiclassical and fully quantal treatments of masers before any unessential approximations were made. The expansion of the field in the diagonal coherent state representation seemed appropriate, since a coherent state is the closest the uncertainty principle will allow a quantal description of a field to approach the usual classical description. The accompanying expansion for the atoms part of the density operator was arrived at intuitively to satisfy our expectation that each atom would have an identical description, but that statistical correlations among the atoms would remain within the coherent state field expansion because the present state of each atom depends not only on the present state of the field but also on its dynamically determined past history. For an atoms-field system described by $\sigma(\beta,\beta^*,\mathfrak{X}_j)$ the mean value of the complex field is proportional to β, the mean populations of atom levels 1 and 2 are $\mathfrak{N}_1$ and $\mathfrak{N}_2$, respectively, and the mean macroscopic complex polarization is proportional to $\mathfrak{M}$. A complete statistical relationship between moments of field-atom operators and the parameters of σ is given in Appendix C, but we will not need it at this point. In order to ensure that the eigenvalues of σ_m remain non-negative, the ranges of its parameters are restricted by

$$\begin{aligned}&\mathfrak{N}_1,\mathfrak{N}_2\geq 0,\\ &\mathfrak{N}_1+\mathfrak{N}_2\leq N,\\ &\mathfrak{M}\mathfrak{M}^*\leq\mathfrak{N}_1\mathfrak{N}_2.\end{aligned} \tag{3.7}$$

All of the component density operators of σ are normalized to unit trace, as therefore are σ_a and σ itself. If we trace (3.1) over all the atoms, we obtain

$$\rho_b\equiv \mathrm{Tr}_a\rho=\int^{(6)}P_1(\beta,\beta^*,\mathfrak{X}_j,t)\mid\beta\rangle\langle\beta\mid d^{(2)}\beta\,d^{(4)}\mathfrak{X}, \tag{3.8}$$

which is the reduced field density operator in the diagonal coherent state expansion. If we trace (3.1) over everything, and assume that ρ is also normalized to unit trace, we obtain

$$1=\int^{(6)}P_1(\beta,\beta^*,\mathfrak{X}_j,t)\,d^{(2)}\beta\,d^{(4)}\mathfrak{X}.$$

Thus P_1 has the form of a probability density; however, there is no *a priori* reason to suppose or expect that P_1 is everywhere positive. In mathematical terminology, P_1 is properly defined as a "distribution"[8] rather than as a function.

For later use we note that if $\mathbf{O}$ is any function of the quantum operators of the problem, then from (3.1) we may write the expression for the mean value of that operator function as

$$\langle\mathbf{O}\rangle\equiv\mathrm{Tr}(\rho\mathbf{O})=\int^{(6)}P_1[\mathrm{Tr}\sigma\mathbf{O}]d^{(2)}\beta\,d^{(4)}\mathfrak{X}. \tag{3.9}$$

To consolidate notation, let us define the angular brackets with subscript 1 by

$$\langle F\rangle_1\equiv\int^{(6)}P_1F\,d^{(2)}\beta\,d^{(4)}\mathfrak{X}, \tag{3.10}$$

where F may be any function of the variables of P_1. Then we can write (3.9) more briefly as

$$\langle\mathbf{O}\rangle=\langle\mathrm{Tr}(\sigma\mathbf{O})\rangle_1. \tag{3.11}$$

While (3.1) is not the most general atom-field density operator, a sensible model starting condition, i.e., the equilibrium state with V removed, may be so represented. If the starting condition is consistent with (3.1) then so will be the subsequent evolution of the density operator, as we shall show.

We shall confine our attention to situations wherein the vast majority of the atoms are continually in the ground-level 0, so that

$$\mathfrak{N}_1,\mathfrak{N}_2,\mathfrak{M},\mathfrak{M}^*\lll N. \tag{3.12}$$

Thus we treat the ground level as an infinite reservoir of atoms; the number of atoms in the two upper levels remain finite. We now show that upon insertion of (3.1) in the density operator equation (2.6), we obtain an equation of the Fokker–Planck form for the evolution of the probability density P_1. This is the basic equation upon which the remainder of our work is built.

To make use of (3.1), we need to know the effect of the various operators in (2.6) on the elementary density operator $\sigma(\beta,\beta^*,\mathfrak{X}_j)$. Using the identity

$$\sigma_b=\mid\beta\rangle\langle\beta\mid=\exp(-\beta^*\beta)\,\exp(\beta\mathbf{b}^\dagger)\mid 0\rangle\langle 0\mid\exp(\beta^*\mathbf{b}), \tag{3.13}$$

where $\mid 0\rangle$ represents the ground (vacuum) state of the field, we obtain

$$\mathbf{b}^\dagger\sigma_b=(\beta^*+\partial/\partial\beta)\,\sigma_b, \tag{3.14a}$$

$$\mathbf{b}\sigma_b=\beta\sigma_b. \tag{3.14b}$$

The first (3.14a) of these results may be seen by

performing the indicated differentiation[11] on (3.13). The second follows from (3.13) by virtue of the identities

$$[\mathbf{b}, \exp(\beta\mathbf{b}^\dagger)]=\beta\exp(\beta b^\dagger); \qquad \mathbf{b}\,|\,0\rangle=0.$$

Alternatively, it is well known that the coherent state $|\,\beta\rangle$ is an eigenstate of $\mathbf{b}$ with eigenvalue β. The adjoints of (3.14a) and (3.14b) also apply.

Further, we have the identities

$$(a_1^\dagger a_2)_m=N(\partial/\partial\mathfrak{M}^*)\sigma_m, \tag{3.15a}$$

$$(a_j^\dagger a_j)_m=(1+N(\partial/\partial\mathfrak{N}_j))\sigma_m; \qquad j=1,2 \tag{3.15b}$$

along with the adjoint of (3.15a). These follow directly from (3.3); in (3.15b) we have applied the limit (3.12). Using (3.15a) and (3.15b), and the properties of the atomic operators discussed in Appendix A, we find that the macroscopic atomic operators satisfy the relations

$$\mathbf{M}\sigma_a=\{\mathfrak{M}(1+(\partial/\partial\mathfrak{N}_1))+\mathfrak{N}_2(\partial/\partial\mathfrak{M}^*)\}\sigma_a, \tag{3.16a}$$

$$\mathbf{M}^\dagger\sigma_a=\{\mathfrak{M}^*(1+(\partial/\partial\mathfrak{N}_2))+\mathfrak{N}_1(\partial/\partial\mathfrak{M})\}\sigma_a, \tag{3.16b}$$

$$\mathbf{N}_1\sigma_a=\{\mathfrak{N}_1(1+(\partial/\partial\mathfrak{N}_1))+\mathfrak{M}^*(\partial/\partial\mathfrak{M}^*)\}\sigma_a, \tag{3.16c}$$

$$\mathbf{N}_2\sigma_a=\{\mathfrak{N}_2(1+(\partial/\partial\mathfrak{N}_2))+\mathfrak{M}(\partial/\partial\mathfrak{M})\}\sigma_a, \tag{3.16d}$$

along with their adjoints. Using (3.14a), (3.14b), and (3.16a)–(3.16d) we can express the first two terms of the right-hand side of (2.6), with σ replacing ρ, in terms of partial differential operators acting on σ. Using (3.14a), (3.14b), and (3.15a)–(3.15b), we can do the same for the reservoir interaction terms.

When we do all this, we find that insertion of (3.1) into (2.6) yields the following equation:

$$\begin{aligned}\int^{(6)}\frac{\partial P_1}{\partial t}\sigma\,d^{(2)}\beta\,d^{(4)}\mathfrak{X}=\int^{(6)}P_1\Big\{&[\mu\mathfrak{M}-(\tfrac{1}{2}\gamma+i\omega_b)\beta]\frac{\partial}{\partial\beta}+[\mu\mathfrak{M}^*-(\tfrac{1}{2}\gamma-i\omega_b)\beta]\frac{\partial}{\partial\beta^*}\\&+[\mu\beta(\mathfrak{N}_2-\mathfrak{N}_1)-(\Gamma_{12}+i\omega_a)\mathfrak{M}]\frac{\partial}{\partial\mathfrak{M}}+[\mu\beta^*(\mathfrak{N}_2-\mathfrak{N}_1)-(\Gamma_{12}-i\omega_a)\mathfrak{M}^*]\frac{\partial}{\partial\mathfrak{M}^*}\\&+[R_1+w_{12}\mathfrak{N}_2-\Gamma_1\mathfrak{N}_1+\mu(\beta^*\mathfrak{M}+\beta\mathfrak{M}^*)]\frac{\partial}{\partial\mathfrak{N}_1}\\&+[R_2+w_{21}\mathfrak{N}_1-\Gamma_2\mathfrak{N}_2-\mu(\beta^*\mathfrak{M}+\beta\mathfrak{M}^*)]\frac{\partial}{\partial\mathfrak{N}_2}+\bar{n}\gamma\frac{\partial^2}{\partial\beta\partial\beta^*}\\&+\mu\mathfrak{N}_2\left(\frac{\partial^2}{\partial\beta\partial\mathfrak{M}^*}+\frac{\partial^2}{\partial\beta^*\partial\mathfrak{M}}\right)+\mu\mathfrak{M}\frac{\partial^2}{\partial\mathfrak{N}_1\partial\beta}+\mu\mathfrak{M}^*\frac{\partial^2}{\partial\mathfrak{N}_1\partial\beta^*}\Big\}\sigma\,d^{(2)}\beta\,d^{(4)}\mathfrak{X}.\end{aligned} \tag{3.17}$$

In (3.17) the curly bracketed operator operates on σ. Also, $R_j\equiv Nw_{j0}$ is the rate at which atoms are pumped from the ground state to the jth state; $\Gamma_j\equiv\sum_{i\neq j}w_{ij}$ is the decay constant for the population of the jth state, and $\Gamma_{12}\equiv\frac{1}{2}\sum_j(w_{j1}+w_{j2})$ is the decay constant for the polarization. The next step is term by term partial integration of (3.17), bringing all derivatives away from σ. On performing this under the assumption that the surface integrals vanish, we obtain

$$\begin{aligned}\int^{(6)}\frac{\partial P_1}{\partial t}\sigma\,d^{(2)}\beta\,d^{(4)}\mathfrak{X}=\int^{(6)}d^{(2)}\beta\,d^{(4)}\mathfrak{X}\sigma\Big\{&-\frac{\partial}{\partial\beta}[\mu\mathfrak{M}-(\tfrac{1}{2}\gamma+i\omega_b)\beta]-\frac{\partial}{\partial\beta^*}[\mu\mathfrak{M}^*-(\tfrac{1}{2}\gamma-i\omega_b)\beta]\\&-\frac{\partial}{\partial\mathfrak{M}}[\mu\beta(\mathfrak{N}_2-\mathfrak{N}_1)-(\Gamma_{12}+i\omega_a)\mathfrak{M}]-\frac{\partial}{\partial\mathfrak{M}^*}[\mu\beta^*(\mathfrak{N}_2-\mathfrak{N}_1)-(\Gamma_{12}-i\omega_a)\mathfrak{M}^*]\\&-\frac{\partial}{\partial\mathfrak{N}_1}[R_1+w_{12}\mathfrak{N}_2-\Gamma_1\mathfrak{N}_1+\mu(\beta^*\mathfrak{M}+\beta\mathfrak{M}^*)]-\frac{\partial}{\partial\mathfrak{N}_2}[R_2+w_{21}\mathfrak{N}_1-\Gamma_2\mathfrak{N}_2-\mu(\beta^*\mathfrak{M}+\beta\mathfrak{M}^*)]\\&+\frac{\partial^2}{\partial\beta\partial\beta^*}\bar{n}\gamma+\left(\frac{\partial^2}{\partial\beta\partial\mathfrak{M}^*}+\frac{\partial^2}{\partial\beta^*\partial\mathfrak{M}}\right)\mu\mathfrak{N}_2+\frac{\partial^2}{\partial\mathfrak{N}_1\partial\beta}\mu\mathfrak{M}+\frac{\partial^2}{\partial\mathfrak{N}_1\partial\beta^*}\mu\mathfrak{M}^*\Big\}P_1,\end{aligned} \tag{3.18}$$

where the long curly bracketed operator operates on P_1.

[11] Throughout this paper we make use of the simplicity of complex derivative notation. Translation to the real and imaginary parts of the complex variable β are made by the identities.

$$\beta\equiv\mathrm{Re}\beta-i\mathrm{Im}\beta,$$

$$\partial/\partial\beta=\tfrac{1}{2}\{[\partial/\partial(\mathrm{Re}\beta)]+i[\partial/\partial(\mathrm{Im}\beta)]\},$$

and their conjugates. Differentiations of functions of β and β^* with respect to β or β^* proceed as though β and β^* were independent variables, as can be proved by making the above translation, and partial integrations may be performed in the same manner. The area element $d^{(2)}\beta$ in the complex β plane always retains its identity as $d(\mathrm{Re}\beta)d(\mathrm{Im}\beta)$. Similar considerations apply to the complex variable $\mathfrak{M}$.

A sufficient condition for the satisfaction of (3.18) is for P_1 to satisfy the Fokker–Planck equation obtained by extracting the integrand; namely

$$\frac{\partial P_1(\beta,\beta^*,\mathfrak{X}_j,t)}{\partial t}=\Big\{-\frac{\partial}{\partial\beta}[\mu\mathfrak{M}-(\tfrac{1}{2}\gamma+i\omega_b)\beta]-\frac{\partial}{\partial\beta^*}[\mu\mathfrak{M}^*-(\tfrac{1}{2}\gamma-i\omega_b)\beta^*]$$
$$-\frac{\partial}{\partial\mathfrak{M}}[\mu\beta(\mathfrak{N}_2-\mathfrak{N}_1)-(\Gamma_{12}+i\omega_a)\mathfrak{M}]-\frac{\partial}{\partial\mathfrak{M}^*}[\mu\beta^*(\mathfrak{N}_2-\mathfrak{N}_1)-(\Gamma_{12}-i\omega_a)\mathfrak{M}^*]$$
$$-\frac{\partial}{\partial\mathfrak{N}_1}[R_1+w_{12}\mathfrak{N}_2-\Gamma_1\mathfrak{N}_1+\mu(\beta^*\mathfrak{M}+\beta\mathfrak{M}^*)]-\frac{\partial}{\partial\mathfrak{N}_2}[R_2+w_{21}\mathfrak{N}_1-\Gamma_2\mathfrak{N}_2-\mu(\beta^*\mathfrak{M}+\beta\mathfrak{M}^*)]$$
$$+\frac{\partial^2}{\partial\beta\partial\beta^*}\bar{n}\gamma+\Big(\frac{\partial^2}{\partial\beta\partial\mathfrak{M}^*}+\frac{\partial^2}{\partial\beta^*\partial\mathfrak{M}}\Big)\mu\mathfrak{N}_2+\frac{\partial^2}{\partial\mathfrak{N}_1\partial\beta}\mu\mathfrak{M}+\frac{\partial^2}{\partial\mathfrak{N}_1\partial\beta^*}\mu\mathfrak{M}^*\Big\}P_1. \tag{3.19}$$

Equation (3.19) is our basic result. The remainder of this paper consists of an investigation of its properties and utility. Given a solution of (3.19), then $\rho(t)$ as defined by (3.1) satisfies the Schrödinger equation of motion (2.6).

In statistical equations of the Fokker–Planck form, the first-derivative terms on the right-hand side prescribe the mean motions of the variables. They are called drift terms. The second-derivative terms prescribe the fluctuations and are called diffusion terms. In (3.19), each drift coefficient, the square-bracketed term following each first derivative time, is the mean rate of change of the variable in the derivative. To see this mathematically, we can derive from (3.19) the set of equations

$$\frac{d}{dt}\langle\beta\rangle_1=\langle\mu\mathfrak{M}-(\tfrac{1}{2}\gamma+i\omega_b)\beta\rangle_1,$$

$$\frac{d}{dt}\langle\mathfrak{M}\rangle_1=\langle\mu\beta(\mathfrak{N}_2-\mathfrak{N}_1)-(\Gamma_{12}+i\omega_a)\mathfrak{M}\rangle_1,$$

$$\frac{d}{dt}\langle\mathfrak{N}_1\rangle_1=\langle R_1+w_{12}\mathfrak{N}_2-\Gamma_1\mathfrak{N}_1+\mu(\beta^*\mathfrak{M}+\beta\mathfrak{M}^*)\rangle_1,$$

$$\frac{d}{dt}\langle\mathfrak{N}_2\rangle_1=\langle R_2+w_{21}\mathfrak{N}_1-\Gamma_2\mathfrak{N}_2-\mu(\beta^*\mathfrak{M}+\beta\mathfrak{M}^*)\rangle_1, \tag{3.20}$$

along with the conjugates of the first two. We have used here the notation (3.10).

Equations (3.20) are derived by writing out their left sides according to (3.10), inserting (3.19) for $\partial P_1/\partial t$, and then integrating by parts under our usual assumption that all surface integrals vanish.

So long as we can neglect the diffusion terms in (3.19), then delta function solutions[12] to it are possible. The position of any such delta function moves according to (3.20) with the mean value signs removed. Thus, equations (3.20) without the mean value signs may be considered as the set of dynamic equations which describe the maser when fluctuations are neglected, and indeed they are precisely the equations one would derive for this model problem by the methods of semi classical physics, with β playing the role of the classical field strength.

In contrast to the drift terms which, as we have seen, make quite good sense, the diffusion terms of (3.19) have a curious form. If the complex derivatives are re-expressed in real and imaginary parts,[11] the matrix of the diffusion coefficients is mainly off-diagonal and clearly has some negative eigenvalues. If the diffusion terms were diagonalized by a linear change of variables, the resulting equation would have some negative diagonal diffusion coefficients. Such a situation would not arise in the treatment of any sensible problem derived from classical physics.

Partly because of this problem of the diffusion matrix with negative eigenvalues, no solutions to the complex equation (3.19) are presently known. It would seem that a study of such equations is called for, particularly since (3.19), for example, has been derived from a simple physical model which one might reasonably expect to be well behaved.

To observe the effects of the diffusion terms in (3.19), we must evaluate the time derivatives of second moments. For example, in the same way that Eqs. (3.20) were derived, we find from (3.19) that

$$\frac{d}{dt}\langle\beta^*\beta\rangle_1=\langle\mu(\beta^*\mathfrak{M}+\beta\mathfrak{M}^*)+\gamma(\bar{n}-\beta^*\beta)\rangle_1, \tag{3.21}$$

where the $\gamma\bar{n}$ on the right is contributed by the $\partial^2/\partial\beta\partial\beta^*$ diffusion term of (3.19). It is clear that $\gamma\bar{n}$ represents the generation of an incoherent or noise field, since no such term appears in the equation for the mean field. To attach physical significance to (3.21), we may derive from (3.11), (3.14), and (3.16) the relations

$$\langle\mathbf{b}^\dagger\mathbf{b}\rangle=\langle\beta^*\beta\rangle_1, \tag{3.22}$$

$$\mu\langle\mathbf{b}^\dagger\mathbf{M}+\mathbf{b}\mathbf{M}^\dagger\rangle=\mu\langle\beta^*\mathfrak{M}+\beta\mathfrak{M}^*\rangle_1. \tag{3.23}$$

[12] Here we are really speaking of solutions in which the weight P_1 is concentrated in a sufficiently small region that we are interested only in its center of gravity and not in any details of its shape. According to the full equation (3.19) such a situation will persist for a certain time depending on the diffusion coefficients.

Hence $\langle\beta^*\beta\rangle_1$ is the mean number of photons in the field, and we see that $\mu\langle\beta^*\mathfrak{M}+\beta\mathfrak{M}^*\rangle_1$ is the mean rate of transfer of energy from the atoms to the field. Note that this same term (3.23) appears in the mean rate equations (3.20) for $\mathfrak{N}_1$ and $\mathfrak{N}_2$, as it should.

If we now proceed to examine the equation of motion of $\langle\beta^*\mathfrak{M}+\beta\mathfrak{M}^*\rangle_1$, we find positive contributions from the $\partial^2/\partial\beta\partial\mathfrak{M}^*$ and $\partial^2/\partial\beta^*\partial\mathfrak{M}$ diffusion terms, which have the common coefficient $\mu\mathfrak{N}_2$. This represents spontaneous emission, and we see that it affects the field as the result of a two-stage process. The remaining diffusion terms do not seem to have any such immediate physical interpretation.

IV. ADIABATIC ELIMINATION OF THE ATOMIC VARIABLES

Pending the availability of solutions to the full equation (3.19), we can examine its behavior in certain limiting situations. In particular, the equations can be reduced in the case that the atomic decay constants Γ are large compared to the field decay constant. Suppose first that $\Gamma_{12}\gg\Gamma_1$, Γ_2, γ. Then the polarization variable $\mathfrak{M}$ is "instantaneously" (time scale Γ_{12}^{-1}) rather than dynamically determined by the other variables, and may be eliminated from the equation of motion. Such "adiabatic" approximations are commonly useful in physics.

As a first step we remove the high-frequency motion of P_1. In (3.19), β and $\mathfrak{M}$ are tied together in phase, and when conditions for steady oscillation exist, the high-frequency motion of $\langle\beta\rangle_1$ and $\langle\mathfrak{M}\rangle_1$ will be nearly as $\exp(-i\omega_0 t)$, where as is well known, or may be shown directly from the mean motions of the variables (3.20),

$$\omega_0=\omega_b+\alpha(\tfrac{1}{2}\gamma)=\omega_a-\alpha\Gamma_{12}, \tag{4.1}$$

where

$$\alpha\equiv(\omega_a-\omega_b)/(\Gamma_{12}+\tfrac{1}{2}\gamma).$$

Hence it is appropriate to change variables from $\mathfrak{M}$ and β to $\mathfrak{M}'$ and β', where

$$\mathfrak{M}=\mathfrak{M}'\exp(-i\omega_0 t),$$
$$\beta=\beta'\exp(-i\omega_0 t). \tag{4.2}$$

The required transformation of the partial derivatives is

$$\frac{\partial}{\partial\mathfrak{M}}\rightarrow\exp(i\omega_0 t)\frac{\partial}{\partial\mathfrak{M}'},$$

$$\frac{\partial}{\partial\beta}\rightarrow\exp(i\omega_0 t)\frac{\partial}{\partial\beta'},$$

$$\frac{\partial}{\partial t}\rightarrow\frac{\partial}{\partial t}+i\omega_0\left(\frac{\partial}{\partial\mathfrak{M}'}\mathfrak{M}'+\frac{\partial}{\partial\beta'}\beta'\right)$$
$$-i\omega_0\left(\frac{\partial}{\partial\mathfrak{M}'^*}\mathfrak{M}'^*+\frac{\partial}{\partial\beta'^*}\beta'^*\right). \tag{4.3}$$

The conjugates of these equations also apply. Making this change of variables and then omitting the primes for simplicity in writing, we obtain

$$\begin{aligned}\frac{\partial P_1}{\partial t}=\Big\{&-\frac{\partial}{\partial\beta}[\mu\mathfrak{M}-\tfrac{1}{2}\gamma(1-i\alpha)\beta]-\frac{\partial}{\partial\beta^*}[\mu\mathfrak{M}^*-\tfrac{1}{2}\gamma(1+i\alpha)\beta^*]\\&-\frac{\partial}{\partial\mathfrak{M}}[\mu\beta(\mathfrak{N}_2-\mathfrak{N}_1)-\Gamma_{12}(1+i\alpha)\mathfrak{M}]-\frac{\partial}{\partial\mathfrak{M}^*}[\mu\beta^*(\mathfrak{N}_2-\mathfrak{N}_1)-\Gamma_{12}(1-i\alpha)\mathfrak{M}^*]\\&-\frac{\partial}{\partial\mathfrak{N}_1}[R_1+w_{12}\mathfrak{N}_2-\Gamma_1\mathfrak{N}_1+\mu(\beta^*\mathfrak{M}+\beta\mathfrak{M}^*)]-\frac{\partial}{\partial\mathfrak{N}_2}[R_2+w_{21}\mathfrak{N}_1-\Gamma_2\mathfrak{N}_2-\mu(\beta^*\mathfrak{M}+\beta\mathfrak{M}^*)]\\&+\frac{\partial^2}{\partial\beta\partial\beta^*}\bar{n}\gamma+\left(\frac{\partial^2}{\partial\beta\partial\mathfrak{M}^*}+\frac{\partial^2}{\partial\beta^*\partial\mathfrak{M}}\right)\mu\mathfrak{N}_2+\frac{\partial^2}{\partial\mathfrak{N}_1\partial\beta}\mu\mathfrak{M}+\frac{\partial^2}{\partial\mathfrak{N}_1\partial\beta^*}\mu\mathfrak{M}^*\Big\}P_1.\end{aligned} \tag{4.4}$$

The adiabatic elimination of $\mathfrak{M}$ and $\mathfrak{M}^*$ from the problem now involves the assumption that Γ_{12} is large enough so that $\mathfrak{M}$, $\mathfrak{M}^*$ come into statistical equilibrium in a time short compared to that required for any of the other variables to change appreciably. A method of adiabatic elimination of variables from Fokker–Planck equations is discussed in Appendix B. In (4.4), it suffices to pick out the terms involving $\partial/\partial\mathfrak{M}$ and $\partial/\partial\mathfrak{M}^*$ and set them equal to zero. Thus we have

$$\frac{\partial}{\partial\mathfrak{M}}\left\{\mu\beta(\mathfrak{N}_2-\mathfrak{N}_1)-\Gamma_{12}(1+i\alpha)\mathfrak{M}-\frac{\partial}{\partial\beta^*}\mu\mathfrak{N}_2\right\}P_1=0 \tag{4.5}$$

and its conjugate, which are solved by

$$\mathfrak{M}P_1=\frac{\mu}{\Gamma_{12}(1+i\alpha)}\left\{\beta(\mathfrak{N}_2-\mathfrak{N}_1)-\frac{\partial}{\partial\beta^*}\mathfrak{N}_2\right\}P_1 \tag{4.6}$$

and its conjugate. This picking out extracts all terms in (4.4) which are important for time intervals of the order of Γ_{12}^{-1}; the remainder of Eq. (4.4) then holds for time intervals much longer than Γ_{12}^{-1}. We use the adiabatic solution (4.6) to eliminate $\mathfrak{M}$ and $\mathfrak{M}^*$ from (4.4). In accord with the discussion in Appendix B, the variables $\mathfrak{M}$ and $\mathfrak{M}^*$ must be placed to the right of

all other variables when the substitutions (4.6) and its conjugate are made. This last is important since the products $\mathfrak{M}\beta^*$ and $\mathfrak{M}^*\beta$ occur in (4.4), and for example the adiabatic solution for $\mathfrak{M}$ does not commute with β^*. Bringing all the resulting derivatives to the left to re-establish the Fokker–Planck form, and integrating over the $\mathfrak{M}$ plane, we obtain, as the new equation with $\mathfrak{M}$ and $\mathfrak{M}^*$ eliminated,

$$\frac{\partial P_1(\beta,\beta^*,\mathfrak{N}_1,\mathfrak{N}_2,t)}{\partial t}$$
$$=\Big\{-\frac{\partial}{\partial\beta}[\pi(\mathfrak{N}_2-\mathfrak{N}_1)-\gamma](\tfrac{1}{2}(1-i\alpha)\beta)-\frac{\partial}{\partial\beta^*}[\pi(\mathfrak{N}_2-\mathfrak{N}_1)-\gamma](\tfrac{1}{2}(1+i\alpha)\beta^*)$$
$$-\frac{\partial}{\partial\mathfrak{N}_1}[R_1+(w_{12}+\pi+\pi\beta\beta^*)\mathfrak{N}_2-(\Gamma_1+\pi\beta\beta^*)\mathfrak{N}_1]-\frac{\partial}{\partial\mathfrak{N}_2}[R_2-(\Gamma_2+\pi+\pi\beta\beta^*)\mathfrak{N}_2+(w_{21}+\pi\beta\beta^*)\mathfrak{N}_1]$$
$$+\frac{\partial^2}{\partial\beta\partial\beta^*}[\gamma\bar{n}+\pi\mathfrak{N}_2]+\frac{\partial^2}{\partial\mathfrak{N}_1\partial\beta}[\pi\beta\mathfrak{N}_2-\tfrac{1}{2}\pi(1-i\alpha)\beta\mathfrak{N}_1]+\frac{\partial^2}{\partial\mathfrak{N}_1\partial\beta^*}[\pi\beta^*\mathfrak{N}_2-\tfrac{1}{2}\pi(1+i\alpha)\beta^*\mathfrak{N}_1]$$
$$+\frac{\partial^2}{\partial\mathfrak{N}_2\partial\beta}[-\tfrac{1}{2}\pi(1+i\alpha)\beta\mathfrak{N}_2]+\frac{\partial^2}{\partial\mathfrak{N}_2\partial\beta^*}[-\tfrac{1}{2}\pi(1-i\alpha)\beta^*\mathfrak{N}_2]-\frac{\partial^3}{\partial\mathfrak{N}_1\partial\beta\partial\beta^*}[\pi\mathfrak{N}_2]\Big\}P_1, \tag{4.7}$$

where

$$\pi\equiv 2\mu^2/\Gamma_{12}(1+\alpha^2)$$

is the decay constant for spontaneous emission of a photon into the laser mode b by an atom in level 2. By contrast, Γ_2 includes spontaneous emission into all other field modes.

In (4.7) the weight distribution P_1 has been integrated over the $\mathfrak{M}$ plane; that is,

$$P_1(\beta,\beta^*,\mathfrak{N}_1,\mathfrak{N}_2,t)\equiv\int^{(2)}P_1(\beta,\beta^*,\mathfrak{N}_1,\mathfrak{N}_2,\mathfrak{M},\mathfrak{M}^*)\,d^{(2)}\mathfrak{M}.$$

By using (4.6) and its conjugate, any moment of the variables including $\mathfrak{M}$ and $\mathfrak{M}^*$ can be re-expressed in terms of the variables other than $\mathfrak{M}$ and $\mathfrak{M}^*$, so that (4.6) and (4.7) together now describe the complete behavior of the maser.

In (4.7), spontaneous emission into the laser mode has become quite explicit, and appears properly in the drift terms which give the mean motion of the atomic populations, and in the field diffusion term $\partial^2/\partial\beta\partial\beta^*$. The origin of all these spontaneous emission terms can be traced to the $\partial^2/\partial\beta\partial\mathfrak{M}^*$ and $\partial^2/\partial\beta^*\partial\mathfrak{M}$ diffusion terms in (3.19). The group of off-diagonal diffusion terms and the third-derivative term give rise to the correlations between level populations and field which are a result of spontaneous emission.

Since intensity[13] and phase fluctuations have different characteristics in an oscillator, it is interesting to transform (4.7) to polar coordinates in the complex β plane. Accordingly, we transform variables in (4.7) from β and β^* to the real variables I and θ, where

$$\beta\equiv I^{1/2}\exp(-i\theta), \tag{4.8}$$

from which we obtain

$$\frac{\partial}{\partial\beta}\beta=\left(\frac{\partial}{\partial\beta^*}\beta^*\right)^*=\frac{\partial}{\partial I}I+\tfrac{1}{2}i\frac{\partial}{\partial\theta},$$
$$\frac{\partial^2}{\partial\beta\partial\beta^*}=\frac{\partial^2}{\partial I^2}I-\frac{\partial}{\partial I}+\frac{\partial^2}{\partial\theta^2}\frac{1}{4I},$$
$$d^{(2)}\beta=\tfrac{1}{2}dI\,d\theta,$$
$$P_1(\beta,\beta^*,\mathfrak{N}_i,\mathfrak{N}_2,t)=2P_1(I,\theta,\mathfrak{N}_1,\mathfrak{N}_2,t).$$

With these substitutions, (4.7) transforms to

$$\frac{\partial P_1(I,\theta,\mathfrak{N}_1,\mathfrak{N}_2,t)}{\partial t}$$
$$=\Big\{-\frac{\partial}{\partial I}[I\{\pi(\mathfrak{N}_2-\mathfrak{N}_1)-\gamma\}+\gamma\bar{n}+\pi\mathfrak{N}_2]-\frac{\partial}{\partial\theta}[\tfrac{1}{2}\alpha\{\pi(\mathfrak{N}_2-\mathfrak{N}_1)-\gamma\}]$$
$$-\frac{\partial}{\partial\mathfrak{N}_1}[R_1+\{w_{12}+\pi(I+1)\}\mathfrak{N}_2-(\Gamma_1+\pi I)\mathfrak{N}_1]-\frac{\partial}{\partial\mathfrak{N}_2}[R_2-\{\Gamma_2+\pi(I+1)\}\mathfrak{N}_2+(w_{21}+\pi I)\mathfrak{N}_1]$$
$$+\frac{\partial^2}{\partial I^2}[I(\gamma\bar{n}+\pi\mathfrak{N}_2)]+\frac{\partial^2}{\partial\theta^2}\left[\frac{1}{4I}(\gamma\bar{n}+\pi\mathfrak{N}_2)\right]+\frac{\partial^2}{\partial\mathfrak{N}_1\partial I}[\pi(2I+1)\mathfrak{N}_2-\pi I\mathfrak{N}_1]$$
$$+\frac{\partial^2}{\partial\mathfrak{N}_2\partial I}[-\pi I\mathfrak{N}_2]+\frac{\partial^2}{\partial\mathfrak{N}_1\partial\theta}[-\tfrac{1}{2}\pi\alpha\mathfrak{N}_1]+\frac{\partial^2}{\partial\mathfrak{N}_2\partial\theta}[\tfrac{1}{2}\pi\alpha\mathfrak{N}_2]-\frac{\partial^3}{\partial\mathfrak{N}_1\partial I^2}[\pi I\mathfrak{N}_2]-\frac{\partial^3}{\partial\mathfrak{N}_1\partial\theta^2}\left[\frac{\pi\mathfrak{N}}{4I}\right]\Big\}P_1. \tag{4.9}$$

[13] We use intensity (amplitude squared) rather than amplitude as a variable here simply because it is the intensity which appears in the drift and diffusion coefficients. Lax (Ref. 3) has shown that use of intensity leads to "quasilinear" solutions which are fairly accurate even in the region near threshold.

Note that in (4.9) both the intensity and phase diffusion coefficients ($\partial^2/\partial I^2$ and $\partial^2/\partial\theta^2$ terms, respectively) are proportional to the factor $(\gamma\bar{n}+\pi\mathfrak{N}_2)$, which also appears as the spontaneous part of the intensity drift coefficient. Again, there occur a number of off-diagonal diffusion terms whose individual physical significances are not particularly apparent.

Finally, we can eliminate $\mathfrak{N}_1$ and $\mathfrak{N}_2$ from the equation in the case that $\Gamma_1, \Gamma_2 \gg \gamma$, so that the level populations are adiabatically dependent upon the field strength. Because $\mathfrak{N}_2$ and $\mathfrak{N}_1$ appear in the diffusion coefficients, we cannot do this elimination "exactly", as was the case with $\mathfrak{M}$. However, the physically important results are probably contained in the diffusion, or Fokker–Planck, approximation, where the resulting equation is found only up through second derivative terms.

In accordance with the discussion of Appendix B, we accomplish the adiabatic elimination of $\mathfrak{N}_1$ and $\mathfrak{N}_2$ by setting (for $i=1, 2$)

$$\mathfrak{N}_i P_1=[A_i-(\partial/\partial I)B_i-(\partial/\partial\theta)C_i+\cdots]P_1. \quad (4.10)$$

We substitute (4.10) into (4.9) to eliminate $\mathfrak{N}_1$ and $\mathfrak{N}_2$ from the drift and diffusion coefficients, bring all derivatives to the left, and then solve for the A_i, B_i, and C_i so that all first- and second-derivative terms containing the $\partial/\partial\mathfrak{N}_i$ vanish. We find that by order of magnitude

$$C:B:A=\pi:\pi I:\Gamma.$$

With neglect of w_{12} and w_{21}, and also with neglect of π (but not πI) with respect to Γ_1 and Γ_2, we find the resulting equation

$$\frac{\partial P_1(I,\theta,t)}{\partial t}$$

$$=\left\{-\frac{\partial}{\partial I}[\pi(I+1)A_2-\pi IA_1+\gamma(\bar{n}-I)]-\frac{\partial}{\partial\theta}[\tfrac{1}{2}\alpha\{\pi(A_2-A_1)-\gamma\}]+\frac{\partial^2}{\partial I^2}[\gamma\bar{n}I+\mathfrak{D}^{-1}\pi I\Gamma_2\{\Gamma_1A_2-\pi I(A_2-A_1)\}]\right.$$

$$\left.+\frac{\partial^2}{\partial\theta^2}[(4I)^{-1}(\gamma\bar{n}+\pi A_2)+\mathfrak{D}^{-1}\tfrac{1}{2}(\pi\alpha)^2(\Gamma_1A_2+\Gamma_2A_1)]+\frac{\partial^2}{\partial\theta\partial I}[-\mathfrak{D}^{-1}\pi^2\alpha I\Gamma_2(A_2-A_1)]\right\}P, \quad (4.11)$$

where

$$\mathfrak{D}\equiv\Gamma_1\Gamma_2+\pi I(\Gamma_1+\Gamma_2),$$

and where A_2 and A_1 are the adiabatic mean values of $\mathfrak{N}_2$ and $\mathfrak{N}_1$, as may be seen from (4.10), and are given by

$$A_2=\mathfrak{D}^{-1}[\Gamma_1R_1+\pi I(R_1+R_2)],$$
$$A_1=\mathfrak{D}^{-1}[\Gamma_2R_1+\pi I(R_1+R_2)]. \quad (4.12)$$

Equation (4.11) is in agreement with similar results obtained recently by Lax and by Scully using quite different methods. The fact that we all reach consistent results here lends some extra credence to all of the methods.

The effect of saturation on the diffusion terms of (4.11) is a noteworthy feature. The phase diffusion coefficient, at least on resonance ($\alpha=0$) is only affected in so far as A_2 is affected. The intensity diffusion coefficient, on the other hand, tends to be surpressed further, and for very high power, where

$$\pi I\gg\Gamma_1, \Gamma_2,$$

evaluates approximately to

$$\gamma\bar{n}I+(\Gamma_2/(\Gamma_1+\Gamma_2))R_1.$$

It is in principle possible to have this quantity nearly zero, by having $\bar{n}, R_1\to 0$. In such a case P_1 would come to steady state with a very small variance in I.

In terms of $P_1(I, \theta, t)$, the field density operator is given by

$$\rho_b=\int^{(2)}P_1(I,\theta,t)\;|\;I^{1/2}\exp[-i(\omega_0t+\theta)]\rangle$$
$$\times\langle I^{1/2}\exp[-i(\omega_0t+\theta)]\;|\;dI\,d\theta. \quad (4.13)$$

We may conclude that a maser oscillator, operating under highly saturated conditions, can produce a field which is very nearly in a coherent state. The phase of the field will, as may be shown from (4.11), diffuse slowly away from some initially measured value according to the law

$$(d/dt)\langle\exp(-i\theta)\rangle_1=-[(4I)^{-1}(\gamma\bar{n}+\pi A_2)]\langle\exp(-i\theta)\rangle_1 \quad (4.14)$$

for the case $\alpha\approx 0$. The result that the drift of θ is zero for $\alpha=0$ is a result of our original choice for ω_0. In (4.14), we have assumed that I and A_2 are sensibly constant in the average taking and have brought them outside of the mean value sign.[13a] Equation (4.14) im-

[13a] *Note added in proof*: If the maser is operating moderately above threshold, it is appropriate to make an adiabatic elimination of the intensity I from (4.11). Straightforward application of the method discussed in Appendix B results in a phase-only diffusion equation, with the diffusion constant given by

$$[4\langle I\rangle_1]^{-1}(\gamma\bar{n}+\pi A_2)(1+\alpha^2),$$

where $A_2=A_2(\langle I\rangle_1)$. This result generalizes the width of the Lorentzian phase spectrum, multiplying the result given just below in the text by a detuning factor $(1+\alpha^2)$, and specifying that A_2 and I are evaluated with I taken as the adiabatic mean value $\langle I\rangle_1$. This result is consistent with Lax's quasilinear analysis.

plies that the oscillation signal has a Lorentzian power spectrum with a full width at half-power given by

$$\Delta\omega=(\gamma\bar{n}+\pi A_2)/2I.$$

This is a well-known result, which is now justified for arbitrarily high levels of saturation.

One final point is worth mention here. Since we have eliminated $\mathfrak{M}$ and $\mathfrak{M}^*$ from our basic equation first, and then have eliminated $\mathfrak{N}_1$ and $\mathfrak{N}_2$ later, it would seem that the validity of (4.11) should be limited to the regime $\Gamma_{12}\gg\Gamma_1$, $\Gamma_2\gg\gamma$. It turns out otherwise; there is a simplicity in this problem, whose mathematical symptom is that the adiabatic $\mathfrak{M}$ (4.6) has no $\partial/\partial\mathfrak{N}_1$ or $\partial/\partial\mathfrak{N}_2$ terms in it, and whose result is that a simultaneous adiabatic elimination of $\mathfrak{M}$, $\mathfrak{M}^*$, $\mathfrak{N}_1$, and $\mathfrak{N}_2$ in favor of the field variables β and β^* leads precisely to (4.11). Thus (4.11) is in fact valid for any possible relationship of Γ_{12} to Γ_1 and Γ_2.

V. THE GENERALIZED WIGNER DENSITY AND THE APPROXIMATE CLASSICAL MODEL

It is not obvious from our exact basic equation (3.19) for P_1 that there exists a classical model for the maser which approximates the properties of the quantal model including fluctuations. In this section, we show that there is such a classical model, and show quantitatively the relationship between it and the quantal model. To this end we shall use (3.19) to derive the equation of motion of what may be called the generalized Wigner density for the maser.

We symbolize the Wigner density by P_2 and define it from the moment generating function relationship

$$\begin{aligned}&\mathrm{Tr}[\rho(t)\,\exp i(\lambda^*\mathbf{b}+\lambda\mathbf{b}^\dagger+\xi_1\mathbf{N}_1+\xi_2\mathbf{N}_2+\xi_3\mathbf{M}+\xi_4\mathbf{M}^\dagger)]\\&\equiv\int^{(6)}P_2(b,b^*,N_1,N_2,M,M^*,t)\\&\quad\times\exp i(\lambda^*b+\lambda b^*+\xi_1N_1+\xi_2N_2+\xi_3M+\xi_4M^*)\\&\quad\times d^{(2)}b\,dN_1dN_2d^{(2)}M.\end{aligned}\tag{5.1}$$

Here, λ, λ^*, and the ξ's are the expansion parameters of the generating functions. If ξ_1 and ξ_2 are real and $\xi_4=\xi_3^*$, then the exponential factor becomes pure imaginary. In (5.1) we have introduced a new set of c-number variables (b, b^*, N_1, N_2, M, M^*), and their associated weight distribution P_2, having the property that any moment of these variables is equal to the *symmetrized* moment of the corresponding operator variables. It is clear from (5.1) that P_2 is the Fourier transform of the symmetrically ordered quantum characteristic function. If the atomic variables are omitted, one can show that P_2 is the ordinary Wigner density[14] for the field. That P_2 should turn out to behave sensibly from a classical viewpoint is not too surprising, as the correspondence between classical variables and the symmetrized form of quantum variables has long been recognized. Nevertheless, it is interesting that the diffusion terms in the Fokker–Planck equation for P_2 turn out to be remarkably simple even for this dynamical nonlinear problem.

In order to derive the equation of motion of P_2, we first need to know its relationship to P_1 whose equation of motion we already have. We find this by inserting the expansion (3.1) in the left side of (5.1). For convenience in writing this, let us use the definition

$$\mathbf{Q}\equiv\exp i(\lambda^*\mathbf{b}+\lambda\mathbf{b}^\dagger+\xi_1\mathbf{N}_1+\xi_2\mathbf{N}_2+\xi_3\mathbf{M}+\xi_4\mathbf{M}^\dagger).\tag{5.2}$$

By virtue of the definitions (5.2) and (3.9), we can write the left side of (5.1) simply as $\langle\mathbf{Q}\rangle$, and from (3.9) we have

$$\langle\mathbf{Q}\rangle=\int^{(6)}P_1(\mathrm{Tr}\sigma\mathbf{Q})\,d^{(2)}\beta\,d\mathfrak{N}_1d\mathfrak{N}_2d^{(2)}\mathfrak{M}.\tag{5.3}$$

In Appendix C the trace of $\sigma\mathbf{Q}$ is evaluated, and is shown to be given in the limit (3.12) by

$$\begin{aligned}&\mathrm{Tr}\sigma\mathbf{Q}\\&=\exp\{i(\nu_1\mathfrak{N}_1+\nu_2\mathfrak{N}_2+\nu_3\mathfrak{M}+\nu_4\mathfrak{M}^*+\lambda^*\beta+\lambda\beta^*)-\tfrac{1}{2}\lambda\lambda^*\},\end{aligned}\tag{5.4a}$$

where

$$\begin{aligned}i\nu_1&\equiv\exp(i\xi_+)[\cos v-i(\xi_-/v)\,\sin v]-1,\\i\nu_2&\equiv\exp(i\xi_+)[\cos v+i(\xi_-/v)\,\sin v]-1,\\i\nu_3&\equiv i\exp(i\xi_+)\,(\xi_3/v)\,\sin v,\\i\nu_4&\equiv i\exp(i\xi_+)\,(\xi_4/v)\,\sin v,\end{aligned}\tag{5.4b}$$

and where

$$\xi_+\equiv\tfrac{1}{2}(\xi_2+\xi_1);\qquad\xi_-\equiv\tfrac{1}{2}(\xi_2-\xi_1);\qquad v\equiv(\xi_-^2+\xi_3\xi_4)^{1/2}.$$

Comparison of (5.3) and (5.4) with (5.1) yields the required relation between P_2 and P_1, namely

$$\begin{aligned}&\int^{(6)}P_2\exp\{i(\lambda^*b+\lambda b^*+\xi_1N_1+\xi_2N_2+\xi_3M+\xi_4M^*)\}\,d^{(2)}b\,dN_1dN_2d^{(2)}M\\&=\int^{(6)}P_1\exp\{-\tfrac{1}{2}(\lambda\lambda^*)+i(\lambda^*\beta+\lambda\beta^*+\nu_1\mathfrak{N}_1+\nu_2\mathfrak{N}_2+\nu_3\mathfrak{M}+\nu_4\mathfrak{M}^*)\}\,d^{(2)}\beta\,d\mathfrak{N}_1d\mathfrak{N}_2d^{(2)}\mathfrak{M},\end{aligned}\tag{5.5}$$

[14] J. E. Moyal, Proc. Cambridge Phil. Soc. **45**, 99 (1949).

where the ν's and the ξ's are related by Eqs. (5.4b). From (5.5) and the equation of motion (3.19) for P_1, we can derive the equation of motion for P_2. It would appear that there is much labor and little insight to be gained by attempting to do this exactly. However, if the sum $\mathfrak{N}_1+\mathfrak{N}_2$ is large compared to unity over the important range of P_1, then Tr $\sigma\mathbf{Q}$ will be appreciable only for small values of the ξ's (particularly in the combinations ξ_+ and v), and it is appropriate to expand (5.4b) in a power series in the ξ's. Keeping only second power terms, we obtain

$$\begin{aligned} i\nu_1&=i\xi_1-\tfrac{1}{2}(\xi_1^2+\xi_3\xi_4),\\ i\nu_2&=i\xi_2-\tfrac{1}{2}(\xi_2^2+\xi_3\xi_4),\\ i\nu_3&=i\xi_3-\tfrac{1}{2}\xi_3(\xi_1+\xi_2),\\ i\nu_4&=i\xi_4-\tfrac{1}{2}\xi_4(\xi_1+\xi_2). \end{aligned} \tag{5.6}$$

The approximation (5.6) yields P_2 as a Gaussian convolution of P_1, as shown in Appendix D, and gives the diffusion approximation to the equation of motion for P_2. The fact that P_2 is a smoothed version of P_1 explains to some extent why the diffusion equation for P_2 is better behaved than is that for P_1. The method of derivation of the diffusion equation for P_2 is given in Appendix E. The result is as follows:

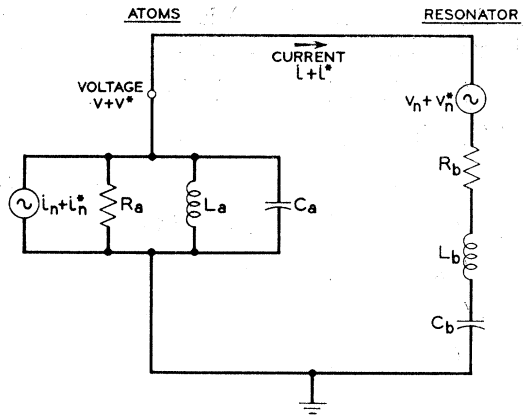

FIG. 2. Circuit equivalent for the maser model.

$$\begin{aligned}
&\frac{\partial P_2(b,b^*,N_1,N_2,M,M^*,t)}{\partial t}\\
&=\Big\{-\frac{\partial}{\partial b}[\mu M-(\tfrac{1}{2}\gamma+i\omega_b)b]-\frac{\partial}{\partial b^*}[\mu M^*-(\tfrac{1}{2}\gamma-i\omega_b)b^*]-\frac{\partial}{\partial M}[\mu b(N_2-N_1)-(\Gamma_{12}+i\omega_a)M]\\
&-\frac{\partial}{\partial M^*}[\mu b^*(N_2-N_1)-(\Gamma_{12}-i\omega_a)M^*]-\frac{\partial}{\partial N_1}[R_1+w_{12}N_2-\Gamma_1N_1+\mu(b^*M+bM^*)]\\
&-\frac{\partial}{\partial N_2}[R_2+w_{21}N_1-\Gamma_2N_2-\mu(b^*M+bM^*)]+\frac{\partial^2}{\partial b\partial b^*}[\gamma(\bar{n}+\tfrac{1}{2})]\\
&+\frac{\partial^2}{\partial M\partial M^*}[\Gamma_{12}(N_1+N_2)+\tfrac{1}{2}(R_1+R_2-\Gamma_1N_1-\Gamma_2N_2+w_{21}N_1+w_{12}N_2)]\\
&+\frac{\partial^2}{\partial N_1^2}[\tfrac{1}{2}(R_1+\Gamma_1N_1+w_{12}N_2)]+\frac{\partial^2}{\partial N_2^2}[\tfrac{1}{2}(R_2+\Gamma_2N_2+w_{21}N_1)]\\
&+\frac{\partial^2}{\partial N_1\partial M}[\tfrac{1}{2}(\Gamma_1-w_{12})M]+\frac{\partial^2}{\partial N_2\partial M^*}[\tfrac{1}{2}(\Gamma_1-w_{12})M^*]\\
&+\frac{\partial^2}{\partial N_2\partial M}[\tfrac{1}{2}(\Gamma_2-w_{21})M]+\frac{\partial^2}{\partial N_2\partial M^*}[\tfrac{1}{2}(\Gamma_2-w_{21})M^*]+\frac{\partial^2}{\partial N_1\partial N_2}[-w_{12}N_2-w_{21}N_1]\Big\}P_2.
\end{aligned} \tag{5.7}$$

Equation (5.7), unlike (3.19), looks quite sensible from a classical viewpoint. The diffusion matrix is positive definite and does not couple the atoms with the field. From (5.7) one can derive a simple classical-model equivalent to the maser. The small price one pays is that (5.7) is a good approximation only when the number of atoms in levels one and two is reasonably large. Of course, it is entirely possible to write the exact form of (5.7), but then its advantage of intuitive simplicity would be lost.

Equation (5.7) is equivalent, in the sense of providing equations of motion for various moments of the variables, to the following set of dynamic Langevin equations of motion:

$$\begin{aligned}
db/dt&=\mu M-(\tfrac{1}{2}\gamma+i\omega_b)b+F_b(t),\\
dM/dt&=\mu b(N_2-N_1)-(\Gamma_{12}+i\omega_a)M+F_M(t),\\
dN_1/dt&=R_1+w_{12}N_2-\Gamma_1N_1+\mu(b^*M+bM^*)+F_1(t),\\
dN_2/dt&=R_2+w_{12}N_1-\Gamma_2N_2-\mu(b^*M+bM^*)+F_2(t),
\end{aligned} \tag{5.8}$$

where the random forces $F(t)$ have zero mean values, broad spectra with respect to the important spectral widths γ and Γ_{12} of our problem, and have the following nonzero correlation functions:

$$\begin{aligned}
&\langle F_b(t')F_{b*}(t)\rangle=\gamma(\bar{n}+\tfrac{1}{2})\delta_s(t-t')\exp[i\omega_b(t-t')],\\
&\langle F_M(t')F_{M*}(t)\rangle=[\Gamma_{12}(N_1+N_2)+\tfrac{1}{2}(R_1+R_2-\Gamma_1N_1-\Gamma_2N_2+w_{12}N_2+w_{21}N_1)]\delta_s(t-t')\exp[i\omega_a(t-t')],\\
&\langle F_1(t)F_1(t')\rangle=(R_1+\Gamma_1N_1+w_{12}N_2)\delta_s(t-t'),\\
&\langle F_2(t)F_2(t')\rangle=(R_2+\Gamma_2N_2+w_{21}N_1)\delta_s(t-t'),\\
&\langle F_M(t)F_1(t')\rangle=\tfrac{1}{2}(\Gamma_1-w_{12})M(t)\delta_s(t-t'),\\
&\langle F_M(t)F_2(t')\rangle=\tfrac{1}{2}(\Gamma_2-w_{21})M(t)\delta_s(t-t'),\\
&\langle F_1(t)F_2(t')\rangle=-(w_{21}N_1+w_{12}N_2)\delta_s(t-t').
\end{aligned}\tag{5.9}$$

In (5.9) the subscript s on the delta function stands for "slow," and indicates consideration of time intervals not too much shorter than the reciprocal bandwidths of the problem; i.e., of the order of γ^{-1} or Γ_{12}^{-1}. The exponential time dependences, and the explicit indication of t in $M(t)$ are included to take care of the high-frequency variation of F_M and F_b. The physical point involved here is that only the spectral region near the resonator and atomic frequencies is important for F_M and F_b, and over this region they have white spectra.

The first two of the dynamic equations (5.8) along with the first two second-moment formulas (5.9) for the random forces are exactly those derivable in the high-Q approximation from the classical equivalent circuit shown in Fig. 2. A series resonant circuit with a noise-voltage source represents the resonator, while a parallel circuit with a noise-current source represents the atoms. The two noise sources are statistically independent. The current in the series circuit is proportional to the field strength, while the voltage across the parallel circuit is proportional to the polarization. If we represent the circuit voltage and current by the sum of positive- and negative-frequency parts

$$(v+v^*),\ (i+i^*);\qquad v,\ i\propto\exp(-i\omega t),$$

then the equivalence relations are

$$\begin{aligned}
(L_bC_b)^{1/2}&=\omega_b,\\
R_b/L_b&=\gamma,\\
i&=(\hbar\omega\gamma/2R_b)^{1/2}b,\\
v_n&=(2\hbar\omega R_b/\gamma)^{1/2}F_b,\\
(L_aC_a)^{-1/2}&=\omega_a,\\
1/R_aC_a&=2\Gamma_{12},\\
v&=\mu(\hbar\omega 2R_b/\gamma)^{1/2}M,\\
i_n&=[\hbar\omega/R_a\Gamma_{12}(N_1-N_2)]^{1/2}F_M,
\end{aligned}$$

and

$$R_a=R_b(2\mu^2/\Gamma_{12}\gamma)(N_1-N_2).\tag{5.10}$$

The proportionality between i and b has been so chosen that power and energy relationships come out correctly; e.g., the power delivered to the resonator is

$$vi^*+iv^*=\mu\hbar\omega(Mb^*+M^*b)\tag{5.11}$$

and the energy stored in the resonator is

$$2L_1i^*i=\hbar\omega b^*b.\tag{5.12}$$

The second moments of the random forces show that in the equivalent circuit

$$\begin{aligned}
&\langle v_n(t)v_n^*(t')\rangle=2R_b\hbar\omega(\bar{n}+\tfrac{1}{2})\delta_s(t-t')\exp[i\omega_b(t-t')]\\
&\langle i_n(t)i_n^*(t')\rangle=\frac{\hbar\omega}{R_a}\frac{(N_1+N_2)+(R_1+R_2-w_{01}N_1-w_{02}N_2)/2\Gamma_{12}}{N_1-N_2}\delta_s(t-t')\exp[i\omega_a(t-t')].
\end{aligned}\tag{5.13}$$

The last two of the dynamic equations (5.8) are the rate equations for the level populations, and the last five of the correlation functions (5.9) are precisely those one should expect as a result of shot noise in the pumping and decay of these level populations. For example, consider the $\langle F_M(t)F_1(t')\rangle$ correlation. A heuristic argument which gives the correct diffusion constant is as follows. The polarization M is proportional to $[N_1N_2]^{1/2}$, hence a fluctuation δN_1 in the decay of N_1 should give rise to a fluctuation δM of M given by

$$\delta M/M=\tfrac{1}{2}\delta N_1/N_1,$$

hence

$$\delta M\delta N_1=(M/2N_1)(\delta N_1)^2.$$

From the $\langle F_1(t)F_1(t')\rangle$ correlation term, the fluctuation $(\delta N_1)^2$ concerned can be seen to be proportional to Γ_1N_1, and hence the correlation term $\langle F_M(t)F_1(t')\rangle$ should contain $(M/2N_1)\Gamma_1N_1=\tfrac{1}{2}\Gamma_1M$, as it indeed does. The w_{12} term in $\langle F_M(t)F_1(t')\rangle$ can be obtained from a similar argument, since a positive fluctuation in the transfer rate $w_{12}N_2$ from level two to level one simultaneously increases N_1 and decreases M, the latter because of the decrease of N_2.

An important physical question to be answered concerns the "number of photons" in the resonator and the fluctuations in that number. This question may be studied through consideration of the moment relation

$$\langle \exp i(\lambda^* \mathbf{b}+\lambda \mathbf{b}^\dagger)\rangle = \langle \exp i(\lambda^* b+\lambda b^*)\rangle_2, \quad (5.14)$$

where on the right-hand side, we are using the angular brackets with subscript 2 to indicate a mean against the probability distribution P_2. Hence this equation is identical with (5.1), with $\xi_k, \nu_k \to 0$. By direct comparison of moments, one can show from (5.14) that

$$\langle \exp(i\chi \mathbf{b}^\dagger \mathbf{b})\rangle = \langle \exp\{i\chi(b^*b-\tfrac{1}{2})+\tfrac{1}{8}\chi^2+\cdots\}\rangle_2, \quad (5.15)$$

where χ is an expansion parameter. For comparison we note that since[15]

$$\langle \beta \mid \exp(i\chi \mathbf{b}^\dagger \mathbf{b}) \mid \beta\rangle = \exp\{[\exp(i\chi)-1]\beta^*\beta\},$$

one finds using (3.11) and (3.2) that

$$\langle \exp(i\chi \mathbf{b}^\dagger \mathbf{b})\rangle = \langle \exp\{[\exp(i\chi)-1]\beta^*\beta\}\rangle_1$$
$$= \langle \exp(i\chi\beta^*\beta-\tfrac{1}{2}\chi^2\beta^*\beta+\cdots)\rangle_1. \quad (5.16)$$

Comparison of (5.15) and (5.16) shows that if $\langle \beta^*\beta\rangle_1$ is greater than $\frac{1}{4}$, then $(b^*b-\frac{1}{2})$ is a significantly better approximation to the photon number than is $\beta^*\beta$. If the mean and variance of the photon number distribution are large compared to unity, then bb^* is a very good approximation to that number. As usual, classical physics is a good approximation to quantal physics in the limit of large quantum numbers.

VI. FURTHER CONSIDERATIONS

There are a number of useful functions (or distributions) other than P_2 which can be derived from our basic weight distribution P_1, and we shall mention a few of these because they shed some further light on our work and, in addition, make further contact with the work of others, notably Lax, and Scully and Lamb, who have treated the same or similar models by quite different methods.

The first of these we label P_{12} because it has the properties of P_1 with respect to the field, and of P_2 with respect to the atoms. To consolidate notation, recall that in (3.6) we abbreviated the atomic variables of P_1 by

$$(\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3, \mathfrak{X}_4) \equiv (\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M}, \mathfrak{M}^*).$$

Likewise, let us now abbreviate the atomic variables of P_2 by

$$(X_1, X_2, X_3, X_4) \equiv (N_1, N_2, M, M^*),$$
$$d^{(4)}X \equiv dN_1 dN_2 d(\mathrm{Re}M)\, d(\mathrm{Im}M). \quad (6.1)$$

The function P_{12} is defined from the general relation

$$\mathrm{Tr}_a(\rho \mathbf{Q}_a) \equiv \int^{(6)} P_{12}(\beta, \beta^*, X_j, t)$$
$$\times \exp(i\textstyle\sum \xi_j X_j) \mid \beta\rangle\langle\beta \mid d^{(2)}\beta\, d^{(4)}X, \quad (6.2)$$

where $\mathbf{Q}_a$ is the atomic part of the generating function (5.2), namely,

$$\mathbf{Q}_a \equiv \exp i(\xi_1 \mathbf{N}_1+\xi_2 \mathbf{N}_2+\xi_3 \mathbf{M}+\xi_4 \mathbf{M}^\dagger). \quad (6.3)$$

If we expand ρ in our P_1 representation, and make use of (5.14) with $\lambda=\lambda^*=0$, we find the relationship between P_1 and P_{12}, namely,

$$\int^{(4)} P_{12}(\beta, \beta^*, X_j, t)\, \exp(i\textstyle\sum \xi_j X_j)\, d^{(4)}X$$
$$= \int^{(4)} P_1(\beta, \beta^*, \mathfrak{X}_j, t)\, \exp(i\textstyle\sum \nu_j \mathfrak{X}_j) d^{(4)}\mathfrak{X}, \quad (6.4)$$

where the ν_j are related to the ξ_j by (5.4b). The passage from P_1 to P_{12} entails the same smoothing with respect to the atomic variables as does the passage from P_1 to P_2. Hence, in the diffusion approximation, the equation of motion for P_{12} can be derived from (3.19) by making the set of substitutions listed in (E5) for the atomic variables only. The resulting equation has many diffusion terms coupling the atoms with the field, and lacks the intuitive simplicity of our P_2 equation. However, like that for P_2, the equation of motion for P_{12} in the diffusion approximation is better behaved than is that for P_1, and it retains the property of directly producing the reduced density operator for the field, as may be seen by setting $\xi_j=0$ in (6.2) and (6.3). The function P_{12} is precisely "the associated classical function" which is being investigated by Lax using Langevin methods.

Other interesting results are obtained by going into the number representation for the field. We do this by taking the n, m field matrix element of (3.1), obtaining

$$\langle n \mid \rho \mid m\rangle = \int^{(4)} P_{n,m}(\mathfrak{X}_j, t)\, \sigma_a d^{(4)}\mathfrak{X}, \quad (6.5)$$

where the function $P_{n,m}(\mathfrak{X}_j, t)$ is defined by

$$P_{n,m}(\mathfrak{X}_j, t) \equiv \int^{(2)} P_1(\beta, \beta^*, \mathfrak{X}_j, t)\, \langle n \mid \sigma_b \mid m\rangle d^{(2)}\beta$$
$$= \int^{(2)} P_1 \exp(-\beta\beta^*)\, \frac{\beta^n \beta^{*m}}{(n!m!)^{1/2}}\, d^{(2)}\beta. \quad (6.6)$$

The equation of motion for $P_{n,m}(\mathfrak{X}_j, t)$ may be derived from (6.6) and the equation of motion for P_1.

A final function of some appeal is formed by going to the atomic variables X_j [see (6.1)] in the number representation for the field. To do this, we use the defining relation

$$\langle n \mid \mathrm{Tr}_a \rho \mathbf{Q}_a \mid m\rangle \equiv \int^{(4)} P_{n,m}(X_j, t)\, \exp(i\textstyle\sum \xi_j X_j) d^{(4)}X. \quad (6.7)$$

[15] This identity follows from reordering the exponential operator into normal ordered form wherein all annihilation operators appear to the right of all creation operators. For an account of such techniques, see W. H. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill Book Company, Inc., New York, 1964); also Ref. 8.

Inserting (6.5) in the left side of (6.7), and again using (5.4), we obtain the relation between $P_{n,m}(X_j, t)$ and $P_{n,m}(\mathfrak{X}_j, t)$, namely,

$$\int^{(4)} P_{n,m}(\mathfrak{X}_j, t)\, \exp(i\sum \nu_j \mathfrak{X}_j)\, d^{(4)}\mathfrak{X}_j$$

$$= \int^{(4)} P_{n,m}(X_j, t)\, \exp(i\sum \xi_j X_j)\, d^{(4)}X. \quad (6.8)$$

Comparing (6.8) with (6.4) we see that the same transformation which takes P_1 to P_{12} takes $P_{n,m}(\mathfrak{X}_j, t)$ to $P_{n,m}(X_j, t)$.

In the course of considering these various functions and their equations of motion, we have found that the equations for P_1-type functions are easier to manipulate (as opposed to solve). Adiabatic eliminations are an example of this. On the other hand, equations for P_2 like functions would appear to have smoother behavior, and in addition have greater intuitive appeal.

To exemplify these remarks, we shall derive the equations of motion for $P_{n,m}(\mathfrak{X}_j, t)$ and $P_{n,m}(X_j, t)$ under conditions appropriate for the adiabatic elimination of the polarization $\mathfrak{M}$. The P_1 equation we shall need is therefore (4.7).

We start with (6.6), transform its right-hand side to the rotating frame using (4.2), and integrate it over the $\mathfrak{M}$ plane, thus obtaining the necessary relation between $P_{n,m}(\mathfrak{N}_1, \mathfrak{N}_2, t)$ and the weight P_1 of (4.7), namely,

$$P_{n,m}(\mathfrak{N}_1, \mathfrak{N}_2, t) = \int^{(2)} P_1(\beta, \beta^*, \mathfrak{N}_1, \mathfrak{N}_2, t)\, \frac{\beta^n \beta^{*m}}{(n!m!)^{1/2}} \times \exp[i(m-n)t]\, d^{(2)}\beta, \quad (6.9)$$

where in (6.9) as in (4.7), the primes on the β's are understood. We derive the equation of motion for $P_{n,m}(\mathfrak{N}_1, \mathfrak{N}_2, t)$ by taking the time derivative of (6.9), inserting the equation of motion (4.7) for

$$P_1(\beta, \beta^*, \mathfrak{N}_1, \mathfrak{N}_2, t),$$

integrating by parts with respect to β and β^* to eliminate derivatives with respect to β and β^*, and then using (6.9) again to reexpress the result in terms of the $P_{n,m}$. We thus obtain the following equation of motion for $P_{n,m}$:

$$\frac{\partial P_{n,m}(\mathfrak{N}_1, \mathfrak{N}_2, t)}{\partial t}$$

$$= [(n+1)(m+1)]^{1/2}[\pi\mathfrak{N}_1 + \gamma(\bar{n}+1)]P_{n+1,m+1} + (nm)^{1/2}[\pi\mathfrak{N}_2 + \gamma\bar{n}]P_{n-1,m-1}$$

$$- [\tfrac{1}{2}\pi(n+m+2)\mathfrak{N}_2 + \tfrac{1}{2}\pi(n+m)\mathfrak{N}_1 + i(n-m)(\omega_b + \alpha\tfrac{1}{2}\pi(\mathfrak{N}_2 - \mathfrak{N}_1)) + \gamma\bar{n}(n+m+1) + \tfrac{1}{2}\gamma(n+m)]P_{n,m}$$

$$- (\partial/\partial\mathfrak{N}_1)\{[R_1 + w_{12}\mathfrak{N}_2 - (\Gamma_1 + \tfrac{1}{2}\pi(n+m) - i\alpha\tfrac{1}{2}\pi(n-m))\mathfrak{N}_1]P_{n,m} + (nm)^{1/2}\pi\mathfrak{N}_2 P_{n-1,m-1}\}$$

$$- (\partial/\partial\mathfrak{N}_2)\{[R_2 + w_{21}\mathfrak{N}_1 - (\Gamma_2 + \tfrac{1}{2}\pi(n+m+2) + i\alpha\tfrac{1}{2}\pi(n-m))\mathfrak{N}_2]P_{n,m} + [(n+1)(m+1)]^{1/2}\pi\mathfrak{N}_1 P_{n+1,m+1}\}. \quad (6.10)$$

An interesting property of (6.10) which was not true in the case of (4.9) is that if we neglect the normally small quantities w_{21} and w_{12}, we can "exactly" carry out the adiabatic elimination of $\mathfrak{N}_1$ and $\mathfrak{N}_2$. When we do this, we reproduce precisely the equation of Scully and Lamb,[5] as we shall demonstrate below. But before doing this we would like to derive from (6.10) the corresponding equation of motion of $P_{n,m}(N_1, N_2, t)$.

To find the equation for $P_{n,m}(N_1, N_2, t)$, we multiply (6.10) by $\exp(i\nu_1\mathfrak{N}_1 + i\nu_2\mathfrak{N}_2)$ and integrate over $\mathfrak{N}_1$ and $\mathfrak{N}_2$. We then make use of the identity (6.8) and follow the same method as outlined in Appendix E. Since everything is integrated over the $\mathfrak{M}$ plane, we set $\xi_3 = \xi_4 = 0$ in the relationships (5.4b) between the ν's and the ξ's, obtaining simply

$$i\nu_j = \exp(i\xi_j) - 1; \qquad j = 1, 2. \quad (6.11)$$

Because of the simplicity of (6.11), the passage to $P_{n,m}(N_1, N_2, t)$ can be done exactly in this case. We find that it entails the following set of substitutions into (6.10):

$$\mathfrak{N}_j \rightarrow \exp(\partial/\partial N_j) N_j,$$

$$\partial/\partial\mathfrak{N}_j \rightarrow (1 - \exp(-\partial/\partial N_j)), \qquad j = 1, 2,$$

$$P_{n,m}(\mathfrak{N}_1, \mathfrak{N}_2, t) \rightarrow P_{n,m}(N_1, N_2, t). \quad (6.12)$$

The order of all factors is preserved in these substitutions, so derivatives always come out to the left. After

suitably rearranging the resulting equation, we find the following equation of motion for $P_{n,m}(N_1, N_2, t)$

$$
\begin{aligned}
&(\partial/\partial t)P_{n,m}(N_1, N_2, t)\\
&=[(n+1)(m+1)]^{1/2}\left[\pi\exp\left(\frac{\partial}{\partial N_1}-\frac{\partial}{\partial N_2}\right)N_1+\gamma(\bar{n}+1)\right]P_{n+1,m+1}+(nm)^{1/2}\left[\pi\exp\left(\frac{\partial}{\partial N_2}-\frac{\partial}{\partial N_1}\right)N_2+\gamma\bar{n}\right]P_{n-1,m-1}\\
&\quad-\{(n+m+2)\tfrac{1}{2}\pi N_2+(n+m)\tfrac{1}{2}\pi N_1+i(n-m)[\omega_b+\alpha\tfrac{1}{2}\pi(N_2-N_1)]+\gamma\bar{n}(n+m+1)+\tfrac{1}{2}\gamma(n+m)\}P_{n,m}\\
&\quad+\left[\exp\left(-\frac{\partial}{\partial N_1}\right)-1\right]\left[R_1+w^{12}\exp\left(\frac{\partial}{\partial N_2}\right)N_2-\Gamma_1\exp\left(\frac{\partial}{\partial N_1}\right)N_1\right]P_{n,m}\\
&\quad+\left[\exp\left(-\frac{\partial}{\partial N_2}\right)-1\right]\left[R_2+w_{21}\exp\left(\frac{\partial}{\partial N_1}\right)N_1-\Gamma_2\exp\left(\frac{\partial}{\partial N_2}\right)N_2\right]P_{n,m}. \qquad (6.13)
\end{aligned}
$$

The form of (6.11) makes $P_{n,m}(N_1, N_2, t)$ a Poisson convolution of $P_{n,m}(\mathfrak{N}_1, \mathfrak{N}_2, t)$, and hence N_1 and N_2 in $P_{n,m}(N_1, N_2, t)$ are actually discrete rather than continuous variables. Correspondingly, in view of the identity

$$\{\exp(\pm\partial/\partial N_j)\}f(N_j)=f(N_j\pm1), \qquad (6.14)$$

where f represents an arbitrary function (to see this, expand the right-hand side in a Taylor series), we see that (6.13) is in fact a difference equation in N_1 and N_2. Note that (6.13) keeps accurate account of the atomic-population changes. For example, the first term on the right-hand side gives the increase in $P_{n,m}$ due to absorption processes, and the absorption which is due to the maser atoms originates from a state in which N_1 is larger by 1, and N_2 is smaller by 1. The diagonal ($m=n$) terms of (6.13) form precisely the equation one would intuitively derive from energy flow considerations (rate equations). This last is the type of equation proposed and solved in the limit of no saturation by Shimoda, Takahasi, and Townes.[8] Equation (6.13) has also been derived by Lax[3] as an extrapolation of his results obtained by Langevin methods.

As a final exercise we return to (6.10) to effect the adiabatic elimination of $\mathfrak{N}_1$ and $\mathfrak{N}_2$. The procedure here differs somewhat from that of Appendix B, but the principle is the same. We seek instantaneous statistical equations for $\mathfrak{N}_1$ and $\mathfrak{N}_2$ which eliminate derivatives with respect to $\mathfrak{N}_1$ and $\mathfrak{N}_2$ from the equation of motion. In the case of (6.10), this involves setting the two curly bracketed expressions following $\partial/\partial\mathfrak{N}_1$ and $\partial/\partial\mathfrak{N}_2$ equal to zero and solving the two resulting difference equations simultaneously for $\mathfrak{N}_1P_{n,m}$ and $\mathfrak{N}_2P_{n,m}$. The solution is quite simple if conditions are assumed such that we can neglect w_{12} and w_{21} with respect to Γ_1 and Γ_2. In this case we have to solve simultaneously the two equations

$$
\begin{aligned}
&[R_1-(\Gamma_1+\tfrac{1}{2}\pi(n+m)-i\alpha\tfrac{1}{2}\pi(n-m))\mathfrak{N}_1]P_{n,m}+(nm)^{1/2}\pi\mathfrak{N}_2P_{n-1,m-1}=0\\
&\text{and}\\
&[R_2-(\Gamma_2+\tfrac{1}{2}\pi(n+m)+i\alpha\tfrac{1}{2}\pi(n-m))\mathfrak{N}_2]P_{n-1,m-1}+(nm)^{1/2}\pi\mathfrak{N}_1P_{n,m}=0, \qquad (6.15)
\end{aligned}
$$

where in the second equation we have reduced the indices by 1 for convenience. The solutions to these two equations are

$$
\begin{aligned}
\mathfrak{N}_1P_{n,m}&=D_{n,m}^{-1}\{R_1[\Gamma_2+\tfrac{1}{2}\pi(n+m)+i\alpha\tfrac{1}{2}\pi(n-m)]P_{n,m}+R_2\pi(nm)^{1/2}P_{n-1,m-1}\},\\
\mathfrak{N}_2P_{n-1,m-1}&=D_{n,m}^{-1}\{R_2[\Gamma_1+\tfrac{1}{2}\pi(n+m)-i\alpha\tfrac{1}{2}\pi(n-m)]P_{n-1,m-1}+R_1\pi(nm)^{1/2}P_{n,m}\}, \qquad (6.16)
\end{aligned}
$$

where

$$D_{n,m}=\Gamma_1\Gamma_2+(\Gamma_1+\Gamma_2)\tfrac{1}{2}\pi(n+m)+(1+\alpha^2)[\tfrac{1}{2}\pi(n-m)]^2+i\alpha\tfrac{1}{2}\pi(\Gamma_1-\Gamma_2)(n-m).$$

Inserting (6.16) in (6.10), we find all dependence on $\mathfrak{N}_1$ and $\mathfrak{N}_2$ has disappeared. We may then integrate the $P_{n,m}$ over $\mathfrak{N}_1$ and $\mathfrak{N}_2$, which simply replaces $P_{n,m}$ by the n, m matrix element of the reduced field density operator. Carrying this out and collecting terms, we obtain

$$
\begin{aligned}
&(d/dt)\langle n\mid\rho_b\mid m\rangle\\
&=[(n+1)(m+1)]^{1/2}[\gamma(\bar{n}+1)+\pi D_{n+1,m+1}^{-1}R_1\Gamma_2]\langle n+1\mid\rho_b\mid m+1\rangle+(nm)^{1/2}[\gamma\bar{n}+\pi D_{n,m}^{-1}R_2\Gamma_1]\langle n-1\mid\rho_b\mid m-1\rangle\\
&\quad-\{D_{n+1,m+1}^{-1}R_2[\Gamma_1\tfrac{1}{2}\pi\{(n+m+2)+i\alpha(n-m)\}+(1+\alpha^2)(\tfrac{1}{2}\pi(n-m))^2]\\
&\quad+D_{n,m}^{-1}R_1[\Gamma_2\tfrac{1}{2}\pi\{(n+m)-i\alpha(n-m)\}+(1+\alpha^2)(\tfrac{1}{2}\pi(n-m))^2]\\
&\quad+[\gamma\bar{n}(n+m+1)+\tfrac{1}{2}\gamma(n+m)+i\omega_b(n-m)]\}\langle n\mid\rho_b\mid m\rangle. \qquad (6.17)
\end{aligned}
$$

Specialized to the case considered by Scully and Lamb,[5] i.e., with $\bar{n}=R_1=0$, Eq. (6.17) can be shown to be identical to their result. In our notation, their quantity $R_{n,n'}$ is

$$R_{n,n'}=D_{n+1,n'+1}^{-1}R_2\tfrac{1}{2}\pi[\Gamma_1(1+i\alpha)+\tfrac{1}{2}\pi(n-n')(1+\alpha^2)].$$

If one makes the substitutions (6.12) in (6.16), one obtains adiabatic solutions for N_1 and N_2. These solutions, substituted into (6.13), lead also to (6.17). However, it is clearly easier to see how to accomplish the adiabatic elimination of the atomic populations from (6.10) than from (6.13).

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APPENDIX A: PROPERTIES OF THE ATOMIC OPERATORS

We will have use for the atomic operators only in pairs such as $(\mathbf{a}_j^\dagger\mathbf{a}_k)_m$, which serves to transfer the mth atom from state k to state j. Thus, the total number of atoms is conserved. A convenient basis for matrix representation of such pairs is the set of states with the atom concerned definitely in one particular state. The properties of the atomic operators are specified by

$$(\mathbf{a}_j^\dagger\mathbf{a}_k)_m\,|\,p,m\rangle=|\,j,m\rangle\delta_{k,p}, \tag{A1}$$

where δ is the Kronecker delta, and $|\,p,m\rangle$ represents the normalized pth state of the mth atom. Pairs of operators referring to different atoms commute. From (A1), we can derive the useful relations

$$(\mathbf{a}_j^\dagger\mathbf{a}_k)_m(\mathbf{a}_p^\dagger\mathbf{a}_q)_m=(\mathbf{a}_j^\dagger\mathbf{a}_q)_m\delta_{k,p} \tag{A2}$$

and

$$\mathrm{Tr}(\mathbf{a}_j^\dagger\mathbf{a}_k)_m=\delta_{j,k}. \tag{A3}$$

Thus, any product of pairs of operators referring to the same atom can be reduced to a single pair. Note that

$$[\sum_{j,k}A_{jk}(\mathbf{a}_j^\dagger\mathbf{a}_k)_m][\sum_{p,q}B_{pq}(\mathbf{a}_p^\dagger\mathbf{a}_q)_m]=\sum C_{jq}(\mathbf{a}_j^\dagger\mathbf{a}_q)_m,$$

where

$$C_{jq}=\sum_k A_{jk}B_{kq}.$$

Hence the pairs of atomic operators may be thought of simply as identifying positions in a matrix.

APPENDIX B: ADIABATIC ELIMINATION

Consider a Fokker–Planck or diffusion equation giving the time evolution of a statistical weight distribution in two variables, say x and y, of the general form

$$\frac{\partial P(x,y,t)}{\partial t}=\left\{-\frac{\partial}{\partial x}A_x-\frac{\partial}{\partial y}A_y+\frac{\partial^2}{\partial x^2}D_{xx}+\frac{\partial^2}{\partial y^2}D_{yy}+\frac{\partial^2}{\partial x\partial y}2D_{xy}\right\}P, \tag{B1}$$

where all partial derivatives on the right-hand side operate on P and where the drift coefficients A_x, A_y and the diffusion coefficients D_{xx}, D_{yy}, D_{xy} may be arbitrary functions of x and y. The drift coefficients have dimensions of velocity in the x–y plane, and prescribe the motion of the mean values of the variables according to

$$\begin{aligned}(d/dt)\langle x\rangle&=\langle A_x\rangle,\\(d/dt)\langle y\rangle&=\langle A_y\rangle,\end{aligned} \tag{B2}$$

where the angular brackets indicate the statistical mean of the included quantity. More generally, if $f(x,y)$ is any function of x and y, then (B1) leads to and is equivalent to the equation

$$\frac{d}{dt}\langle f(x,y)\rangle=\left\langle\left\{A_x\frac{\partial}{\partial x}+A_y\frac{\partial}{\partial y}+D_{xx}\frac{\partial^2}{\partial x^2}+D_{yy}\frac{\partial^2}{\partial y^2}+2D_{xy}\frac{\partial^2}{\partial x\partial y}\right\}f\right\rangle, \tag{B3}$$

where

$$\langle f(x,y)\rangle\equiv\int^{(2)}f(x,y)P(x,y,t)\,dx\,dy. \tag{B4}$$

Equation (B3) is derived by taking the time derivative of (B4), inserting (B1), and then integrating by parts with the assumption that P is sufficiently well behaved that all surface integrals vanish. In (B3) the derivatives operate on f but not on P. Equations (B2) are special cases of (B3) with $f(x,y)$ set equal to x and y, respectively.

The variable x, say, may be adiabatically eliminated from (B1) if conditions are such that, given any y distribution in some range under consideration, the x distribution has some conditional equilibrium to which it relaxes before the y distribution can change appreciably. Then the x distribution can be assumed to remain adiabatically in its conditional equilibrium and an equation of motion of the y distribution alone obtained.

If we assume that the x distribution is continually in conditional equilibrium then the mean of any function of x and y must be expressible as the mean of some function of y alone. We can formalize this statement in a useful way by requiring satisfaction of a relation of the form

$$\langle xf(x,y)\rangle=\langle af+b_x(\partial f/\partial x)+b_y(\partial f/\partial y)\rangle, \tag{B5}$$

where $f(x,y)$ is again an arbitrary function of x and y, and the three coefficients a, b_x, and b_y are functions of y alone. Special cases of (B5) are

$$\begin{aligned}\langle x\rangle&=\langle a\rangle,\\\langle xy\rangle&=\langle ay+b_y\rangle,\\\langle x^2\rangle&=\langle ax+b_x\rangle,\\&=\langle a^2+b_y(\partial a/\partial y)+b_x\rangle.\end{aligned} \tag{B5′}$$

In the last example we have had to make use of (B5) twice to eliminate x completely. Thus a is the mean value of x conditional on some particular value of y, b_y provides for correlation between the x and y distributions, while b_x provides for additional variance of the x distribution. Equation (B5) can be used to eliminate x from any function of x and y, because its right-hand side contains one less power of x than does its left. Actually, (B5) is a normally adequate approximation to a more general relation of the same type but including all orders of derivatives.

By partial integration we can transform (B5) to the form

$$\int^{(2)} f(x,y)\left\{x-a+\frac{\partial}{\partial x}b_x+\frac{\partial}{\partial y}b_y\right\}P(x,y,t)\,dx\,dy=0, \tag{B6}$$

where the derivatives operate on P. Because f is an arbitrary function we may extract the integrand, giving an an equivalent to (B5) the relation

$$f(x,y)xP(x,y,t)=f(x,y)\left\{a-\frac{\partial}{\partial x}b_x-\frac{\partial}{\partial y}b_y\right\}P. \tag{B7}$$

We use (B7) to eliminate x from the drift and diffusion coefficients of (B1), and then pull all the resulting derivatives to the left to re-establish the Fokker–Planck form. Generally this process introduces derivatives higher than the second into the result, but if the diffusion approximation is adequate, we can throw away such terms. [If such terms are not negligible, we must go back and extend (B5) to include higher derivatives also.] We then establish values of the three coefficients a, b_x, b_y, and thus establish the adiabatic x distribution, by solving this resulting equation for steady state while ignoring the motion of the y distribution, i.e., ignoring the $\partial/\partial y$ and $\partial^2/\partial y^2$ terms. Hence we set the coefficients of the $\partial/\partial x$, $\partial^2/\partial x^2$, and $\partial^2/\partial x\partial y$ terms equal to zero and solve for a, b_x, and b_y. This eliminates x from everything except P, which we are now free to integrate over x, thus achieving the goal of an equation in y alone.

If the drift coefficients are linear in x while the diffusion coefficients are independent of x, as is so in the elimination of $\mathfrak{M}$ from (4.4) (the extension to more variables is quite straightforward) then our method does not introduce any higher derivatives and so is "exact." As a general example of this sort, suppose that

$$\begin{aligned} A_x&=B-\Gamma x,\\ A_y&=K+Mx, \end{aligned} \tag{B8}$$

where B, Γ, K, and M are functions of y alone. The quantity Γ^{-1} is the pertinent relaxation time for the x distribution and must be positive. Then our method results in the solution

$$\begin{aligned} a&=\Gamma^{-1}[B-\Gamma^{-1}(d\Gamma/dy)\,(2D_{xy}+(M/\Gamma)D_{xx})],\\ b_x&=D_{xx}/\Gamma,\\ b_y&=\Gamma^{-1}(2D_{xy}+(M/\Gamma)D_{xx}),\\ \frac{\partial P(y,t)}{\partial t}&=\left\{-\frac{\partial}{\partial y}\left[K+M\frac{B}{\Gamma}-\frac{d(M/\Gamma)}{dy}\left(2D_{xy}+\frac{M}{\Gamma}D_{xx}\right)\right]+\frac{\partial^2}{\partial y^2}\left(D_{yy}+\frac{M}{\Gamma}2D_{xy}+\frac{M^2}{\Gamma^2}D_{xx}\right)\right\}P, \end{aligned}$$

where

$$P(y,t)=\int P(x,y,t)\,dx. \tag{B9}$$

Results obtained by this method correspond precisely with those obtained by careful application of the Langevin method discussed by Lax.[3] We have included this discussion because it was not at first clear how to accomplish the adiabatic elimination directly in a Fokker–Planck equation.

APPENDIX C. EVALUATION OF TRACE ($\sigma\mathbf{Q}$)

We need to evaluate [see (5.3)]

$$\mathrm{Tr}\sigma\mathbf{Q}, \tag{C1}$$

where

$$\mathbf{Q}=\exp i\{\lambda^*\mathbf{b}+\lambda\mathbf{b}^\dagger+\xi_1\mathbf{N}_1+\xi_2\mathbf{N}_2+\xi_3\mathbf{M}+\xi_4\mathbf{M}^\dagger\},\quad(\xi_4=\xi_3^*); \tag{C2}$$

and σ is the product density operator

$$\sigma=\sigma_b\prod_{m=1}^{N}\sigma_m \tag{C3}$$

[see (3.5), (3.6)]. Since $\mathbf{b}$ and $\mathbf{b}^\dagger$ commute with all the atomic operators, and since pairs of atomic operators pertaining to different atoms also commute, (C1) may be factored; i.e.,

$$\mathrm{Tr}\sigma\mathbf{Q}=[\mathrm{Tr}\sigma_b\mathbf{Q}_b][\mathrm{Tr}\sigma_m\mathbf{Q}_m]^N, \tag{C4}$$

where

$$\mathbf{Q}_b=\exp i(\lambda^*\mathbf{b}+\lambda\mathbf{b}^\dagger) \tag{C5}$$

is the part of $\mathbf{Q}$ referring to the field, and

$$\mathbf{Q}_m=\exp i\{\xi_1(\mathbf{a}_1^\dagger\mathbf{a}_1)_m+\xi_2(\mathbf{a}_2^\dagger\mathbf{a}_2)_m+\xi_3(\mathbf{a}_1^\dagger\mathbf{a}_2)_m+\xi_4(\mathbf{a}_2^\dagger\mathbf{a}_1)_m\} \tag{C6}$$

is the part of $\mathbf{Q}$ referring to the mth atom. In (C4) we

have used the fact that Tr $\sigma_m \mathbf{Q}_m$ is independent of the particular atom, i.e., of m.

First, consider the field part: we have

$$\begin{aligned}\mathrm{Tr}\sigma_b \mathbf{Q}_b &= \langle\beta \mid \exp i(\lambda^* \mathbf{b} + \lambda \mathbf{b}^\dagger) \mid \beta\rangle, \\ &= \exp(-\tfrac{1}{2}\lambda\lambda^*)\,\langle\beta \mid (\exp i\lambda \mathbf{b}^\dagger)(\exp i\lambda \mathbf{b}) \mid \beta\rangle, \\ &= \exp(i\lambda^*\beta + i\lambda\beta^* - \tfrac{1}{2}\lambda\lambda^*), \end{aligned} \tag{C7}$$

where we have first reordered[15] the exponential to take advantage of the fact that $\mid\beta\rangle$ is an eigenstate of $\mathbf{b}$ with eigenvalue β.

The atoms part of $\mathbf{Q}$ is slightly more complicated. By virtue of (A2), we can reduce $\mathbf{Q}_m$ to a linear sum of pairs of atomic operators, plus unity. This may be done efficiently by rewriting (C6) in the form

$$\mathbf{Q}_m = \exp i\{\xi_+ \mathbf{n}_+ + \xi_- \mathbf{n}_- + \xi_3 \mathbf{a}_1^\dagger \mathbf{a}_2 + \xi_4 \mathbf{a}_2^\dagger \mathbf{a}_1\}, \tag{C8}$$

where

$$\begin{aligned}\mathbf{n}_+ &\equiv \mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{a}_1^\dagger \mathbf{a}_1, \\ \mathbf{n}_- &\equiv \mathbf{a}_2^\dagger \mathbf{a}_2 - \mathbf{a}_1^\dagger \mathbf{a}_1, \\ \xi_+ &\equiv \tfrac{1}{2}(\xi_2 + \xi_1), \\ \xi_- &\equiv \tfrac{1}{2}(\xi_2 - \xi_1). \end{aligned}$$

Now by virtue of the relations [easily derived from (A2)]

$$\mathbf{n}_+ \mathbf{O} = \mathbf{O}\mathbf{n}_+ = \mathbf{O}, \tag{C9}$$

where $\mathbf{O}$ may be any of the four operators in the exponential of (C8), and

$$(\xi_- \mathbf{n}_- + \xi_3 \mathbf{a}_1^\dagger \mathbf{a}_2 + \xi_4 \mathbf{a}_2^\dagger \mathbf{a}_1)^2 = v^2(\mathbf{n}_+), \tag{C10}$$

where

$$v^2 = \xi_-^2 + \xi_3 \xi_4.$$

we can easily reduce $\mathbf{Q}_m$ through the series of self-explanatory steps

$$\begin{aligned}\mathbf{Q}_m &= (\exp i\xi_+ \mathbf{n}_+)(\exp i\{\xi_- \mathbf{n}_- + \xi_3 \mathbf{a}_1^\dagger \mathbf{a}_2 + \xi_4 \mathbf{a}_2^\dagger \mathbf{a}_1\}) \\ &= \{1 + [\exp(i\xi_+) - 1]\mathbf{n}_+\}[1 + (\cos v - 1)\mathbf{n}_+ + i(\sin v/v)(\xi_- \mathbf{n}_- + \xi_3 \mathbf{a}_1^\dagger \mathbf{a}_2 + \xi_4 a_2^\dagger a_1)] \\ &\quad 1 + [\exp(i\xi_+)\cos v - 1]\mathbf{n}_+ + i\exp(i\xi_+)(\sin v/v)(\xi_- \mathbf{n}_- + \xi_3 \mathbf{a}_1^\dagger \mathbf{a}_2 + \xi_4 \mathbf{a}_2^\dagger \mathbf{a}_1). \end{aligned} \tag{C11}$$

We can now evaluate Tr $(\sigma_m \mathbf{Q}_m)$, with σ_m given by (3.3). Using (A2) and (A3), we obtain

$$\mathrm{Tr}\sigma_m \mathbf{Q}_m = \{1 + [\exp(i\xi_+)\cos v - 1](\mathfrak{N}_+/N) + i\exp(i\xi_+)(\sin v/vN)(\xi_- \mathfrak{N}_- + \xi_3 \mathfrak{M} + \xi_4 \mathfrak{M}^*)\}, \tag{C12}$$

where

$$\mathfrak{N}_+ \equiv \mathfrak{N}_2 + \mathfrak{N}_1; \qquad \mathfrak{N}_- \equiv \mathfrak{N}_2 - \mathfrak{N}_1.$$

The atoms part of (C4) is (C12) raised to the Nth power; in the limit of large N [see (3.12)] we obtain

$$[\mathrm{Tr}\sigma_m \mathbf{Q}_m]^N = \exp\{[\exp(i\xi_+)\cos v - 1]\mathfrak{N}_+ + i\exp(i\xi_+)(\sin v/v)(\xi_- \mathfrak{N}_- + \xi_3 \mathfrak{M} + \xi_4 \mathfrak{M}^*)\}. \tag{C13}$$

Equations (C13) and (C7) combine according to (C4) to yield the desired result

$$\mathrm{Tr}\sigma \mathbf{Q} = \exp\{-\tfrac{1}{2}(\lambda\lambda^*) + i(\lambda^*\beta + \lambda\beta^* + \textstyle\sum \nu_j \mathfrak{X}_j)\}, \tag{C14}$$

where the ν_j are as listed in (5.4b) of the text.

APPENDIX D: EVALUATION OF P_2 FROM P_1 IN THE GAUSSIAN APPROXIMATION

In the approximation, valid for large $\mathfrak{N}_1 + \mathfrak{N}_2$, that the ν_j in (5.4) need only be expanded to second power in the ξ_j, we obtain a direct solution for P_2 in terms of P_1. We have from (5.5), the transform relationship

$$\begin{aligned}\int P_2(b, b^*, X_j, t)\,\exp[i(\lambda^* b + \lambda b^* + \textstyle\sum \xi_j X_j)]\, d^{(2)}b\, d^{(4)}X \\ = \int P_1(\beta, \beta^*, \mathfrak{X}_j, t)\,\exp\{[-\tfrac{1}{2}\lambda\lambda^* + i(\lambda^*\beta + \lambda\beta^* + \textstyle\sum \nu_j \mathfrak{X}_j)]\}\, d^{(2)}\beta\, d^{(4)}\mathfrak{X}, \end{aligned} \tag{D1}$$

where the ν_j and the ξ_j are related by (5.6), i.e.,

$$\begin{aligned} i\nu_1 &= i\xi_1 - \tfrac{1}{2}(\xi_1^2 + \xi_3\xi_4), \\ i\nu_2 &= i\xi_2 - \tfrac{1}{2}(\xi_2^2 + \xi_3\xi_4), \\ i\nu_3 &= i\xi_3 - \tfrac{1}{2}\xi_3(\xi_1 + \xi_2), \\ i\nu_4 &= i\xi_4 - \tfrac{1}{2}\xi_4(\xi_1 + \xi_2). \end{aligned}$$

To extract P_2 from (D1) by Fourier transformation we make ξ_1 and ξ_2 real and set $\xi_3=\xi_4^*$ to make the exponential of the left-hand side pure imaginary, then multiply (D1) by the factor

$$(1/4\pi^6)\exp\{-i(\lambda^*b'+\lambda b'^*+\sum\xi_jX_j')\}d(\mathrm{Re}\lambda)\,d(\mathrm{Im}\lambda)\,d\xi_1d\xi_2d(\mathrm{Re}\xi_4)\,d(\mathrm{Im}\xi_4)$$

and integrate over all the parameters. On the left, this extracts P_2; and so transforms (D1) to

$$P_2(b,b^*,X_j,t)=\int^{(6)}KP_1(\beta,\beta^*,\mathfrak{X}_j,t)\,d^{(2)}\beta\,d^{(4)}\mathfrak{X}, \tag{D2}$$

where the convolution kernel K is given by

$$K=\frac{1}{4\pi^6}\int^{(6)}\exp\{i[\lambda^*(\beta-b)+\lambda(\beta^*-b^*)+\sum\xi_j(\mathfrak{X}_j-X_j)]$$

$$-\tfrac{1}{2}[\lambda\lambda^*+\xi_1^2\mathfrak{N}_1+\xi_2^2\mathfrak{N}_2+\xi_4^*\xi_4(\mathfrak{N}_1+\mathfrak{N}_2)]-\tfrac{1}{2}(\xi_1+\xi_2)(\xi_4^*\mathfrak{M}+\xi_4\mathfrak{M}^*)\}d(\mathrm{Re}\lambda)\,d(\mathrm{Im}\lambda)\,d\xi_1d\xi_2d(\mathrm{Re}\xi_4)\,d(\mathrm{Im}\xi_4). \tag{D3}$$

Evaluation of the integral of (D3) is facilitated by the following linear transformation which diagonalizes the quadratic terms:

$$\xi_1=\zeta_+[2\mathfrak{N}_2/(\mathfrak{N}_1+\mathfrak{N}_2)]-\tfrac{1}{2}\zeta_-,$$

$$\xi_2=\zeta_+[2\mathfrak{N}_1/(\mathfrak{N}_1+\mathfrak{N}_2)]+\tfrac{1}{2}\zeta_-,$$

$$\xi_4=\zeta_4-\zeta_+[2\mathfrak{M}/(\mathfrak{N}_1+\mathfrak{N}_2)], \tag{D4}$$

along with the conjugate of the last. After this transformation, K factors into the product of six independent integrals, and evaluates to

$$K=\frac{2}{\pi^3(\mathfrak{N}_1+\mathfrak{N}_2)(\mathfrak{N}_1\mathfrak{N}_2-\mathfrak{M}\mathfrak{M}^*)^{1/2}}\exp\left\{-2\,|\,\beta-b\,|^2-\frac{2\,|\,\mathfrak{M}-M\,|^2}{(\mathfrak{N}_1+\mathfrak{N}_2)}-\frac{[(\mathfrak{N}_2-\mathfrak{N}_1)-(N_2-N_1)]^2}{2(\mathfrak{N}_1+\mathfrak{N}_2)}\right.$$

$$\left.-\frac{[\mathfrak{N}_2(\mathfrak{N}_1-N_1)+\mathfrak{N}_1(\mathfrak{N}_2-N_2)-\mathfrak{M}(\mathfrak{M}^*-M^*)-\mathfrak{M}^*(\mathfrak{M}-M)]^2}{2(\mathfrak{N}_1+\mathfrak{N}_2)(\mathfrak{N}_1\mathfrak{N}_2-\mathfrak{M}\mathfrak{M}^*)}\right\}. \tag{D5}$$

APPENDIX E. DERIVATION OF THE "CLASSICAL" FOKKER-PLANCK EQ. (5.7)

We start from the equation of motion (3.19) for P_1, and the relationship between P_1 and P_2 given in (5.5) along with the approximate relations (5.6) between the ν's and the ξ's. We first differentiate the right side of (5.5) with respect to time, and insert the equation of motion (3.19) for $\partial P_1/\partial t$. This yields an equation of motion for $\langle\mathbf{Q}\rangle$ [note that (5.5) equates two expressions for $\langle\mathbf{Q}\rangle$] of the form

$$\frac{d}{dt}\langle\mathbf{Q}\rangle=\int^{(6)}\exp\{-\tfrac{1}{2}(\lambda\lambda^*)+i(\lambda^*\beta+\lambda\beta^*+)\sum\nu_j\mathfrak{X}_j)\}$$

$$\times\mathfrak{F}_1\left(\frac{\partial}{\partial\beta},\frac{\partial}{\partial\beta^*},\frac{\partial}{\partial\mathfrak{X}_j};\beta,\beta^*,\mathfrak{X}_j\right)P_1d^{(2)}\beta d^{(4)}\mathfrak{X}, \tag{E1}$$

where $\mathfrak{F}_1$ stands for the complete curly bracketed Fokker-Planck operator in the right-hand side of (3.19). The semicolon in $\mathfrak{F}$ indicates that all derivatives are to the left. We integrate (E1) by parts, applying the derivatives to the exponential, where they bring down the parameters λ^*, λ, and the ν_j. Next we express the variables in $\mathfrak{F}_1$ as partial derivatives of the exponential with respect to the parameters. An example of this is

$$\beta^*\longrightarrow-i(\partial/\partial\lambda+\tfrac{1}{2}\lambda^*).$$

This procedure yields an equation of the form

$$\frac{d}{dt}\langle\mathbf{Q}\rangle=\mathfrak{L}_1\left(\lambda,\lambda^*,\nu_j;\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\lambda^*},\frac{\partial}{\partial\nu_j}\right)\langle\mathbf{Q}\rangle. \tag{E2}$$

In $\mathfrak{L}_1$, all derivatives are to the right, again indicated by the semicolon. Now we use the relations (5.6) to

express $\mathfrak{L}_1$ in terms of the ξ_j, transforming (E2) to the form

$$\frac{d}{dt}\langle Q\rangle=\mathfrak{L}_2\left(\lambda,\lambda^*,\xi_j;\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\lambda^*},\frac{\partial}{\partial\xi_j}\right)\langle Q\rangle. \quad \text{(E3)}$$

The final stage is to express $\langle Q\rangle$ as the left side of (5.5) and to reverse the procedure which led from (E1) to (E2), obtaining an equation of the form

$$\frac{d}{dt}\langle Q\rangle=\int^{(6)}\exp i(\lambda^*b+\lambda b^*+\sum\xi_jX_j)$$

$$\times\mathfrak{F}_2\left(\frac{\partial}{\partial b},\frac{\partial}{\partial b^*},\frac{\partial}{\partial X_j};b,b^*,X_j\right)P_2d^{(2)}bd^{(4)}X, \quad \text{(E4)}$$

where [see also (6.1)] we have used the notation

$$X_j(j=1,2,3,4)\equiv(N_1,N_2,M,M^*)$$

$$d^{(4)}X\equiv dN_1dN_2d(\mathrm{Re}M)d(\mathrm{Im}M).$$

In (E4), the operator $\mathfrak{F}_2$ comes out to be the complete curly bracketed Fokker–Planck operator on the right-hand side of (5.7) after all derivatives higher than the second have been discarded. Extraction of the integrand of (E4) by Fourier transform yields (5.7).

Carrying out the above procedure, we find that passage from $\mathfrak{F}_1$ to $\mathfrak{F}_2$ entails the following set of substitutions:

$$\beta\rightarrow b+\frac{1}{2}\frac{\partial}{\partial b^*},$$

$$\frac{\partial}{\partial\beta}\rightarrow\frac{\partial}{\partial b},$$

$$\mathfrak{M}\rightarrow\left[1+\frac{1}{2}\left(\frac{\partial}{\partial N_2}+\frac{\partial}{\partial N_1}\right)\right]M+\frac{1}{2}\frac{\partial}{\partial M^*}(N_1+N_2),$$

$$\frac{\partial}{\partial\mathfrak{M}}\rightarrow\frac{\partial}{\partial M}\left[1-\frac{1}{2}\left(\frac{\partial}{\partial N_2}+\frac{\partial}{\partial N_1}\right)\right],$$

$$\mathfrak{N}_j\rightarrow\left(1+\frac{\partial}{\partial N_j}\right)N_j+\frac{1}{2}\left(\frac{\partial}{\partial M}M+\frac{\partial}{\partial M^*}M^*\right),$$

$$j=1,2$$

$$\frac{\partial}{\partial\mathfrak{N}_j}\rightarrow\frac{\partial}{\partial N_j}-\frac{1}{2}\left(\frac{\partial^2}{\partial N_j^2}+\frac{\partial^2}{\partial M\partial M^*}\right)\qquad j=1,2 \quad \text{(E5)}$$

along with the conjugates of the first four. These replacements are made without disturbing the order of factors. In $\mathfrak{F}_1$, products of field quantities times atomic quantities such as $\beta^*\mathfrak{M}$ and $\beta\mathfrak{N}_1$ occur, but these give no ordering problems since their replacements commute. After the replacements, all derivatives higher than the second are discarded.

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Paramagnetic Relaxation Measurements on Ce, Nd, and Yb in $CaWO_4$ by an Electron Spin-Echo Method

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Lattice relaxation measurements have been made for Ce^{3+}, Nd^{3+}, and Yb^{3+} at axial sites in $CaWO_4$ by a novel spin-echo method. In this method the spin system is inverted by adiabatic rapid passage, and the magnetization is later sampled by a two-pulse echo sequence. Temperatures were in the range from 1.4 to 9°K, the transfer of cold helium gas being used to maintain temperatures of 4.7°K and above. The relaxation behavior of all the ions appeared to be best described by a direct process ($w\propto T$) going over into a Raman process ($w\propto T^n$) in the vicinity of 4°K. The results could not, however, be fitted to the T^9 law, which arises in simple theoretical treatments of the Raman process, but yielded powers of the absolute temperature between 10 and 11. Orbach processes did not seem to be relevant. Two causes for this behavior are considered: dispersion in the phonon spectrum, and the inadequacy of an approximation which is commonly made in treating the denominator of the Raman integral. The Yb relaxation rate showed a marked angular variation in the Raman region. This is discussed in terms of a possible admixture of the two processes which, in the simple theory, lead to T^7 and T^9 laws.

I. INTRODUCTION

We have made lattice relaxation measurements for the three Kramers doublet ions Ce^{3+}, Nd^{3+}, and Yb^{3+}, in axial sites in a $CaWO_4$ lattice. The experimental frequency was 9.4 Gc/sec, and the temperature range 1.8 to 9°K. The measurements were made by a method analogous to that often used in nuclear relaxation studies, in which the spin system is inverted and its magnetization sampled after a variable time interval by means of a two-pulse spin-echo sequence. In our experiments, an adiabatic rapid passage was used rather than a 180° pulse to invert the spins, since it was virtually impossible to realize the latter pulse condition with the broad resonance lines, which we encountered in the materials studied. The relaxation behavior of all three ions appeared to be best described by a direct process going over into a Raman process above ~4°K. Although some problems arise in this interpretation, we believe that it is essentially correct,