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Properties of Dilute Mixtures of He^3 in Liquid He^4 at Low Temperatures*

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Properties of dilute mixtures of He^3 in liquid He^4 are calculated on the assumption that, for the most part, they behave as low-density fermion systems with an effective He^3 Hamiltonian. Expressions for He^3 scattering amplitudes in terms of phase shifts are obtained, and the measured spin diffusion coefficient D for temperature T below 1°K is used to determine the effective interaction between He^3 atoms. It is found to have a strong repulsive region with a longer-range attraction which is deeper but of shorter range than the corresponding part of the Van der Waals force. At low momenta, the phase shifts are small, and at low temperatures, the effects are weak. At higher temperatures, larger momenta are probed, and the scattering is more significant. The phase-separation curve and the low-temperature viscosity, thermal conductivity, spin susceptibility, compressibility, and specific heat are calculated and agree with existing experimental evidence. It is found that the mixtures should undergo a fermion superfluid phase transition, but the maximum transition temperature is 10^{-6} °K. Evidence for the transition should be found in a 4% decrease of DT^2 between 10^{-2} ° and 2×10^{-3} °K.

I. INTRODUCTION

DILUTE mixtures of He^3 in liquid He^4 are of special interest because they provide a new class of strongly interacting, low-density fermion systems. This situation arises when the temperature is below about 1°K and the He^4 has few real excitations,[1] so that it is mainly a medium which keeps the He^3 atoms apart and modifies their Hamiltonian.[2]

It has been found[3] that solutions with He^3 concentration less than 6% do not undergo phase separation. They have a maximum degeneracy temperature of about 0.35°K, and so experiments can be carried out from the semiclassical to the Fermi-degenerate regions. The He^3 number density is low enough for good microscopic calculations to be feasible, and it may be varied over a wide range by changing the concentration.

On the other hand, the interaction between the He^3 atoms, as modified by the He^4, is not known. Average properties have been discussed,[4,5] but it has not yet proved possible to determine the shape of the interaction in detail from the coupling between the He^3 and the He^4, since a microscopic calculation is difficult and there is not enough information to put into a tractable, partly phenomenological theory.[2]

The purpose of this paper is to discuss the existing experiments, to use some of them to determine the effective He^3 Hamiltonian, and to consider some of the consequences. The general conclusion will be that the experiments are consistent with the assumption of a He^3–He^3 interaction v with a strong repulsive region and a longer-range attraction. The scattering length is about -1 Å, and although the potential is strong, its phase shifts are small at low momenta, so that its effects are weak at low temperatures. As the temperature is raised, high momenta are given increasing weight and the strength of the potential is revealed. In particular, it will be seen that this picture gives a natural explanation of the temperature dependence of the spin-diffusion coefficient in the semiclassical regime.

In comparison with the Van der Waals potential v_0 for two isolated He^3 atoms, the attractive part of v is deeper and of shorter range, although the repulsive regions are the same. The net result is that v is less attractive than v_0, which agrees with a previous estimate.[2]

Since the scattering length is negative, the potential is effectively attractive at low momenta. It follows that spin waves should propagate instead of zero sound, and there should be a fermion superfluid phase transition, although the small phase shifts lead to transition temperatures of the order of 10^{-6} °K. However, it will be shown that evidence for the transition may be found in transport coefficients at temperatures which are accessible at present. As the transition temperature T_c is approached from above, the scattering rate for pairs of particles with total momentum near to zero becomes very large and, in the limit, the transport mean-free times tend to zero. In He^3 or superconductors, this effect would be seen only very close to T_c, since it has to compete with strong scattering for total momentum not equal to zero, or with impurity scattering. In dilute mixtures, the total scattering rate is small and a large fraction of it comes from the neighborhood of total momentum zero, so that the decrease in transport coefficients becomes apparent at a relatively high temperature. For example, it will be shown that in the case of the spin diffusion coefficient D in a 1.24% mixture, for which T_c is a maximum, there is a 4% drop in DT^2 as the temperature T is decreased from 0.01 to 0.002°K.

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[1] At higher temperatures, it is possible to estimate the effects of collisions between He^3 atoms and phonons, but we wish to avoid this complication here.

[2] V. J. Emery, Phys. Rev. **148**, A138 (1966).

[3] D. O. Edwards, D. F. Brewer, P. Seligman, M. Skertic, and M. Yaqub, Phys. Rev. Letters **15**, 773 (1965).

[4] G. Baym, Phys. Rev. Letters **17**, 956 (1966).

[5] J. Bardeen, G. Baym, and D. Pines, Phys. Rev. Letters **17**, 372 (1966); Phys. Rev **156**, 207 (1967).

In Sec. II, expressions for the spin diffusion coefficient in the semiclassical region and for transport coefficients, susceptibility, effective mass, and compressibility, in the degenerate region will be given and the calculation of the chemical potential will be discussed. The framework of Landau's theory of a Fermi liquid[6] will be used, although in this case it is a convenience rather than a necessity, since it is possible to carry out good microscopic calculations of the scattering amplitudes, and in Sec. III it will be shown that, to a good approximation, they may be expressed directly in terms of free-space phase shifts. From this point of view, the weak potential of Bardeen, Baym, and Pines[5] and of Ebner[7] is a local approximation to the scattering amplitude.

In Sec. IV, for simplicity, it will be assumed that the repulsive part of $v(r)$ is a hard core. The attractive region will be represented, first, by an effective range and scattering length to obtain shape-independent information for low momenta; and second, by a square well, as a particular assumption which may be used at higher momenta also. Since the core radius is larger than the scattering length, this form of effective-range theory is significantly better than an unmodified application to the entire interaction.

The experiments which will be discussed are measurements of the phase-separation curve of Edwards, Brewer, Seligman, Skertic, and Yaqub,[3] the thermal and magnetic properties at 1.3% and 5% concentration of Anderson, Edwards, Roach, Sarwinski, and Wheatley,[8] and the sound attenuation at 5% concentration of Abraham, Eckstein, Ketterson, and Vignos.[9] The potential parameters will be determined from the spin diffusion coefficient, and then it will be shown that the predicted values of the other properties agree well with experiment. In Sec. V, the fermion superfluid phase transition and the related behavior of the transport coefficients will be considered.

II. THE TWO-PARTICLE-COLLISION APPROXIMATION

The effective interaction may be written

$$v = v_0 + v_i, \qquad (2.1)$$

where v_0 is the potential between two He^3 atoms in free space, and v_i is the change induced by the He^4 as a consequence of the fact that the interaction of each He^3 atom with the He^4 is modified by the presence of the other He^3 atom.

One form of $v_0(r)$ is

$$v_0(r) = V_0[(r_0/r)^{12} - 2(r_0/r)^6], \qquad (2.2)$$

with[10] $V_0 = 10.2°\text{K}$ and $r_0 = 2.87$ Å. For $r < r_0$, this potential is strongly repulsive and is not significantly changed by v_i. For larger r, v_i is important, and its role[2] is to make v less attractive than v_0. There is a *qualitatively* similar effective force between any pair of particles in pure He^3, pure He^4, and mixtures. In the case of dilute mixtures, it has been shown[2] that v_i is essentially independent of velocity, spin, and concentration, and is not retarded.

For momenta of interest, the energy of a single He^3 atom in the He^4 is, to a good approximation,[2] a kinetic energy with effective mass m_0^*. In Sec. IV its value will be determined from experiment, and it will turn out to be 2.33 times the mass m_3 of a bare He^3 atom.

Thus, it will be assumed that the effective He^3 Hamiltonian is

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_0^*} + \tfrac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} v(\mathbf{r}_i - \mathbf{r}_j). \qquad (2.3)$$

If n_3 is the number density of the He^3, and R is the range of the force, then $n_3^{1/3} R$ is less than about $\frac{1}{3}$ for He^3 concentrations less than 6%, and the properties of the system may be calculated in the two-body-collision approximation. The situation is very similar to nuclear matter, for which it is known that the importance of three-body collisions is not large.[11] The approximation is even better for a mixture, since the third He^3 atom has to displace a He^4 atom whose effect is included in v_i. The error, then, is the difference in v_i caused by substituting a He^3 atom for a He^4 atom in an already rare three-body collision, and this should be quite small.

A. The Semiclassical Region

When the temperature T is greater than the degeneracy temperature T_F, transport coefficients may be calculated with Boltzmann distribution functions and quantum-mechanical scattering cross sections. The spin diffusion coefficient D is given by[12]

$$D = (3/8n_3 m_0^*)(k_B T/\Omega_{12}^{(1,1)}), \qquad (2.4)$$

where $\Omega_{12}^{(1,1)}$ is a special case of

$$\Omega_{12}^{(n,t)} = \left(\frac{k_B T}{\pi m_0^*}\right)^{1/2} \int_0^\infty d\gamma\, e^{-\gamma^2} \gamma^{2t+3} Q^{(n)}(\gamma), \qquad (2.5)$$

and

$$Q^{(1)}(\gamma) = (4\pi\hbar^2/m_0^* k_B T) \sum_l (l+1) \sin^2(\delta_{l+1} - \delta_l). \qquad (2.6)$$

The notation is standard in the kinetic theory of gases, and δ_l is the phase shift for scattering at the appropriate relative kinetic energy $\gamma^2 k_B T$ and angular momentum l. (k_B is Boltzmann's constant and $\hbar$ is Planck's constant

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[6] A. A. Abrikosov and I. M. Khalatnikov, Rept. Progr. Phys. **22**, 329 (1959).

[7] C. Ebner, Phys. Rev. **156**, 222 (1967).

[8] A. C. Anderson, D. O. Edwards, W. R. Roach, R. E. Sarwinski, and J. C. Wheatley, Phys. Rev. Letters **17**, 367 (1966).

[9] B. M. Abraham, Y. Eckstein, J. B. Ketterson, and J. H. Vignos, Phys. Rev. Letters **17**, 1254 (1966).

[10] J. de Boer, Physica Suppl. **24**, 90 (1958).

[11] H. A. Bethe, Phys. Rev. **138**, B804 (1965).

[12] V. J. Emery, Phys. Rev. **133**, A661 (1964).

divided by 2π.) The leading correction for Fermi statistics is of order $(T_F/T)^3$, and it becomes important when $T \lesssim 2T_F$.

The corresponding equations for the viscosity η and thermal conductivity κ will not be written here, since the degeneracy correction is of order[13] $(T/T_F)^{3/2}$. (This order vanishes identically for the spin diffusion.) At present, κ has not been measured and the only experiment from which η may be determined concerns a 5% mixture for which T_F is 0.33°K, so that the nondegenerate limit applies only above 1°K, where scattering from real phonons must be taken into account. This introduces an additional uncertainty, which we do not wish to consider at present.

B. The Degenerate Region

When $T \ll T_F$, it is convenient to use the framework of Landau's theory of a Fermi liquid, since much of the formal manipulation has been carried out already.

The transport coefficients are given by[6,14]

$$D=(\chi_0/\chi)(\tfrac{1}{3}v_F^2\tau_D), \tag{2.7}$$

$$\eta=\tfrac{3}{5}n_3m^*(\tfrac{1}{3}v_F^2\tau_\eta), \tag{2.8}$$

$$\kappa=C_v(\tfrac{1}{3}v_F^2\tau_\kappa), \tag{2.9}$$

where χ and χ_0 are, respectively, the spin susceptibilities of the mixture and the corresponding ideal gas, m^* is the effective mass at concentration x (as modified by interactions), C_v is the specific heat, and v_F is the Fermi velocity. The τ's are mean free times given by

$$\hbar/\tau_D=(k_BT^2/T_F)W^{(1,1)}, \tag{2.10}$$

$$\hbar/\tau_\eta=(k_BT^2/T_F)\tfrac{3}{2}W_s^{(2,2)}, \tag{2.11}$$

$$\hbar/\tau_\kappa=(k_BT^2/T_F)\tfrac{2}{3}W_s^{(0,1)}, \tag{2.12}$$

where

$$W^{(n,t)}=\left(\frac{m^*k_F}{4\pi\hbar^2}\right)^2\int_0^{2\pi}d\phi\,(1-\cos^n\phi)\times\int_0^{k_F}\frac{dk}{k_F}\frac{(k/k_F)^{2t+1}}{(1-k^2/k_F^2)^{1/2}}\,|\,A_E(k,\phi)+A_O(k,\phi)\,|^2, \tag{2.13}$$

and $W_s^{(nt)}$ is given by the same expression with $|\,A_E+A_O\,|^2$ replaced by $(^1A_E\,|^2+3\,|\,A_O\,|^2)$. Here $\hbar k_F$ is the Fermi momentum and $A_E(k,\phi)$ and $A_O(k,\phi)$ are even- and odd-state scattering amplitudes for two particles on the Fermi surface with relative momentum $\hbar\mathbf{k}$ and scattering angle ϕ.

Using the relation[6] between $A_E(k,\phi)$, $A_O(k,\phi)$ and Landau's forward scattering amplitude f, χ and m^* may be written[6]

$$\frac{\chi}{\chi_0}=1+\left(\frac{m^*}{k_F\pi^2}\right)\int_0^{k_F}dk\,k[A_E(k,0)-A_O(k,0)], \tag{2.14}$$

$$\frac{m_0^*}{m^*}=1-\left(\frac{m^*}{k_F\pi^2}\right)\int_0^{k_F}dk\,k\left(1-\frac{2k^2}{k_F^2}\right)\times[A_E(k,0)+3A_O(k,0)]. \tag{2.15}$$

The specific heat may be obtained[6] from m^*. The isothermal compressibility κ_T, calculated as if H were the entire Hamiltonian, is given by

$$\kappa_T=\frac{3m^*}{n_3\hbar^2k_F^2}\times\left\{1-\left(\frac{m^*}{k_F\pi^2}\right)\int_0^{k_F}dk\,k[A_E(k,0)+3A_O(k,0)]\right\}, \tag{2.16}$$

and is used in the discussion of zero sound.[6]

These expressions are the leading terms of an expansion in powers of T/T_F. In Sec. III, the scattering amplitudes $A_E(k,\phi)$ and $A_O(k,\phi)$ will be determined from the effective He^3 Hamiltonian.

C. Phase Separation Curve and the Effective He^3 Chemical Potential

In a two-phase mixture, the chemical potentials of the He^3 atoms in each phase must be equal. At very low temperatures, the He^3-rich phase is essentially pure He^3, and its chemical potential is known from experiment; thus the phase separation curve can be determined from a calculation of the chemical potential of the He^4-rich phase.

For this purpose, it is necessary to take account of the energy of interaction of the He^3 atoms with the He^4, which is not included in H in Eq. (2.3), and to remember that the measurements of the phase separation curves[3] have been carried out at essentially constant pressure so that the total number density n will vary with concentration.[4]

Now the assumption upon which the use of the Fermi-liquid analysis is based is that the temperature dependence of the free energy comes mainly from the contribution of the effective Hamiltonian H. Then

$$\mu_3(n,T,x)=\mu_3(n,0,x)+\tilde{\mu}_3(n_3,T)-\tilde{\mu}_3(n_3,0), \tag{2.17}$$

where $\tilde{\mu}_3(n_3,T)$ is the chemical potential calculated from H, and n_3 is nx. In a low-density system, $\tilde{\mu}_3(n_3,0)$ may be written in terms of a single-particle energy[15]

$$e_k=(k^2/2m_0^*)+V_k, \tag{2.18}$$

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[13] J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, *Molecular Theory of Gases and Liquids* (John Wiley & Sons, Inc., New York, 1954).

[14] D. Hone, Phys. Rev. **121**, 669 (1961).

[15] K. A. Brueckner, in *The Many-Body Problem*, edited by C. DeWitt and P. Nozieres (John Wiley & Sons, Inc., New York, 1959).

as

$$\tilde{\mu}_3(n_3, 0) = e_{k_F}$$
$$= E_F + V_0, \tag{2.19}$$

which defines the Fermi energy E_F.

For low concentrations, it is sufficient to calculate $\mu_3(n, 0, x)$ to lowest order in x. Since E_F is proportional to $x^{2/3}$, it will be necessary to expand $\mu_3(n, 0, x) - E_F$. Let the total free energy per particle be $f(n, T, x)$. The pressure P is given by

$$P = n^2(\partial f/\partial n)_{T,x}, \tag{2.20}$$

and

$$\mu_3(n, T, x) = [(\partial/\partial N_3) Nf]_{T,N_4}$$
$$= f + (1-x)(\partial f/\partial x)_{T,n} + P/n, \tag{2.21}$$

where N_3 and N_4 are, respectively, the total number of He^3 and He^4 atoms and $N = N_3 + N_4$. From Eqs. (2.20) and (2.21) and the chain rule

$$(\partial P/\partial x)_{n,T} = -(\partial P/\partial n)_{T,x}(\partial n/\partial x)_{P,T}, \tag{2.22}$$

it follows that

$$\lim_{x\to 0}[(\partial \mu_3(n, 0, x)/\partial x)_P - (\partial E_F/\partial x)_P]$$
$$= [(\partial^2 f/\partial x^2)_{n,T} - n(\partial E_F/\partial n_3) - (1/n^2)$$
$$\times (\partial P/\partial n)_{T,x}(\partial n/\partial x)_{P,T}^{2}]_{T=0,x=0}. \tag{2.23}$$

Now in the limit $T \to 0$, $x \to 0$, $(\partial^2 f/\partial x^2)_{n,T} - n(\partial E_F/\partial n_3)$ is the same as $(\partial^2 E_B/\partial x^2)_n$, where E_B is the total ground-state energy calculated as if the He^3 atoms were bosons. If E_B is estimated from the variation principle using the ground state of N He^4 atoms as a trial function,[4] then $(\partial^2 E_B/\partial x^2)_n$ is zero, so that it is likely to be small in a more accurate calculation, and it will be neglected. The phase separation curve calculated on this assumption agrees with experiment.

Using Eq. (2.23) and expressing $(\partial P/\partial n)_{T,x}$ in terms of the mass m_4 and sound velocity c of He^4, Eq. (2.17) may be written

$$\mu_3(n, T, x) \approx \mu_3(n_0, 0, 0)$$
$$- (x m_4 c^2/n_0^2)(\partial n/\partial x)_{P,T}^{2}|_{T=0,x=0}$$
$$+ E_F + \tilde{\mu}_3(n_3, T) - \tilde{\mu}_3(n_3, 0). \tag{2.24}$$

It turns out that with less than about 3% error, V_k is quadratic in k for $k < k_F$ and E_F is $k_F^2/2m^*$. Also, in the range of temperatures of interest $(T \lesssim T_F/2)$, the temperature dependence of m^* is insignificant; therefore, to a good approximation, the last three terms on the right-hand side of Eq. (2.24) are the chemical potential of an ideal gas with mass m^* calculated for the appropriate concentration. For small x, this could be expanded in powers of x, but it is simpler not to do so. All other terms in Eq. (2.24) are known from experiment,[4] except for $\mu_3(n, 0, 0)$, which will be chosen as a single parameter to fit the phase separation curve.

The existence of the second term on the right-hand side of Eq. (2.24) in the variation of μ_3 with x at constant P was first pointed out by Baym,[4] although it has been used in a rather different manner here.

III. CALCULATION OF SCATTERING AMPLITUDES

In the two-body-collision approximation,[16]

$$A_E(k, \phi) = (4\pi/k^2) \sum_{(l\ \text{even})} (2l+1) P_l(\cos\phi)$$
$$\times \langle k, l \,|\, v \,|\, \psi(k, l) \rangle, \tag{3.1}$$

and $A_O(k, \phi)$ is given by a similar expression with the sum carried over odd values of l. For $k < k_F$ and $k_F < 0.35$ Å, the d-state impact parameter lies outside the range of the force, and so s and p states dominate. In these circumstances, it has been shown[16] that the equations determining $|\psi(k, l)\rangle$ decouple, and they may be rearranged to give

$$\langle k, l \,|\, v \,|\, \psi(k, l) \rangle = \frac{\langle k, l \,|\, v \,|\, \psi_0(k, l) \rangle}{1 + \bar{\Lambda}(k, l) \langle k, l \,|\, v \,|\, \psi_0(k, l) \rangle}. \tag{3.2}$$

Here,

$$\langle r \,|\, k, l \rangle = krj_l(kr), \tag{3.3}$$

with $j_l(kr)$ the spherical Bessel function of order l, and $|\psi_0(k, l)\rangle$ is the solution of

$$|\psi_0(k, l)\rangle = |k, l\rangle + G_0(k, l) v \,|\, \psi_0(k, l)\rangle, \tag{3.4}$$

with

$$G_0(k, l) = -\frac{2l+1}{2\pi^2} \int d\mathbf{k}' \frac{\Lambda_K(k')[P_l(\hat{k}' \cdot \hat{k})]^2}{k'^2}$$
$$\times (|k', l\rangle\langle k' l| - |k, l\rangle\langle k, l|) \tag{3.5}$$

and

$$\Lambda_K(\mathbf{k}) = \frac{1}{2} \frac{\tanh\frac{1}{2}\beta\epsilon_{1/2\mathbf{K}+\mathbf{k}} + \tanh\frac{1}{2}\beta\epsilon_{1/2\mathbf{K}-\mathbf{k}}}{\epsilon_{(1/2)\mathbf{K}+\mathbf{k}} + \epsilon_{(1/2)\mathbf{K}-\mathbf{k}}}. \tag{3.6}$$

Finally,

$$\bar{\Lambda}(k, l) = \frac{2l+1}{2\pi^2} \int \frac{d\mathbf{k}'}{k'^2} \Lambda_K(\mathbf{k}')(P_l(\hat{k}' \cdot \hat{k}))^2. \tag{3.7}$$

Here $\frac{1}{2}\mathbf{K} \pm \mathbf{k}$ are the momenta of the scattering particles which lie on the Fermi surface, so that

$$K^2/4 = k_F^2 - k^2 \tag{3.8}$$

and ϵ_k is $e_k - e_{k_F}$.

The reason for writing $|\psi(k, l)\rangle$ in terms of $|\psi_0(k, l)\rangle$ is that $G_0(k, l)$ is a slowly varying function of T and the most significant temperature dependence of $\langle k, l \,|\, v \,|\, \psi(k, l)\rangle$ is in $\Lambda_K(\mathbf{k}')$ in the denominator of the right-hand side of Eq. (3.2).

It is straightforward to solve Eqs. (3.2) to (3.7) numerically, but for the present problem it is possible to obtain a good approximation for $\langle k, l \,|\, v \,|\, \psi(k, l)\rangle$

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[16] V. J. Emery, Ann, Phys. (N.Y.) **28**, 1 (1964).

in terms of the free-space phase shifts $\delta_l(k)$, which is useful both as a method of calculation and as a means of obtaining an intuitive feeling for the problem.

For this purpose, define the free-space function

$$|\psi_F(k,l)\rangle = |k,l\rangle + G_F(k,l)v|\psi_F(k,l)\rangle, \quad (3.9)$$

with

$$G_F(k,l) = -\frac{2(2l+1)}{\pi\hbar^2} \times \int_0^\infty dk' \frac{|k',l\rangle\langle k',l| - |k,l\rangle\langle k,l|}{(k'^2/m_0^*)-(k^2/m_0^*)}. \quad (3.10)$$

Subtracting Eq. (3.9) from Eq. (3.4) gives

$$|\psi_0(k,l)\rangle = |\psi_F(k,l)\rangle + [1-G_F(k,l)v]^{-1}[G_0(k,l)-G_F(k,l)]v|\psi_0(k,l)\rangle. \quad (3.11)$$

Then multiplying on the left by $\langle k,l|v$ and using the conjugate of Eq. (3.9),

$$\langle k,l|v|\psi_0(k,l)\rangle = \langle k,l|v|\psi_F(k,l)\rangle + \langle\psi_F(k,l)|v[G_0(k,l)-G_F(k,l)]v|\psi_0(k,l)\rangle. \quad (3.12)$$

In the next section it will be seen that, except for k near k_F, the second term on the right-hand side of this equation is about 10% of the first. In evaluating $G_0(k,l)$, it will be assumed that ϵ_k is a constant plus $k^2/2m_0^*$. In practice, ϵ_k differs from this form only in the Fermi sea and just outside, and even there the difference is only a few percent. At very low temperatures, the integrands in $G_0(k,l)$ and $G_F(k,l)$ are the same for $k' \gtrsim k_F + K/2$, and so cancel in Eq. (3.12). For the other values of k', $|k',l\rangle$ will be replaced by $(k'/k)^{l+1}|k,l\rangle$. This approximation is good for small k_F. In the worst case of a 6% mixture and s states, assuming the range of force to be found in the next section, the error in this approximation is, on average, less than 20% in $G_0(k,l)-G_F(k,l)$, and it leads to a rather small error in $\langle k,l|v|\psi_0(k,l)\rangle$.

With this assumption, Eqs. (3.5), (3.10), and (3.12) give

$$\langle k,l|v|\psi_0(k,l)\rangle = \frac{\langle k,l|v|\psi_F(k,l)\rangle}{1+\lambda(k,l)\langle k,l|v|\psi_F(k,l)\rangle},$$

where, using Eq. (3.8),

$$\lambda(k,l) = \frac{(2l+1)}{2\pi^2}\int\frac{d\mathbf{k}'}{k'^2}\left[\Lambda_K(\mathbf{k}') - \frac{m_0^*/\hbar^2}{k'^2-k^2}\right] \quad (3.13)$$

$$\times[P_l(\hat{k}\cdot\hat{k}')]^2\left[\left(\frac{k'}{k}\right)^{2(l+1)}-1\right]. \quad (3.14)$$

Since[17]

$$\langle k,l|v|\psi_F(k,l)\rangle = -\hbar^2 k/m_0^* \tan\delta_l(k), \quad (3.15)$$

Eqs. (3.2), (3.6), and (3.13) give

$$\langle k,l|v|\psi(k,l)\rangle = \frac{-(\hbar^2 k/m_0^*)\tan\delta_l(k)}{1-[\bar{\Lambda}(k,l)+\lambda(k,l)](\hbar^2 k/m_0^*)\tan\delta_l(k)}. \quad (3.16)$$

It will be seen that the denominator on the right-hand side of this equation differs most significantly from unity for $l=0$. When $(k_F-k)/k_F \gg T/T_F$, it is possible to set $T=0$ and find

$$\bar{\Lambda}(k,0)+\lambda(k,0) = \frac{m^*}{\pi\hbar^2 k}\left[\ln\left(\frac{k_F+k}{k_F-k}\right) - \frac{2k_F}{k}\right], \quad (3.17)$$

whereas for $(k_F-k)/k_F \ll T/T_F$, the leading term is[18]

$$\bar{\Lambda}(k_F,0)+\lambda(k_F,0) = \frac{2m^*}{\pi\hbar^2 k_F}\left(\ln\frac{4.56T_F}{T} - 2\right). \quad (3.18)$$

These approximations will be used in Sec. V in the discussion of a fermion superfluid phase transition.

IV. DETERMINATION OF THE EFFECTIVE INTERACTION

It will be seen that the existing experiments are not sufficient to determine the shape of $v(r)$ completely, so to simplify the calculation of the phase shifts, the short-range repulsive region will be replaced by a hard core of radius r_c. Later in this section, a square well will be assumed for the attractive part of $v(r)$, but first, in order to get a feeling for what is going on, the effective-range theory of Blatt and Jackson[19] will be used.

A. Effective-Range Theory

The s-state phase shift will be written in the form

$$\delta_0(k) = -kr_c + \arctan\left(-ka'/(1-\tfrac{1}{2}r_0'a'k^2)\right), \quad (4.1)$$

where a' is the scattering length and r_0' the effective range of the *attractive part* of the interaction. This equation may be obtained by changing origin to r_c in the s-state projection of the Schrödinger equation and then using the effective-range approximation.[19]

For k sufficiently small, s states dominate, and the expansion of Eq. (4.1)

$$\delta_0(k) \approx -ak - b^3k^3, \quad (4.2)$$

with

$$a = a' + r_c,$$

$$b^3 = \tfrac{1}{2}r_0'a'^2 - \tfrac{1}{3}a'^3, \quad (4.3)$$

may be used. Then $\delta_0(k)$ depends upon only two numbers a and b which do not give enough information to determine r_c, r_0', and a', although it will be seen that the possible values of r_c are restricted.

In the degenerate region, momenta $\hbar\mathbf{k}$ for $k<k_F$ contribute. The lowest concentration for which detailed

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[17] V. J. Emery, Nucl. Phys. **19**, 154 (1960).

[18] V. J. Emery and A. M. Sessler, Phys. Rev. **119**, 43 (1960).

[19] J. M. Blatt and J. D. Jackson, Phys. Rev. **26**, 18 (1949).

measurements have been made[8] is 1.3% for which $k_F=0.204$ Å^{-1} and they will be used to determine a and b. It will be seen that Eq. (4.2) is a good approximation in this case, but there is some inaccuracy because p states, though small, are not entirely negligible.

If Eq. (4.2) is used in Eqs. (2.7), (2.10), (2.13), and (2.14), it is found that the measurements[8] of D and χ give a between -0.8 and -1.0 Å, and b between 1.8 and 2.4 Å. The reason for the uncertainty is that the force contributes about 10% to χ and so the 3% experimental error is very significant. The specific heat (and hence m^*) has been measured, but m_0^* has to be known before Eq. (2.15) can be used. Later, it will be shown that $m_0^*=2.33m_3$. This additional information then requires $a=-1$ Å and $b=2.4$ Å. A better determination could be made if there were more complete measurements at lower concentration.

Now, in its usual form, effective-range theory[19] is applied to the entire interaction to obtain Eq. (4.1) with $r_c=0$. Then $r_0'=27$ Å, and it was inconsistent to omit higher-angular-momentum states, and corrections to Eq. (4.1) become important.

If r_c is now included, then, as it increases, a' increases in magnitude and r_0' decreases. The maximum value of r_c, for $r_0'>0$, is 2.47 Å. Later in this section, it will be seen that the best fit to all of the existing experiments is given by $r_c=1.8$Å, and if this value is used here, $a'=-2.8$ Å and $r_0'=1.7$ Å. Thus, since $r_c>|a|$, the modification of the effective-range theory makes a significant difference in the interpretation of the parameters. The situation is quite different with the nuclear force[19] for which $r_c<|a|$.

If the same analysis is applied to the free-space phase shifts calculated[20] from v_0, Eq. (4.1) gives a poor fit with $r_c=0$, but if $r_c=1.8$, it fits for k up to about 1 Å^{-1} with $a'=-7.1$ Å and $r_0'=2.7$ Å. Thus, since the free-space phase shifts were calculated with mass m_3, the attractive part of v is weaker but of shorter range than the corresponding part of v_0.

Given r_c, the values of a' and r_0' are independent of the shape of the attractive part of v, and this analysis complements the remainder of this section where a square well will be used and higher-angular-momentum states will be included. With this modification, it will be seen that a' is -2.9 Å and r_0' is 0.9 Å, so that it is desirable to work at lower concentrations if p states are to be neglected.

B. Hard Core Plus Square Well

It will now be assumed that

$$v(r)=-\tfrac{1}{4}(s\pi^2)(\hbar^2/m_0^*b^2),\qquad r_c<r<b+r_c$$
$$=0,\qquad b+r_c<r.\qquad(4.4)$$

Here b is the range of the force and s is the well-depth parameter which is defined so that when $s=1$, $v(r)$ has a bound s state of zero energy, and is a very convenient measure of the strength of the interaction. The square well is chosen because its phase shifts can be calculated analytically, which reduces the calculation even though the integrals have to be performed numerically. The fit to the data is reasonably good and, at present, it is not reasonable to attempt to improve it by changing the shape of the well or the repulsive region, since the existing data are not sufficient to give a unique result.

The potential parameters are given most accurately by D since the experimental error is also the error in the mean-square scattering amplitude. Given r_c, the low-temperature limits of DT^2 for 1.3 and 5% concentrations,[8] which have an experimental error of 10%, determine b to about 5% and, given r_c and b, they determine s with less than 1% error. The reason is that the phase shifts are small as a result of a close cancellation of the core and attractive contributions, so a small change in s has a large effect.

If the low-temperature measurements are used to find b and s as functions of r_c, and then the semiclassical results are used to determine r_c, it is found that the best fit is given by

$$r_c=1.8\text{ Å},$$
$$b=0.8\text{ Å},$$
$$s=0.81.\qquad(4.5)$$

Figure 1 shows $Dx/T^{3/2}$ as a function of T. From Eq. (2.4), it can be seen that Dn_3 (and hence Dx) is independent of concentration in lowest order, provided real phonon scattering is unimportant, so Dx can be obtained from the measurements at 1.3 or 5% concentration. However, in order that $T\gtrsim 2T_F$, which is the condition for Eq. (2.4) to be valid, the experimental points in Fig. 1 come from 1.3% concentration between 0.3 and 0.6°K and from 5% concentration at 0.8°K.

The low value of D at 0.3°K is (at least partly) a consequence of the omission of the degeneracy correction which is of the order of 10% at this temperature. The curve is larger than the experimental value at 0.8°K and this is likely to be a result of the inadequacy of a hard-core repulsion. Improvement of the fit requires a change in the shape of the potential rather than an adjustment of parameters, but this has not been attempted, since it seems that the existing data would not determine $v(r)$ uniquely.

From Eqs. (2.4), (2.5), and (2.6), the temperature dependence of $D/T^{3/2}$ comes from the fact that the phase shifts are evaluated at relative kinetic energy γ^2k_BT. At low temperatures, the relevant kinetic energies are low and, as shown in Fig. 2, phase shifts are small because the over-all scattering length a is small. As T increases, higher relative kinetic energies

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[20] J. de Boer, J. Van Kranendonk, and D. Compaan, Physica **16**, 545 (1950).

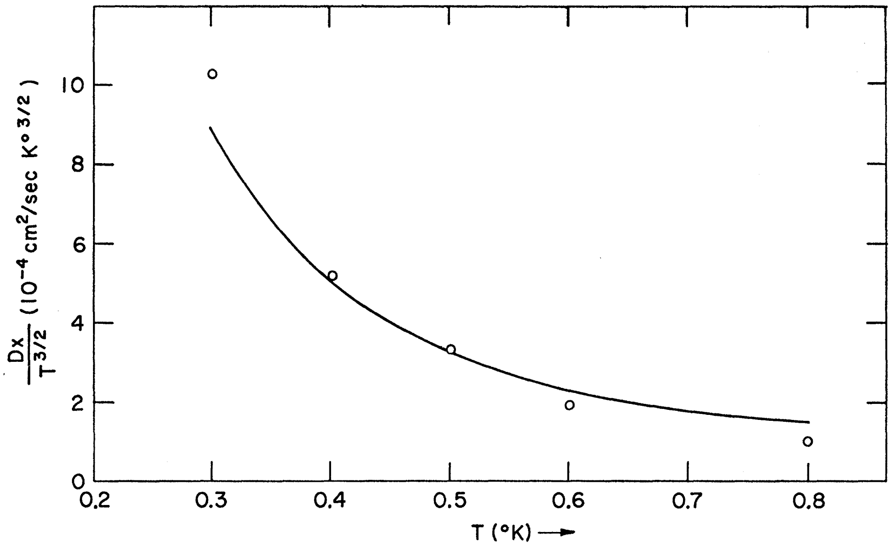

FIG. 1. Spin diffusion D as a function of temperature T in the semiclassical region. The curve shows $Dx/T^{3/2}$ calculated from the effective interaction (x is the concentration). The experimental points are from Anderson *et al.* (Ref. 8).

are sampled, and the mean of the phase shifts increases as $D/T^{3/2}$ decreases.

If the hard-core and square-well potential is fitted to the free-space values of r_0' and a', with $r_c=1.8$ Å, the potential parameters are

$$r_0=1.8\text{ Å},$$
$$b=2.4\text{ Å},$$
$$s=0.79, \tag{4.6}$$

which give v_0 provided m_0^* is replaced by m_3 in Eq. (4.4).

It can be seen that, in comparison with v_0, v is deeper and has a shorter range, so that the potential v_i induced by the He^4 is repulsive for r greater than about 2.6 Å and attractive between 1.8 and 2.6 Å. It would be surprising if the induced potential exactly cancelled $v_0(r)$ at large distances, and it is possible that a more accurate determination of $v(r)$ would reveal a repulsive tail or even an undulating tail. The values of r_0' and a' and possibly s are reasonably shape-independent, but the intrinsic range and hence the depth could change somewhat, and we do not consider them to be well determined at present.

Table I shows the properties calculated from $v(r)$. The values of DT^2 fit the experimental results which are the same as those quoted. It can be seen that $(m^*/m_0^*)-1$ and $(\chi_0/\chi)-1$ also fit the experiments but this is not a particularly stringent test of the potential since the experimental errors are an appreciable fraction of its contribution. The sound-attenuation measurements of Abraham *et al.*[9] give a relaxation time which can be shown to be τ_η by means of an analysis similar to that carried out[6] for pure He^3. Their measured values of τ_η extrapolate reasonably well with the limiting value shown in the table. No measurements of the thermal conductivity have been reported at present.

The measured phase-separation curve[3] is well fitted if $\mu_3(n_0, 0, 0)$ is chosen to be -2.78°K. With this value, the maximum stable concentration at $T=0$, obtained by solving Eq. (2.24) at $T=0$, is 6%.

The results generally are quite similar to those obtained by Bardeen, Baym, and Pines,[5] and by Ebner.[7] They assume that the scattering amplitudes A_E and A_O can be calculated from a velocity-independent potential in first-order perturbation theory. The scattering amplitudes which were calculated in Sec. III have a different behavior from this in general, although the separation method,[21] in the form used by Moszkowski

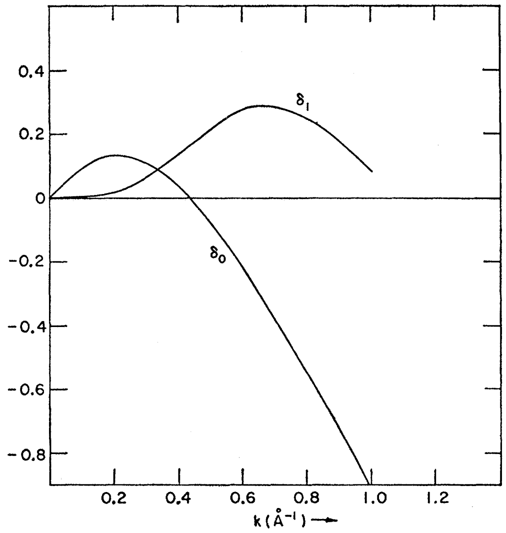

FIG. 2. Phase shifts for angular momentum $l=0$ and $l=1$ for the relative momentum $\hbar k$.

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[21] V. J. Emery and R. J. Eden, Proc. Roy. Soc. (London) **A248**, 266 (1958).

[22] S. Moszkowski and B. L. Scott, Ann. Phys. (N.Y.) **11**, 65 (1960).

and Scott,[22] suggests that it may be a good approximation in the degenerate limit, but is less likely to be so in the semiclassical region. At present, there is not enough experimental information to distinguish between the various possibilities.

The restoring force which produces the analog of zero sound[6] in the mixture is $A_E(k, 0)+3A_O(k, 0)$, and from Eqs. (3.1), (3.16), and Fig. 2, it can be seen that it is negative for all $k<k_F$. (Figure 2 does not show phase shifts for $l>1$, but they are positive for $k<k_F$ and small.) Therefore zero sound will not propagate. The restoring force for spin waves[23] is $A_O(k, 0)-A_E(k, 0)$, and this is positive for all $k<k_F$ in a 1.3% mixture and negative only for k near to k_F in a 5% mixture; and spin waves will propagate. In the approximation that $A_O(k, 0)-A_E(k, 0)$ is constant, the velocity u of the spin waves is determined[23] by $(\chi_0/\chi)-1$, which is so small that u is very close to v_F. For this reason, the generation and detection of spin waves will be rather difficult.

V. FERMION SUPERFLUID PHASE TRANSITION

Since the effective interaction is attractive at small k (i.e., $\delta_0(k)$ and $\delta_1(k)$ are positive) the mixtures should undergo a fermion superfluid phase transition. At the transition temperature $T_c(l)$ for angular momentum l, $\langle k_F l \mid v \mid \psi(k_F, l)\rangle$ has a singularity[16,17,24] and the scattering amplitude for particles on the Fermi surface with total momentum zero diverges. Thus, from Eq. (3.16), T_c is given by

$$\bar{\Lambda}(k_F, l)+\lambda(k_F, l)=(m^*/\hbar^2 k_F)\cot\delta_l(k_F). \qquad (5.1)$$

Figure 2 shows that $\delta_0>\delta_1$ for $k<0.33$, so that $l=0$ gives the larger transition temperatures. The maximum occurs at $k=0.2$ Å^{-1}, where $\delta_0=0.134$. For $l=0$, Eqs. (3.17), (3.18), and (5.1) give

$$T_c=4.56T_F\exp[-(\pi/2)\cot\delta_0(k_F)-2]. \qquad (5.2)$$

The factor e^{-2} is an improvement of the low-density phase-shift approximation,[17] and an equation of this type has been found before[25] in the special case of a separable potential.

The maximum value of T_c occurs at $k_F=0.2$ Å^{-1}, or $x=1.24\%$, and is 0.79×10^{-6}°K. This is too small for the transition to be observed at present, but now it will be shown that evidence should be found in the transport coefficients above a millidegree as a result of the way in which the divergence of $\langle k_F, 0 \mid v \mid \psi(k_F, 0)\rangle$ affects the scattering rates. In order to estimate the magnitude of this effect, $\bar{\Lambda}(k, 0)+\lambda(k, 0)$ will be approximated by Eq. (3.17) for $k<k_0$ and Eq. (3.18) for $k>k_0$, where k_0 is chosen so that the right-hand sides of Eqs. (3.17) and (3.18) are equal at $k=k_0$. The error caused by this approximation is very small. Then, from Eqs. (2.13), (3.1), (3.16), and (5.1), the change in $W^{(1,1)}$ when the temperature is changed from T_1 to T_2 may be reduced to

$$\Delta W^{(1,1)}=\frac{\pi^3 e}{4.56}\frac{T_c}{T_F}\int_{T_2/T_c}^{T_1/T_c}\frac{dx}{(\log x)^3}\left(1-\tfrac{3}{2}\tan\delta_1(k_F)\log x\right). \qquad (5.3)$$

Here the leading order in k_F-k has been taken.

TABLE I. Calculated properties of solutions at 1.3 and 5% He^3 concentration. The potential was chosen to fit the measured value of DT^2 which is the same as quoted here. The experimental values[a] of m^*/m_0^* and χ_0/χ are given. $\kappa_T{}^0$ is the ideal gas compressibility.

	1.3%	5%
DT^2 (10^{-6} cm^2 K^{02}/sec)	17.2	90
$\tau_\eta T^2$ (10^{-11} sec K^{02})	1.7	2.2
KT (erg/cm sec)	19.1	64
$\kappa_T{}^0/\kappa_T$	0.86	0.74
$(m^*/m_0^*)-1$	0.017	0.061
Experiment (assuming $m_0^*=2.33m_3$)	0.021±0.017	0.056±0.017
$(\chi_0/\chi)-1$	0.09	0.05
Experiment	0.09±0.03	0.08±0.03

[a] Reference 8.

The transport coefficients are close to their limiting values at 0.01°K, so we take $T_1=0.01$°K, $T_2=0.002$°K, and choose $k_F=0.2$ Å^{-1} which gives the largest T_c. Then $\Delta W^{(1,1)}/W$ is 0.04, and DT^2 should decrease by 4% over this temperature range. This should be observable but is within the quoted experimental error at present.[8] The decrease is about 3% for KT and ηT^2, since their scattering rates are larger than for D.

The reason that the effect is observable so far above T_c is that it does not have to compete with a large scattering rate for k far from k_F as in pure He^3, or impurity scattering as in superconductors.[16] The value of W for $k_F=0.2$ Å^{-1} is about $2\pi\times10^{-2}$.

The prediction of the decrease of DT^2 depends upon very much the same assumptions as the prediction of the transition; therefore its observation would strongly suggest that a transition would be seen at low enough temperatures.

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[23] L. D. Landau, Zh. Eksperim. i Teor. Fiz. **32**, 59 (1957) [English transl.: Soviet Phys.—JETP **5**, 101 (1957)].

[24] D. Thouless, Ann. Phys. (N.Y.) **10**, 533 (1960).

[25] P. C. Sood and S. A. Moszkowski, Nucl. Phys. **21**, 582 (1960).