Delta-Function Fermi Gas with Two-Spin Deviates*

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The problem of one-dimensional particles interacting via a delta-function potential has been a useful model for the many-body problem. While it can be solved for bosons, it has so far not been solved for fermions except for the special cases of either all spins parallel or all but one spin parallel. The full difficulty of the problem first manifests itself when two spins are down (i.e., $S=\frac{1}{2}N-2$), and this we solve here. We are confident that our method can be extended to the general problem.

INTRODUCTION

MANY authors have sought an exactly soluble model of interacting fermions (with ordinary pair potentials and in a "box" at a finite density) which would be an asset in the study of the many-body problem that pervades so much of physics.[1] Such a model—the delta-function model—exists for bosons,[2] but so far has defied solution for fermions. While there does exist one exactly soluble model of interacting fermions having physically reasonable properties,[3] the Hamiltonian of that model is sufficiently unphysical that it is difficult to have great confidence in its predictions.

In this paper we show how to solve the delta-function model for fermions with 2 spins down and N–2 spins up. We are fully confident that our method will solve the general problem, although its solution may be complicated. It must not be forgotten that one requires more of an exactly soluble model than a mere formal solution; one also needs an interpretation of the results. Since it is difficult to infer very much by comparison of the 2-spin deviate problem with the zero-spin deviate problem, we have put aside this important task until we are able to exhibit the full solution to the general problem.

The Hamiltonian[4] of the delta-function model for N particles is

$$H=-\sum_{1}^{N}\partial^2/\partial x_i^2+2c\sum_{1=i<j}^{N}\delta(x_i-x_j). \tag{1.1}$$

Relative to bosons, fermions possess a simplifying as well as a complicating feature, the latter vastly outweighing the former. The simplification comes from the fact that an antisymmetric spatial wave function automatically vanishes when two particles "touch," so that the delta-function interaction plays no role. Thus, fermions with all spins parallel do not interact with each other. For other values of the total spin there will, indeed, be an interaction between the "spin-up" particles and the "spin-down" particles. In the latter case the problem is quite similar to the problem of a two-component Bose gas having the property that particles of each species interact only with particles of the other species. For particles of only one species (i.e., either a totally symmetric or a totally antisymmetric wave function), it is only necessary to consider the subregion of configuration space

$$R_1:\quad x_1\leq x_2\leq\cdots\leq x_N \tag{1.2}$$

instead of the full configuration space because knowledge of the wave function in R_1 is, by symmetry, sufficient to define the wave function everywhere.

The complication mentioned above is that we are forced to consider a region much more complicated than R_1. To be explicit, suppose we wish to consider the eigenfunctions of H when j spins are down and $N-j$ spins are up [i.e., the value of S_z is $M_j=\frac{1}{2}(N-2j)$]. As was explained elsewhere,[5] it is necessary and sufficient to consider wave functions of the form $\Psi(x_1, x_2, \cdots, x_j \mid x_{j+1}, \cdots, x_N)$, where the "bar" means that Ψ is antisymmetric in the first j variables and in the last $N-j$ variables, but has no particular symmetry with respect to interchange of variables across the bar. Such a function could belong to a total $S\geq M_j$. Indeed,

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[1] For a review of exactly soluble many-body problems see E. Lieb and D. C. Mattis, *Mathematical Physics in One-Dimension* (Academic Press Inc., New York, 1966).
[2] E. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963); E. Lieb, *ibid*. **130**, 1616 (1963).
[3] This model, based on the Thirring model of field theory was introduced by J. Luttinger, J. Math. Phys **4**, 1154 (1963), and was solved by D. C. Mattis and E. Lieb, *ibid*. **6**, 304 (1965).
[4] $\hbar=1$, $2m=1$.
[5] E. Lieb and D. C. Mattis, Phys. Rev. **125**, 164 (1962).

all functions belonging to $S \geq M_j$ have a representative in the M_j subspace by application of S_-, the spin-lowering operator. Thus, if we specify j (in this paper we shall consider $j=0$, 1, 2) there are two ways to determine the total S value of any particular wave function. The first, and complicated, way is to investigate the symmetry properties across the bar (i.e., to decide to which irreducible representation of the symmetric group the function belongs). The second, and simpler way, is to inquire whether the energy of Ψ is among the energies belonging to the $j-1$ subspace. If it does, then, barring accidental degeneracy, the S value of Ψ is greater than M_j.

To find eigenfunctions of the j type we must consider the configurational subspace

$$R^j:\quad x_1 \leq x_2 \leq \cdots \leq x_j,\qquad x_{j+1} \leq \cdots \leq x_N. \tag{1.3}$$

Equation (1.3) defines only a partial ordering. Plainly the region R^j contains $\binom{N}{j}$ physically distinguishable, completely ordered subregions of the type R_1. Herein lies the difficulty.

The imposition of periodic boundary conditions (PBC) does mitigate the difficulty somewhat, and we shall henceforth adopt these boundary conditions. Clearly, in this case, only the *relative* ordering of the "up" variables and the "down" variables is of any consequence. Thus, we need consider only $\binom{N-1}{j-1}$ essentially inequivalent subregions. For $j=1$, the case solved by McGuire,[6,7] it is only necessary to consider one instead of N regions. The full difficulty of the problem does not manifest itself until $j=2$. By solving this last case, we believe we shall be able to undertake the general problem of arbitrary j.

To solve the problem, we observe that the delta functions in (1.1) can be replaced by the $\binom{N}{2}$ boundary conditions:

$$\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)\Psi\,|_{x_i=x_j^+}-\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)\Psi\,|_{x_i=x_j^-}=2c\Psi\,|_{x_i=x_j}, \tag{1.4a}$$

together with the continuity conditions

$$\Psi\,|_{x_i=x_j^+}=\Psi\,|_{x_i=x_j^-}. \tag{1.4b}$$

These are in addition to the periodic boundary conditions (PBC):

$$\Psi(x_1,\cdots,x_j,\cdots,x_N)=\Psi(x_1,\cdots,x_j+L,\cdots,x_N) \tag{1.5}$$

for $j=1, 2, \cdots, N$. Here, L is the length of the "box." It is obviously sufficient to restrict our attention, whatever the order of the x's, to

$$x_{\max}-x_{\min}\leq L, \tag{1.6}$$

and we shall do so henceforth. We then make the following *Ansatz* for Ψ: For a suitable choice of N *distinct* numbers $\{k\}=k_1, k_2, \cdots, k_N$, we note that Ψ can be written in every subregion R_α of the type R_1 as

$$\psi_\alpha(x_1,\cdots,x_N)=\sum_P A_\alpha(P)\exp\Big(i\sum_1^N k_{j(P)}x_j\Big), \tag{1.7}$$

where $\sum_P$ stands for a sum on $N!$ permutations. The important point to note is that $A_\alpha(P)$ is a set of coefficients which depend upon the particular region R_α, but that the set $\{k\}$ is fixed and does not depend upon R_α. The problem is to find the allowed sets $\{k\}$, together with their accompanying coefficients $\{A_\alpha(P)\}$ such that the boundary conditions (1.4) and (1.5) are satisfied.

Clearly, the Schrödinger differential equation is satisfied with an energy

$$E=\sum_1^N k_i^2. \tag{1.8}$$

Moreover, the total momentum of the state will be

$$P=\sum_1^N k_i, \tag{1.9}$$

and we know that any allowed set $\{k\}$ will automatically satisfy

$$P=(2\pi/L)\times\text{integer}. \tag{1.10}$$

Before investigating the $j=2$ case, let us see how the program works for $j=0$ and $j=1$. For $j=0$, we need consider only R_1 (1.2) and impose the boundary condition that $\psi=0$ on the boundary of R_1. In an arbitrary subregion, R_α, $\psi_\alpha=\pm\psi_1$ and it is seen that the continuity conditions and delta-function conditions (1.4) will be automatically satisfied. The PBC, (1.5), reads

$$\begin{aligned}\Psi(x_1,&\cdots,x_j,\cdots,x_N)\\ &=\Psi(x_1,\cdots,x_j+L,\cdots,x_N)\\ &=(-1)^{N-j}\Psi(x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_N,x_j+L)\end{aligned} \tag{1.11}$$

for all $j=1,\cdots,N$. The point of Eq. (1.11) is that the PBC has been reduced to a statement about Ψ in R_1 alone, i.e., about ψ_1. Now, the fact that ψ_1 vanishes on the boundary of R_1 implies that $A_1(P)=\epsilon_P$, whence

$$\psi_1(x_1,\cdots,x_N)=\mathrm{Det}\,|\exp ik_ix_j\,|. \tag{1.12}$$

Equation (1.11) then fixes the k_i as

$$k_i=(2\pi/L)n_i, \tag{1.13}$$

where $n_1,\cdots,n_N$ is any set of distinct integers. The

[6] J. B. McGuire, J. Math. Phys. **6**, 432 (1965).
[7] J. B. McGuire, J. Math. Phys. **7**, 123 (1966).

result then, for the $j=0$ case, is that the eigenfunctions and eigenvalues are the same as for the noninteracting case. In fact, as we could have seen in advance, (1.12) represents Ψ everywhere, not only in R_1, but this is fortuitous; it will not be true for $j>0$.

For $j=1$, the region R^1, Eq. (1.3) consists of N regions $R_1, R_2, \cdots, R_N$ where R_k is defined by the fact that $x_k \leq x_1 \leq x_{k+1}$. As we remarked above, the PBC imply that these N regions are effectively equivalent. Using PBC we can find enough conditions in R_1 alone to determine $\{k\}$. Denoting the value of Ψ in R_i by ψ_i, and using PBC on the "up" particles, we have

$$\begin{aligned}\psi_2(x_1 \mid x_2, x_3, \cdots, x_N) &= \psi(x_1 \mid x_2+L, x_3, \cdots, x_N) \\ &= (-1)^{N-2}\psi_1(x_1 \mid x_3, x_4, \cdots, x_N, x_2+L).\end{aligned} \tag{1.14}$$

It is here that the value of PBC manifests itself, for we now have two conditions connecting R_1 and R_2; (1.4) and (1.14). These, together with (1.10), are sufficient to determine $\{k\}$.

Since ψ_1 vanishes whenever $x_{j+1}=x_j$ $(j=2, \cdots, N-1)$, we obviously infer

$$\psi_1 = \begin{vmatrix} f_1\chi_{11} & f_2\chi_{12} & \cdots & f_N\chi_{1N} \\ \chi_{21} & \chi_{22} & \cdots & \chi_{2N} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \chi_{N1} & \chi_{N2} & \cdots & \chi_{NN} \end{vmatrix}, \tag{1.15}$$

where $f_1, \cdots, f_N$ are some coefficients and

$$\chi_{ij} \equiv \exp(ix_ik_j).$$

If we denote by $D_{mn}(x_3, \cdots, x_N)$ the cofactor of $f_m\chi_{1m}\chi_{2n}$ in the determinant of Eq. (1.15), and use the fact that ψ_2 vanishes whenever $x_{j+1}=x_j$ $(j=3, \cdots, N-1)$, then

$$\psi_2 = \sum_{m,n=1} A_{mn}\chi_{1m}\chi_{2n}D_{mn}(x_3, \cdots, x_N). \tag{1.16}$$

In this language, the $j=0$ case is retrieved by the choice $f_j=1$ and $A_{mn}=1$ (all j, m, n).

The delta-function condition, (1.4a), reads

$$i(k_n-k_m)(f_m+f_n-A_{mn}-A_{nm}) = 2c(f_m-f_n), \tag{1.17a}$$

while the continuity condition, (1.4b), reads

$$f_m-f_n = A_{mn}-A_{nm} \tag{1.17b}$$

for all $m\neq n$. Finally, the PBC reads

$$A_{mn} = f_m \exp(ik_nL). \tag{1.18}$$

Inserting Eq. (1.18) into Eq. (1.17), we find

$$f_m(1-\chi_n) = ic(f_n-f_m)/(k_n-k_m), \tag{1.19}$$

where $\chi_n = \exp(ik_nL)$. One solution is to have $\chi_n=1$ and $f_n=1$ (all n), thereby recovering the $j=0$ solution. Otherwise, by comparing Eq. (1.19) to the same equation with n and m interchanged, we find

$$f_n = (1-\chi_n). \tag{1.20}$$

Upon insertion of Eq. (1.20) into Eq. (1.19) and a small amount of algebraic manipulation, we have

$$\tfrac{1}{2}c \cot\tfrac{1}{2}(k_jL) - k_j = \lambda \tag{1.21}$$

(λ a constant), with the constraint that $k_n\neq k_m$. This is the famous equation first derived by McGuire[6] and discussed extensively by him.[6,7] It is useful to rewrite Eq. (1.21) as

$$\chi_j = \frac{k_j+\lambda+\frac{1}{2}(ic)}{k_j+\lambda-\frac{1}{2}(ic)}. \tag{1.22}$$

The reader will note that for every pair of neighboring regions, R_j and R_{j+1}, there is a similar pair of boundary conditions [i.e., Eq. (1.4) and the analog of Eq. (1.14)], and he can easily verify that these will be satisfied by the same choice of $\{k\}$ as in (1.21). Where the final condition, Eq. (1.10), enters is in the PBC on the "down" particle, x_1:

$$\begin{aligned}\psi_1(x_1 \mid x_2, \cdots, x_N) &= \psi(x_1+L \mid x_2, \cdots, x_N) \\ &= \psi_N(x_1+L \mid x_2, \cdots, x_N),\end{aligned} \tag{1.23}$$

an equation connecting R_1 to R_N. This will be satisfied for the choice (1.21) if and only if (1.10) is obeyed. It is clear from Eq. (1.22) that Eq. (1.10) may be rewritten as

$$\prod_{j=1}^{N}\left(\frac{k_j+\lambda+\frac{1}{2}(ic)}{k_j+\lambda-\frac{1}{2}(ic)}\right) = 1 \tag{1.24}$$

Finally, it must be remembered that the solutions to Eq. (1.22) and Eq. (1.24) give only the $S_{\text{total}} \geq \frac{1}{2}N-1$ states for the Hamiltonian (1.1). The $S_{\text{total}}=\frac{1}{2}N$ states correspond to $\lambda=\infty$.

The final result is startling. It is that each k_j satisfies an "independent particle" equation, (1.21), which is only slightly more complicated than that for the noninteracting gas, (1.13). True, these equations are not really independent because the separation constant λ in Eq. (1.21) must be chosen to satisfy the momentum condition (1.10). Nevertheless, the situation is much simpler than for the Bose gas[2] where each k is a function of all the other k's.

When $j=2$, the algebraic problem will be vastly more complicated than the foregoing because we must consider $(N-1)$ inequivalent subregions rather than one subregion. Nevertheless, the final equations for $\{k\}$ bear a striking similarity to Eqs. (1.22) and (1.24).

It is the main purpose of this paper to prove that

$$\chi_j \equiv \exp(ik_jL) = \left(\frac{k_j+\delta+\frac{1}{2}(ic)}{k_j+\delta-\frac{1}{2}(ic)}\right)\left(\frac{k_j+\epsilon+\frac{1}{2}(ic)}{k_j+\epsilon-\frac{1}{2}(ic)}\right), \tag{1.25}$$

where

$$\prod_{j=1}^{N}\left(\frac{k_j+\delta+\frac{1}{2}(ic)}{k_j+\delta-\frac{1}{2}(ic)}\right)=\frac{\epsilon-\delta-ic}{\epsilon-\delta+ic}, \tag{1.26a}$$

$$\prod_{j=1}^{N}\left(\frac{k_j+\epsilon+\frac{1}{2}(ic)}{k_j+\epsilon-\frac{1}{2}(ic)}\right)=\frac{\delta-\epsilon-ic}{\delta-\epsilon+ic}. \tag{1.26b}$$

Here, δ and ϵ are two constants. Once again, solutions to (1.25) and (1.26) give us only the $S_{\text{total}}\geq\frac{1}{2}N-2$ states.

II. THE TWO-SPIN DEVIATE PROBLEM—DETERMINATION OF $\{k\}$

The region R^2, Eq. (1.3), contains $\binom{N}{2}$ primitive subregions which we shall denote by R_{jk}, the index j signifying the position of x_1 among the N variables and the index k signifying the position of x_2 measured from x_1. Thus

$$\begin{aligned}
R_{11}&:\quad x_1<x_2<x_3<x_4<\cdots<x_N,\\
R_{12}&:\quad x_1<x_3<x_2<x_4<\cdots<x_N,\\
R_{21}&:\quad x_3<x_1<x_2<x_4<\cdots<x_N, \quad \text{etc.}
\end{aligned} \tag{2.1}$$

The regions R_{1k} $(k=1,\cdots,N-1)$ are the fundamental inequivalent regions, for our intuition tells us, although we must carefully verify it explicitly, that all regions with a common value of the second index are equivalent to each other. Each region R_{jk} is connected to neighboring regions through Eq. (1.4). For $k>1$ and $j>1$ there are four: $R_{j-1,k+1}$, $R_{j+1,k-1}$, $R_{j,k+1}$, and $R_{j,k-1}$. For $k=1$ and $j>1$ there are two: $R_{j-1,k+1}$ and $R_{j,k+1}$.

Our first step will be to take a "down" particle through the "up" particles. That is, if we assume we know $\psi_{1,1}$ then Eq. (1.4) will determine $\psi_{1,2}$ and then $\psi_{1,3}$, etc. Finally, $\psi_{1,1}$ is related to $\psi_{1,N-1}$ by PBC:

$$\begin{aligned}
\psi_{1,1}(x_1x_2\mid x_3,\cdots,x_N)&=\psi(x_1+L,x_2\mid x_3,\cdots,x_N)\\
&=-\psi(x_2,x_1+L\mid x_3,\cdots,x_N)\\
&=-\psi_{1,N-1}(x_2,x_1+L\mid x_3,\cdots,x_N).
\end{aligned} \tag{2.2}$$

We shall then be able to determine a great deal, but not everything, about $\{k\}$. The remaining conditions will come later from the second step—taking an "up" particle through the "down" particles.

Since $\psi_{1,1}$ vanishes when $x_1=x_2$ or when $x_{j-1}=x_j$ $(j=4,\cdots,N)$, we have

$$\psi_{1,1}=\tfrac{1}{2}\sum_{s,t=1,s\neq t}^{N}A(st)f_{st}(x_1,x_2)D_{st}(x_3,\cdots,x_N), \tag{2.3}$$

where

$$f_{st}(x_1x_2)=\chi_{1s}\chi_{2t}-\chi_{1t}\chi_{2s} \tag{2.4}$$

and D_{st} is as previously defined [Eq. (1.15) *et seq.*]. Obviously, $A(st)=A(ts)$. Likewise, define

$$D_{stu\cdots}=\text{cofactor of } \chi_{1s}\chi_{2t}\chi_{3u}\cdots \tag{2.5}$$

in the determinant of Eq. (1.12). The wave function in $R_{1,j-1}$ can be written (for $j=3,\cdots,N$) as

$$\psi_{1,j-1}=\sum_{\substack{\alpha_1\cdots\alpha_j=1\\ \text{(all distinct)}}}^{N}A^{j-1}(\alpha_1,\cdots,\alpha_j)\times\prod_{m=1}^{j}\chi_{m,\alpha_m}D_{\alpha_1,\cdots,\alpha_j}(x_{j+1},\cdots,x_N). \tag{2.6}$$

The fact that $\psi_{1,j-1}$ vanishes when $x_{m-1}=x_m$ (for $m=4,5,\cdots,j-2$) requires that $A^{j-1}(\alpha_1,\cdots,\alpha_j)$ be *symmetric* in the last $j-2$ variables. Beyond that there is no other symmetry requirement on A^j.

Just as we derived Eq. (1.17),[8] we can use (1.4) to express A^{j-1} in terms of A^{j-2}, viz.,

$$\begin{aligned}
A^{j-1}(\alpha_1,\cdots,\alpha_j)&=A^{j-2}(\alpha_1,\cdots,\alpha_{j-1})[1-ic(k_{\alpha_2}-k_{\alpha_j})^{-1}]\\
&\quad+A^{j-2}(\alpha_1,\alpha_j,\alpha_3,\cdots,\alpha_{j-1})(ic)(k_{\alpha_2}-k_{\alpha_j})^{-1},
\end{aligned} \tag{2.7}$$

where $A^1(\alpha_1,\alpha_2)\equiv A(\alpha_1,\alpha_2)$. Defining

$$t(r,s)=-ic(k_r-k_s)^{-1} \quad\text{and}\quad m(r,s)=1+t(r,s), \tag{2.8}$$

we deduce that

$$A^{j-1}(\alpha_1,\cdots,\alpha_j)=A(\alpha_1,\alpha_2)\prod_{k=3}^{j}m(\alpha_2,\alpha_k)+\sum_{k=3}^{j}A(\alpha_1,\alpha_k)t(\alpha_k,\alpha_2)\prod_{l=3,l\neq k}^{j}m(\alpha_k,\alpha_l). \tag{2.9}$$

The last product in Eq. (2.9) is to be omitted for $j=3$. The derivation of Eq. (2.9) from Eq. (2.7) is a simple exercise in induction if one observes that

$$t(k,l)t(m,k)+t(l,k)t(m,l)+t(m,k)t(l,m)=0. \tag{2.10}$$

Note that A^{j-1} satisfies the symmetry requirements mentioned above.

To determine $A(s,t)$ we turn to the PBC relation, (2.2), which states that

$$A(s,t)=\chi_sA^{N-1}(t,s,\alpha_3,\cdots,\alpha_N) \tag{2.11}$$

[8] For details such as this, see Michael Flicker, Ph.D. thesis, Yeshiva University, 1966 (unpublished).

for t, s not in the set $\alpha_3, \cdots, \alpha_N$, and with $\chi_s = \exp(ik_sL)$ as before. We insert (2.9) into (2.11) and using (2.8) and a bit of algebraic manipulation we obtain the key equation

$$\sum_{j=1}^{N} (K_{sj} - K_{rj}) B_{jr} = 0 \tag{2.12}$$

for all $s, r = 1, \cdots, N$. Here,

$$K_{sj} = \delta_{sj}\chi_s^{-1} - ic\eta_j(k_s - k_j + ic)^{-1}, \tag{2.13a}$$

$$\eta_j = \prod_{k=1, k\neq j}^{N} m_{jk}, \tag{2.13b}$$

$$B_{sj} = A(s, j)(k_s - k_j). \tag{2.13c}$$

A necessary and sufficient condition for $A(s,j)$ to be symmetric is that B_{sj} be antisymmetric. Even though $A(s, s)$ is not defined, we have defined $B_{ss} = 0$ in obtaining (2.12). Furthermore, if $k_j = k_s + ic$ for some pair s and j, then K_{sj} is not well defined by (2.13a). In such a case, however, $\eta_j = 0$ and it is correct to use L'Hospital's rule whence

$$-ic\eta_j(k_s - k_j + ic)^{-1} \text{ is replaced by } \prod_{k=1, k\neq j, k\neq s}^{N} m_{jk}.$$

Equation (2.12) can be regarded as an eigenvalue equation for the antisymmetric matrix B. Only for special choices of K, and hence of $\{k\}$, will there be a solution. To solve (2.12) for B we first regard the N-dimensional vector $\mathbf{P}$, defined by

$$P_r = \sum_j K_{rj} B_{jr}, \tag{2.14}$$

as a known vector, and then determine B from the matrix equation

$$\sum_j K_{sj} B_{jr} = P_r. \tag{2.15}$$

If $\mathbf{P} = 0$, then (2.15) is homogeneous. Let $\mathbf{V}^1, \mathbf{V}^2, \cdots, \mathbf{V}^\mu$ be an orthonormal basis for the null space, V, of K, which is defined as the set of vectors satisfying

$$K\mathbf{V} = 0. \tag{2.16}$$

Then, for a fixed r, we can write

$$B_{jr} = \sum_{\alpha=1}^{\mu} V_j^\alpha S_r^\alpha, \tag{2.17}$$

where the S_r^α are a set of μ coefficients which depend upon r. But, since B is antisymmetric, it is easy to prove that (for $\mathbf{P} = 0$)

$$B_{jr} = \sum_{\alpha=1}^{\mu} \sum_{\beta=1}^{\mu} V_j^\alpha V_r^\beta C_{\alpha\beta}, \tag{2.18}$$

where C is *any* antisymmetric μ-square matrix. Of course, $\mu = d(V) =$ dimension of V is unknown. Unless $\mu \geq 2$, there is no solution to (2.15) for $\mathbf{P} = 0$.

If $\mathbf{P} \neq 0$, then multiply (2.15) by P_r^* and sum on r. Defining the vector $\mathbf{G} = B\mathbf{P}^*$, we have

$$K\mathbf{G} = \rho^2 \mathbf{I}, \tag{2.19}$$

where $\mathbf{I}$ is the vector $I_j = 1$ and $\rho^2 = \sum_r |P_r|^2 \neq 0$. This means that the matrix K must have a "unit vector" $\mathbf{g}$:

$$K\mathbf{g} = \mathbf{I}. \tag{2.20}$$

This unit vector can be defined to be orthogonal to the space V and hence is unique. By the same argument as that leading to (2.18), it is easy to show that

$$B_{jr} = \sum_\alpha \sum_\beta V_j^\alpha V_r^\beta C_{\alpha\beta} + g_j P_r - P_j g_r. \tag{2.21}$$

If this form is inserted into (2.15), it follows at once that $\mathbf{P}$ must itself be a null vector which may be taken to be $\rho \mathbf{V}^1$. Hence, the general solution to (2.15) is

$$B_{jr} = \sum_{\alpha=1}^{\mu} \sum_{\beta=1}^{\mu} V_j^\alpha V_r^\beta C_{\alpha\beta} + \rho(g_j V_r^1 - V_j^1 g_r). \tag{2.22}$$

If now we recall the definition of $\mathbf{P}$ [Eq. (2.14)], we see that for arbitrary ρ and arbitrary antisymmetric C (2.22) is indeed a solution to (2.12).

In conclusion, we may state that the necessary and sufficient conditions for a nonzero antisymmetric solution, B, to (2.12) to exist is that the matrix K have either (a) a "unit vector" and one or more null vectors or (b) no unit vector but two or more null vectors. In both cases, the general solution is (2.22).

At this point we could inquire into the properties of $\{k\}$ necessary to insure that the above requirement on K is fulfilled. Since it turns out that the set $\{k\}$ is *not* thereby completely determined, we shall instead turn to additional properties of B which will completely determine $\{k\}$. These are provided by the PBC on the "up" particles.

Consider first the *full* statement of PBC for the "down" particles which is that

$$\begin{aligned}\psi_{1j}(x_1, x_2 \mid x_3, \cdots, x_N) &= \psi(x_1 + L, x_2 \mid x_3, \cdots, x_N) \\ &= -\psi(x_2, x_1 + L \mid x_3, \cdots, x_N) \\ &= -\psi_{j,N-j}(x_2, x_1 + L \mid x_3, \cdots, x_N)\end{aligned} \tag{2.23}$$

for $j = 1, \cdots, N-1$. Equation (2.23) leads to Eq. (2.12) for $j = 1$, as we have already discussed, but for $j > 1$ it is first necessary to define $\psi_{j,N-j}$. This can be done from the statement of PBC on the "up" particles which reads

$$\begin{aligned}\psi_{ij}&(x_1, x_2 \mid x_3, \cdots, x_N) \\ &= \psi(x_1, x_2 \mid x_3 + L, x_4, \cdots x_N) \\ &= (-1)^{N-1}\psi_{i-1,j}(x_1, x_2 \mid x_4, \cdots, x_N, x_3 + L)\end{aligned} \tag{2.24}$$

for $i=2, \cdots, N-1$. If we iterate (2.24) $i-1$ times, we obtain

$$\psi_{ij}(x_1, x_2 \mid x_3, \cdots, x_N) = (-1)^{(N-1)(i-1)}\psi_{1j}(x_1, x_2 \mid x_{i+2}, \cdots, x_N, x_3+L, \cdots, x_{i+1}+L). \quad (2.25)$$

Equation (2.25) gives ψ_{ij} in terms of ψ_{1j} (for $i>1$) and, in fact, may be taken as the definition of ψ_{ij}. With this definition of ψ_{ij} it is trivial to verify that (a) PBC for the "up" particles, (2.24), is automatically satisfied and (b) the delta-function and continuity conditions, (1.4), relating the ψ_{ij} to each other are automatically fulfilled if the sequence ψ_{1j} satisfies these conditions. This latter requirement has already been met through (2.12).

Thus, the only outstanding condition to be satisfied is (2.23) which, using (2.25), reads

$$\psi_{1j}(x_1, x_2 \mid x_3, \cdots, x_N) = -(-1)^{(N-1)(j-1)}\psi_{1,N-j}(x_2, x_1+L \mid x_{j+2}, \cdots, x_N, x_3+L, \cdots, x_{j+1}+L). \quad (2.26)$$

In terms of the functions A^j defined in (2.6), Eq. (2.26) becomes

$$A^j(\alpha_1, \cdots, \alpha_{j+1}) = A^{N-j}(\alpha_2, \alpha_1, \alpha_{j+2}, \cdots, \alpha_N)\chi_{\alpha_1}\prod_{i=3}^{j+1}\chi_{\alpha_i}. \quad (2.27)$$

Equation (2.27) must be satisfied for $j=1, \cdots, N-1$. It seems intuitively reasonable, however, that satisfying it for $j=1$ (as we have done) and for $j=2$ should be sufficient, because in some sense we will have thereby satisfied PBC for an "up" and a "down" particle. Unfortunately, we have not found a general proof with which to make our intuition rigorous. Instead, we give a tedious, but rigorous, inductive proof in Appendix A, using the defining Eq. (2.9), that (2.27) is automatically satisfied for all j if it is satisfied for $j=1$ and $j=2$.

To evaluate (2.27) for $j=2$, we insert the definitions (2.8), (2.9), and (2.13) and obtain

$$\frac{1}{\chi_r\chi_j}\left[\frac{B_{rs}m_{sj}}{k_r-k_s}+\frac{B_{rj}t_{js}}{k_r-k_j}\right] = ic\sum_{e=1}^{N}\frac{B_{se}(k_j-k_e)\eta_e}{(k_r-k_e+ic)(k_s-k_e+ic)(k_j-k_e+ic)} \quad (2.28)$$

for every triplet of distinct integers r, s, j. If we now make a partial fraction expansion of the triple product in the denominator of (2.28) and make use of the condition (2.12), we obtain the alternative condition

$$B_{sr}\{k_sk_r\alpha_j+k_s\beta_j+k_r\gamma_j+\delta_j\} = ic(k_r-k_s)\{B_{rj}+\chi_rB_{sj}\}, \quad (2.29)$$

where

$$\alpha_j=1-\chi_j,$$
$$\beta_j=\chi_j(k_j-ic)-k_j,$$
$$\gamma_j=k_j\chi_j-(k_j+ic),$$
$$\delta_j=k_j^2(1-\chi_j)+ick_j(1+\chi_j). \quad (2.30)$$

It is a simple matter to show that (2.29) is true for *all* values of r, s, and j not only distinct values. Our problem is then completely solved if we can choose $\{k\}$ so that (2.22) and (2.29) are both satisfied.

Consider the four vectors, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\delta}$ which are defined in (2.30). Note that $\boldsymbol{\gamma}=\boldsymbol{\beta}-ic\boldsymbol{\alpha}$. We wish to show that these vectors are in the space $U=V\oplus\mathbf{g}$ which is the set of all vectors $\mathbf{u}$ such that $K\mathbf{u}=$constant. To do this we consider the space Q orthogonal to U and show that $\boldsymbol{\alpha}\cdot\mathbf{q}=\boldsymbol{\beta}\cdot\mathbf{q}=\boldsymbol{\gamma}\cdot\mathbf{q}=\boldsymbol{\delta}\cdot\mathbf{q}=0$ for all $\mathbf{q}$ in Q. It is easy to show that Q is nonempty but it is not necessary to do so. If $\mathbf{q}$ is in Q, multiply (2.29) by q_j^* and sum on j. From (2.22) $B\mathbf{q}=0$, whence

$$B_{sr}\xi_{sr}=0 \qquad (\text{all } s, r), \quad (2.31)$$

where

$$\xi_{sr}=k_sk_r(\boldsymbol{\alpha}\cdot\mathbf{q})+k_s(\boldsymbol{\beta}\cdot\mathbf{q})+k_r(\boldsymbol{\gamma}\cdot\mathbf{q})+\boldsymbol{\delta}\cdot\mathbf{q}=0. \quad (2.32)$$

Now, B_{sr} can not be zero for all s and r (otherwise the wave function vanishes), and we can therefore state that $B_{\sigma\rho}\neq0$ for some σ and ρ. Since B is antisymmetric it follows that $B_{\rho\sigma}\neq0$ as well. We can also state that $B_{\tau\rho}\neq0$ for some $\tau\neq\sigma$ otherwise, from (2.15) $K_{s\sigma}B_{\sigma\rho}=P_\rho$ so that $K_{s\sigma}$ is independent of s—an impossible conclusion by inspection of (2.13a) and use of the fact that the k's are distinct. Hence, (2.31) says that $0=\xi_{\sigma\rho}=\xi_{\rho\sigma}=\xi_{\tau\rho}=\xi_{\rho\tau}$ for σ, ρ, and τ distinct. Again, using the fact that the k's are distinct, we conclude that the factors in parentheses in (2.32) are zero and we establish that $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\delta}$ are in U. Note that if $\mathbf{P}=0$, then $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\delta}$ are in V.

The above exploited only the properties of (2.29) and did not take into account the fact that the matrix K contains χ and is therefore not totally unrelated to the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\delta}$. We now make use of this latter fact and find the stronger result that $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ (and hence $\boldsymbol{\beta}$ also) must be contained in the null space V whether or not $\mathbf{P}=0$. To prove this we rewrite (2.13a) as $K_{sj}=\delta_{sj}\chi_s^{-1}+D_{sj}$ with

$$D_{sj}=-ic\eta_j(k_s-k_j+ic)^{-1}. \quad (2.33)$$

The matrix D has two interesting properties:

(a) $$\sum_j D_{sj} = -1; \quad (2.34)$$

(b) if $$\sum_j D_{sj}u_j = w_s, \quad (2.35a)$$

then
$$\sum_j D_{sj}u_jk_j = ic(\sum_j \eta_j u_j) + (k_s+ic)w_s. \quad (2.35b)$$

To prove (a) consider
$$F(x) \equiv \prod_{j=1}^{N} [1-ic(x-k_j)^{-1}]$$
$$= 1 - ic \sum_{j=1}^{N} \eta_j (x-k_j)^{-1}, \quad (2.36)$$

the latter form having been obtained by partial fraction expansion. The first expression yields $F(k_s+ic)=0$, while the second yields $F(k_s+ic)=1+\sum_j D_{sj}$, Q.E.D. The proof of (b) follows immediately from (2.33) and the observation that
$$u_jk_j = -u_j(k_s-k_j+ic)+u_j(k_s+ic).$$

Note that the first term on the right side of (2.35b) is a constant, independent of s, and is therefore proportional to the vector $\mathbf{I}$ introduced in (2.19).

To proceed, we introduce an abbreviated, self-explanatory, notation whereby a quantity such as $\mathbf{k}\boldsymbol{\alpha}$ stands for the vector with components $k_j\alpha_j$, while $\boldsymbol{\chi}^{-1}$ stands for the vector with components $(\chi_j)^{-1}$. Now, since $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\delta}$ are in $V \oplus \mathbf{g}$ we have
$$K\boldsymbol{\alpha} = b^1\mathbf{I},$$
$$K\boldsymbol{\gamma} = b^2\mathbf{I},$$
$$K\boldsymbol{\delta} = b^3\mathbf{I}, \quad (2.37)$$

where b^1, b^2, and b^3 are certain constants. To prove that $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are in V we must show that $0=b^1=b^2$. Using the explicit forms (2.30) we have
$$D\boldsymbol{\alpha} = (b^1+1)\mathbf{I} - \boldsymbol{\chi}^{-1}, \quad (2.38a)$$
$$D\boldsymbol{\gamma} = b^2\mathbf{I} - \mathbf{k} + (\mathbf{k}+ic)\boldsymbol{\chi}^{-1}, \quad (2.38b)$$
$$D\boldsymbol{\delta} = b^3\mathbf{I} + \mathbf{k}(\mathbf{k}-ic) - \mathbf{k}(\mathbf{k}+ic)\boldsymbol{\chi}^{-1}. \quad (2.38c)$$

But note that $\boldsymbol{\gamma} = -\mathbf{k}\boldsymbol{\alpha} - ic\mathbf{I}$ and, inserting this expression into (2.38b) and using lemmata (a) and (b) above, we have
$$D\boldsymbol{\gamma} = -ic(b^1 + \sum_{j=1}^{N} \eta_j\alpha_j)\mathbf{I} - \mathbf{k} + (\mathbf{k}+ic)\boldsymbol{\chi}^{-1} - b^1\mathbf{k}. \quad (2.39)$$

Since the right sides of (2.38b) and (2.39) must be equal for all the N different values of k, we must conclude that $b^1=0$. Likewise, $\boldsymbol{\delta} = -\mathbf{k}\boldsymbol{\gamma} + ic\mathbf{k} - ic\mathbf{k}\boldsymbol{\alpha}$. Inserting this into (2.38c) and again using the two lemmata, we have
$$D\boldsymbol{\delta} = -\mathbf{k}(\mathbf{k}+ic)\boldsymbol{\chi}^{-1} + \mathbf{k}(\mathbf{k}-ic) - b^2\mathbf{k}$$
$$+ ic[ic\sum \eta_j\chi_j - \sum \eta_j\gamma_j - 2ic - b^2]\mathbf{I}. \quad (2.40)$$

Upon comparison with the right side of (2.38c), we see that b^2 must also vanish, Q.E.D.

We note also that $b_1=b_2=0$ implies, from (2.38b) and (2.39), that
$$\sum_{j=1}^{N} \eta_j\alpha_j = 0. \quad (2.41)$$

The condition that $K\boldsymbol{\alpha} = K\boldsymbol{\beta} = 0$ is thus equivalent to
$$K\boldsymbol{\alpha} = 0 \text{ and } (2.41).$$

The next fact to be proved is that, for $S_T = \frac{1}{2}N - 2$, $B_{sr} = E_1(\alpha_s\beta_r - \alpha_r\beta_s)$ where E_1 depends on the normalization of the wave function. At the very end of the proof we make the important observation that $\boldsymbol{\delta}$ is a linear combination of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and also that $\mathbf{P}=0$. It will become clear during the course of the proof exactly how the possibilities $S_T = \frac{1}{2}N$, $S_T = \frac{1}{2}N - 1$, and $S_T = \frac{1}{2}N - 2$ come out of the formalism.

When we combine (2.29) and the equation obtained from it by interchange of r and s we obtain
$$B_{sr}\alpha_j = [B_{sj}\alpha_r - B_{rj}\alpha_s]. \quad (2.42)$$

If we now substitute the explicit form of B (2.22) into the right-hand side and also observe that when (2.42) is multiplied by K_{ls} and summed on s one obtains
$$\rho V_r^1\alpha_j = \rho V_j^1\alpha_r, \quad (2.43)$$

it then follows that
$$B_{sr}\alpha_j = \sum_i C_{ij}'(V_s^i\alpha_r - V_r^i\alpha_s) + \rho V_j^1(\alpha_r g_s - \alpha_s g_r), \quad (2.44)$$

where $C_{ij}' = \sum_k C_{ik}V_j^k$. Clearly there are now two possibilities for the α_j; either $\alpha_j=0$ for all j or $\alpha_j \neq 0$ for some j. If $\alpha_j = 0$ for all j, then $S_T = \frac{1}{2}N$; we will not say any more about this possibility. If $\alpha_j \neq 0$ for some j, then from (2.44) we have
$$B_{sr} = E_1(\alpha_r\beta_s - \alpha_s\beta_r) + E_2(\alpha_r v_s - \alpha_s v_r) + E_3(\alpha_r g_s - \alpha_s g_r), \quad (2.45)$$

where $\mathbf{v}$ is a null vector orthogonal to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, and E_1, E_2, and E_3 are constants. Upon substitution of (2.45) into (2.29) it follows, after some algebra, that
$$B_{sr}(\mathbf{v}\cdot\boldsymbol{\delta}) = icE_2 \mid v \mid^2 \{\alpha_r k_s + \beta_r + \chi_r(ic\alpha_s - \beta_s) - k_r\chi_r\alpha_s\}. \quad (2.46)$$

Recalling that $KB = \mathbf{P}$, $K\boldsymbol{\alpha}=0$, and $K\boldsymbol{\beta}=0$, we obtain from (2.46)
$$P_r(\mathbf{v}\cdot\boldsymbol{\delta}) = icE_2 \mid v \mid^2 \{\alpha_r \sum_s K_{ls}k_s + \beta_r \sum_s K_{ls}\}. \quad (2.47)$$

Since a consequence of (2.43) is that $\mathbf{P}=\rho\sigma\boldsymbol{\alpha}(\sigma\neq 0)$, either $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are linearly dependent or

$$icE_2\,|\,v\,|^2\beta_r\sum_s K_{ls}=0\quad\text{for all}\quad r,\ l.$$

If the former is the case, then $S_T=\frac{1}{2}N-1$ (this is the solution of McGuire[6]). The remaining possibility corresponds to $S_T=\frac{1}{2}N-2$. It is a simple exercise to show that, for some l, $\sum_s K_{ls}\neq 0$. Also $\beta_r\neq 0$ for some r; if $\beta_r=0$ for all r, then $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are linearly dependent and we have the McGuire problem. Hence $E_2\,|\,v\,|^2=0$. In a similar way we conclude that $E_3=0$. Consequently $B_{sr}=E_1(\alpha_r\beta_s-\alpha_s\beta_r)$. Direct substitution of this relation into (2.29) verifies that $\boldsymbol{\delta}$ is a linear combination of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$; from (2.22) it is obvious that $\mathbf{P}=0$.

To summarize the results for $S_T=\frac{1}{2}N-2$:

(i) $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are null vectors;

(ii) $\boldsymbol{\delta}$ is a linear combination of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$;

(iii) $B_{sr}=E_1(\alpha_r\beta_s-\alpha_s\beta_r)$.

What remains to be done is to put (i) and (ii) into a form more amenable to the calculation of the set $\{k\}$. From (ii) and (2.30) we obtain an equation that when solved for χ yields (1.25), i.e.,

$$\chi_j\equiv\exp(ik_jL)=\frac{[k_j+\delta+\frac{1}{2}(ic)]\,[k_j+\epsilon+\frac{1}{2}(ic)]}{[k_j+\delta-\frac{1}{2}(ic)]\,[k\,+\epsilon-\frac{1}{2}(ic)]},\qquad(2.48)$$

where δ and ϵ are constants to be determined. The assumption $\delta<\epsilon$ can be partially justified[8] by observing that the set $\{k_j(c)\}$ is one to one with the known set $\{k_j(0)\}$.

Having obtained the expression (2.48) for χ_j (and thus $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$) in terms of k_j, our problem will be completely solved if we can satisfy condition (i), namely that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be null vectors. This requirement leads to the conditions (1.26) on the constants δ and ϵ. The details are given in Appendix B.

III. THE ENERGY LEVELS

In this section we will derive the explicit form, in the thermodynamic limit, of the sets $\{k_j\}$. Once the $\{k_j\}$ are known, the determination of the energy levels is straightforward. As an illustration of this we will calculate the ground-state energy.

In order to calculate the sets $\{k_j\}$ it is helpful to write Eqs. (1.25) and (1.26) in a slightly different form. With the observation that

$$\frac{[k_j+\delta+\frac{1}{2}(ic)]}{[k_j+\delta-\frac{1}{2}(ic)]}=-\exp\{-2i\tan^{-1}(2/c)(k_j+\delta)\},\qquad(3.1)$$

we see that (1.25) is equivalent to

$$k_j+\frac{2}{L}\tan^{-1}\frac{2}{c}\,(k_j+\delta)+\frac{2}{L}\tan^{-1}\frac{2}{c}\,(k_j+\epsilon)=\frac{2\pi n_j}{L}\qquad(3.2)$$

and (1.26) to

$$2\tan^{-1}\left(\frac{\epsilon-\delta}{c}\right)+2\sum\tan^{-1}\frac{2}{c}\,(k_j+\delta)=2\pi\nu+\pi[(N-1)\bmod 2],\qquad(3.3\text{a})$$

$$2\tan^{-1}\left(\frac{\delta-\epsilon}{c}\right)+2\sum\tan^{-1}\frac{2}{c}\,(k_j+\epsilon)=2\pi\mu+\pi[(N-1)\bmod 2].\qquad(3.3\text{b})$$

In particular, we are interested in determining the energy levels in the limit of a large system. This means N, $L\to\infty$ such that $\rho=N/L=$fixed constant. Under these conditions the set $\{k_j\}$ can be determined by iteration. To determine the energy and momentum of a state correct to $O(1)$ it is sufficient that k be known to $O(1/L)$. It is clear from (3.2) that it is only necessary to know ϵ and δ to $O(1)$ to get the desired accuracy for k. However, from (3.3) it follows that it is only necessary to know the k's to $O(1)$ in order to get ϵ and δ to $O(1)$. This is because μ, ν will be $O(N)$ and the summation on the left hand side of (3.3) will be $O(N)$. But to $O(1)$, $k_j=2\pi n_j/L$, therefore ϵ and δ are the solutions to

$$\sum\tan^{-1}\frac{2}{c}\left(\frac{2\pi n_j}{L}+\delta\right)=\pi\nu,\qquad(3.4\text{a})$$

$$\sum\tan^{-1}\frac{2}{c}\left(\frac{2\pi n_j}{L}+\epsilon\right)=\pi\mu;\qquad(3.4\text{b})$$

and, to the desired accuracy,

$$k_j=\frac{2\pi n_j}{L}-\frac{2}{L}\tan^{-1}\frac{2}{c}\left(\frac{2\pi n_j}{L}+\delta\right)-\frac{2}{L}\tan^{-1}\frac{2}{c}\left(\frac{2\pi n_j}{L}+\epsilon\right).\qquad(3.5)$$

Note: Since, for $c\neq 0$, the k's are distinct, the n's are also distinct.

The numbers $\{n_j\}$, μ, and ν, which clearly are the quantum numbers of the system and specify all the states, must be known in order to calculate the energy of any state. Therefore, the first step in calculating the ground-state energy must be to determine its quantum numbers. Since the k_j's are given essentially by $2\pi n_j/L$, the problem is similar to that of the one-dimensional noninteracting, spinless Fermi gas. Thus, we would expect $n_{j+1}-n_j=1$, $n_1\sim-[\frac{1}{2}N]$ and $n_N\sim[\frac{1}{2}N]$. The proof that this is actually the case is quite simple. Consider two states, a and b, such that

$$\begin{aligned}&n_i^a=n_i^b && i=1,\cdots,j;\\ &n_{i+1}^a=n_i^b && i=j+1,\cdots,N-1;\\ &\nu_a=\nu_b;\quad \mu_a=\mu_b; && |\,n_j^a\,|<|\,n_N^b\,|.\end{aligned}\qquad(3.6)$$

From (3.4) it is clear that, to the order of interest, $\delta^a=\delta^b$ and $\epsilon^a=\epsilon^b$. Hence, from (3.5) it is obvious that

$|k_j^a|<|k_N^b|$. Since $E^b-E^a=(k_N^b)^2-(k_j^a)^2$ we conclude $E^a<E^b$. By induction it follows that in the ground state $n_{j+1}-n_j=1$. Thus,

$$n_j=-\tfrac{1}{2}(N-2j+1)+\lambda, \tag{3.7}$$

where $\lambda=0$ for N odd and $\lambda=\pm\frac{1}{2}$ for N even.

Define

$$q_j\equiv 2\pi n_j/L \tag{3.8}$$

then from (3.5)

$$k_j=q_j-(2/L)[\tan^{-1}(2/c)(q_j+\delta)+\tan^{-1}(2/c)(q_j+\epsilon)]. \tag{3.9}$$

Therefore, the energy, (1.8), of the state is given to $O(1)$ by

$$E=\sum_{j=1}^{N}\left\{q_j^2-\frac{4}{L}q_j\left[\tan^{-1}\frac{2}{c}(q_j+\delta)+\tan^{-1}\frac{2}{c}(q_j+\epsilon)\right]\right\}. \tag{3.10}$$

The term $\sum q_j^2$ is just the ground state for $S=N/2$. Thus, changing the remaining sum to an integral, (3.10) can be written as

$$E=E_g(S=\tfrac{1}{2}N)-\frac{2}{\pi}\int_{-K}^{K}q[\tan^{-1}(2/c)(q+\delta)+\tan^{-1}(2/c)(q+\epsilon)]dq, \tag{3.11}$$

where $K=\pi\rho$. When E is minimized with respect to ϵ and δ, we obtain the ground state energy for $S=\frac{1}{2}N-2$.

$$E_g(S=\tfrac{1}{2}N-2)=E_g(S=\tfrac{1}{2}N)+2c\rho-4\pi\rho^2\tan^{-1}\frac{2\pi\rho}{c}-\frac{c^2}{\pi}\tan^{-1}\frac{2\pi\rho}{c}. \tag{3.12}$$

The result is just what one would expect. The deviation from $E_g(S=N/2)$ is just twice the deviation of the $S=\frac{1}{2}N-1$ problem.[6]

APPENDIX A

Theorem: If

$$A^m(\alpha_1,\cdots,\alpha_{m+1})=A^{N-m}(\alpha_2,\alpha_1,\alpha_{m+2},\cdots,\alpha_N)\chi_{\alpha_1}\prod_{i=3}^{m+1}\chi_{\alpha_i} \tag{A1}$$

for $m=1$, 2 then it is true for $m>2$.

Proof: The proof is by induction. We will assume (A1) is true for $m=j-2$, $j-1$ and prove it is true for $m=j$.

The natural starting place is Eqs. (2.7) that relate various A^j's. To proceed: From (2.7) we obtain two equations, one that "steps down" the A^j's and another that "steps up" the A^j's, that is,

$$A^{j-1}(\alpha_1,\cdots,\alpha_{j+1})=A^{j-2}(\alpha_1,\cdots,\alpha_{j-1})m(\alpha_2,\alpha_j)-A^{j-2}(\alpha_1,\alpha_j,\alpha_3,\cdots,\alpha_{j-1})t(\alpha_2,\alpha_j), \tag{A2a}$$

$$A^{j-2}(\alpha_1,\cdots,\alpha_{j-1})=A^{j-1}(\alpha_1,\cdots,\alpha_j)m(\alpha_j,\alpha_2)-A^{j-1}(\alpha_1,\alpha_j,\alpha_3,\cdots,\alpha_{j-1},\alpha_2)t(\alpha_j,\alpha_2). \tag{A2b}$$

If we "step up" $A^{N-j}(\alpha_1,\cdots,\alpha_{N-j+1})$ and change the indices as follows

$$\begin{aligned}&\alpha_1\to\alpha_2\\&\alpha_2\to\alpha_1\\&\alpha_3\to\alpha_{j+2}\\&\quad\vdots\\&\alpha_{N-j+1}\to\alpha_N\\&\alpha_{N-j+2}\to\alpha_{j+1},\end{aligned}$$

we obtain

$$A^{N-j}(\alpha_2,\alpha_1,\alpha_{j+2},\cdots,\alpha_N)=A^{N-j+1}(\alpha_2,\alpha_1,\alpha_{j+1},\cdots,\alpha_N)m(\alpha_{j+1},\alpha_1)-A^{N-j+1}(\alpha_2,\alpha_{j+1},\alpha_{j+2},\cdots,\alpha_N,\alpha_1)t(\alpha_{j+1},\alpha_1). \tag{A3}$$

Since (A1) is true for $m=j-1$, (A3) yields

$$\chi_{\alpha_1}\prod_{i=3}^{j+1}\chi_{\alpha_i}A^{N-j}(\alpha_2,\alpha_1,\alpha_{j+2},\cdots,\alpha_N)=\chi_{\alpha_{j+1}}m(\alpha_{j+1},\alpha_1)A^{j-1}(\alpha_1,\cdots,\alpha_j)-\chi_{\alpha_1}t(\alpha_{j+1},\alpha_1)A^{j-1}(\alpha_{j+1},\alpha_2,\cdots,\alpha_j). \tag{A4}$$

But, when we step down $A^j(\alpha_1,\cdots,\alpha_{j+1})$ we find

$$A^j(\alpha_1,\cdots,\alpha_{j+1})=A^{j-1}(\alpha_1,\cdots,\alpha_j)m(\alpha_2,\alpha_{j+1})-A^{j-1}(\alpha_1,\alpha_{j+1},\alpha_3,\cdots,\alpha_j)t(\alpha_2,\alpha_{j+1}). \tag{A5}$$

Thus, if (A1) is to be true for $m=j$ we must prove that

$$\text{r.h.s. (A4)}-\text{r.h.s. (A5)}\equiv\Lambda_{j-1}=0. \tag{A6}$$

If in (A6) we change variables to

$$\begin{aligned}\alpha_1&=r,\\\alpha_2&=s,\\\alpha_j&=u,\\\alpha_{j+1}&=v,\\\{(\alpha_3,\cdots,\alpha_{j+1})\}&=\alpha,\end{aligned}$$

and "step down" all A^{j-1}'s to A^{j-2}'s we obtain

$$\chi_v m(v,r)\{A^{j-2}(rs\alpha)m(s,u)-A^{j-2}(ru\alpha)t(s,u)\}$$
$$-\chi_r t(v,r)\{A^{j-2}(vs\alpha)m(s,u)-A^{j-2}(vu\alpha)t(s,u)\}$$
$$-m(s,v)\{A^{j-2}(rs\alpha)m(s,u)-A^{j-2}(ru\alpha)t(s,u)\}$$
$$+t(s,v)\{A^{j-2}(rv\alpha)m(v,u)-A^{j-2}(ru\alpha)t(v,u)\}$$
$$=\Lambda_{j-1}. \quad \text{(A7)}$$

Since (A1) is assumed true for $m=j-2, j-1$, it is obvious that $\Lambda_{j-2}=0$. As a consequence we have

$$\chi_v m(v,r)A^{j-2}(rs\alpha)-\chi_r t(v,r)A^{j-2}(vs\alpha)$$
$$-m(s,v)A^{j-2}(rs\alpha)=-A^{j-2}(rv\alpha)t(s,v) \quad \text{(A8a)}$$

and

$$\chi_v m(v,r)A^{j-2}(ru\alpha)-\chi_r t(v,r)A^{j-2}(vu\alpha)$$
$$=m(u,v)A^{j-2}(ru\alpha)-t(u,v)A^{j-2}(rv\alpha). \quad \text{(A8b)}$$

Direct substitution of (A8) into (A7) coupled with (2.10) require $\Lambda_{j-1}=0$ Q.E.D.

APPENDIX B

The condition that α is a null vector is

$$\sum_{j=1}^{N} K_{ij}(1-\chi_j)=0. \quad \text{(B1)}$$

In terms of the matrix D_{ij} defined in (2.33) plus the property (2.34) (B1) can be written as

$$\sum_{j=1}^{N} D_{ij}\chi_j=\chi_i^{-1}-2. \quad \text{(B2)}$$

But from (2.48)

$$\sum_{j=1}^{N} D_{ij}\chi_j=-ic\sum_{j=1}^{N}\frac{\eta_j}{(k_i-k_j+ic)}$$
$$\times\frac{[k_j+\delta+(\tfrac{1}{2}ic)]}{[k_j+\delta-(\tfrac{1}{2}ic)]}\frac{[k_j+\epsilon+(\tfrac{1}{2}ic)]}{[k_j+\epsilon-(\tfrac{1}{2}ic)]}. \quad \text{(B3)}$$

Assuming $\epsilon\neq\delta$, we can make a partial fraction expansion of the triple product in (B3):

$$\frac{1}{(k_i-k_j+ic)}\frac{[k_j+\delta+(\tfrac{1}{2}ic)]}{[k_j+\delta-(\tfrac{1}{2}ic)]}\frac{[k_j+\epsilon+(\tfrac{1}{2}ic)]}{[k_j+\epsilon-(\tfrac{1}{2}ic)]}$$
$$=\frac{[k_i+\delta+\tfrac{3}{2}(ic)][k_i+\epsilon+\tfrac{3}{2}(ic)]}{[k_i+\delta+(\tfrac{1}{2}ic)][k_i+\epsilon+(\tfrac{1}{2}ic)]}\frac{1}{(k_i-k_j+ic)}$$
$$+\frac{ic(ic+\epsilon-\delta)}{(\epsilon-\delta)[k_i+\delta+(\tfrac{1}{2}ic)]}\frac{1}{[k_j+\delta-(\tfrac{1}{2}ic)]}$$
$$+\frac{ic(ic+\delta-\epsilon)}{(\delta-\epsilon)[k_i+\epsilon+(\tfrac{1}{2}ic)]}\frac{1}{[k_j+\epsilon-(\tfrac{1}{2}ic)]}. \quad \text{(B4)}$$

If we define

$$a\equiv-ic\sum\eta_j[k_j+\delta-(\tfrac{1}{2}ic)]^{-1}$$

and

$$b\equiv-ic\sum\eta_j[k_j+\epsilon-(\tfrac{1}{2}ic)]^{-1}, \quad \text{(B5)}$$

and again use (2.34), then from (B3)

$$\sum_{j=1}^{N} D_{ij}\chi_j=-\frac{[k_i+\delta+(\tfrac{3}{2}ic)]}{[k_i+\delta+(\tfrac{1}{2}ic)]}\frac{[k_i+\epsilon+(\tfrac{3}{2}ic)]}{[k_i+\epsilon+(\tfrac{1}{2}ic)]}$$
$$+\frac{aic(ic+\epsilon-\delta)}{(\epsilon-\delta)[k_i+\delta+(\tfrac{1}{2}ic)]}+\frac{bic(ic+\delta-\epsilon)}{(\delta-\epsilon)[k_i+\epsilon+(\tfrac{1}{2}ic)]}. \quad \text{(B6)}$$

If we substitute (B6) on the left-hand side of (B2), put the explicit k dependence of χ, (2.48), in on the right-hand side and simplify we obtain

$$k_i[a(ic+\epsilon-\delta)-b(ic+\delta-\epsilon)]$$
$$+a(ic+\epsilon-\delta)[\epsilon+(\tfrac{1}{2}ic)]-b(ic+\delta-\epsilon)[\delta+(\tfrac{1}{2}ic)]$$
$$-2ic(\epsilon-\delta)=0. \quad \text{(B7)}$$

Since this must be true for all values of k_i, both the coefficient of the k_i and the constant term must be zero. Manipulation of these two terms yields

$$a(\epsilon-\delta+ic)=2ic, \quad \text{(B8a)}$$

$$b(\delta-\epsilon+ic)=2ic. \quad \text{(B8b)}$$

If in (2.36) we let $x=-\delta+\frac{1}{2}(ic)$, we find

$$1-a=\prod_{j=1}^{N}\frac{[k_j+\delta+(\tfrac{1}{2}ic)]}{k_j+\delta-(\tfrac{1}{2}ic)}. \quad \text{(B9)}$$

Thus, from (B8a) and (B9) we obtain

$$\prod_{j=1}^{N}\frac{[k_j+\delta+(\tfrac{1}{2}ic)]}{[k_j+\delta-(\tfrac{1}{2}ic)]}=\frac{\epsilon-\delta-ic}{\epsilon-\delta+ic}. \quad \text{(B10a)}$$

From (B8) it follows that (B10) must also be true if ϵ and δ are interchanged. Thus,

$$\prod_{j=1}^{N}\frac{[k_j+\epsilon+(\tfrac{1}{2}ic)]}{[k_j+\epsilon-(\tfrac{1}{2}ic)]}=\frac{\delta-\epsilon-ic}{\delta-\epsilon+ic}. \quad \text{(B10b)}$$

The condition that β be a null vector can be shown not to contain any new information.

Note added in proof. After this work was completed, a letter by M. Gaudin[9] appeared in Physics Letters in which he presented the results of his independent investigation of this problem. His work gives the generalization of the essential results of this paper, Eqs. (1.25) and (1.26), to the case for an arbitrary number of spin deviates. None of the details of the solution were given.

[9] M. Gaudin, Phys. Letters **24A**, 55 (1967).