Symmetry Restrictions on $K \to 2\pi$ Amplitudes*

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The restrictions on the $K \to 2\pi$ amplitudes which follow from the general $SU(3)$ and chiral $SU(3)\otimes SU(3)$ effective weak interactions are investigated. In particular it is found that the peculiar zero-energy limits obtained in the "current-algebra" approach are a consequence of $SU(3)$ invariance for the current-current weak Hamiltonian wherein the symmetry breaking is thought to result from continuation of the external meson masses to their physical values.

I. INTRODUCTION

LET us denote the amplitudes for the three CP-conserving K-meson decays: $K^+ \to \pi^+\pi^0$, $K_1{}^0 \to \pi^+\pi^-$, and $K_1{}^0 \to \pi^0\pi^0$, by A_{+0}, A_{+-}, and A_{00}, respectively. In this article we shall assume that the effects of CP violation are negligible.

The question of major interest is: Which nonleptonic Hamiltonian is responsible for these decays? Generally, the nonleptonic decay Hamiltonian density is taken to be the following universal current-current one:

$$H_1{}^{\rm NL} = -\frac{1}{2}\frac{G}{\sqrt{2}}\cos\theta\,\sin\theta \times\{[(V_2{}^1+P_2{}^1)_\mu,(V_1{}^3+P_1{}^3)_\mu]_+ + (2\leftrightarrow 3)\}\,, \quad (1)$$

where $\sin\theta \simeq \frac{1}{4}$, $G \simeq 10^{-5}/M_p{}^2$, and $V_{b\mu}{}^a$ and $P_{b\mu}{}^a$ are the vector and pseudovector octet currents. $H_1{}^{\rm NL}$ gives rise to both $\Delta I = \frac{1}{2}$ and $\frac{3}{2}$ transitions, so if it is employed it is necessary to explain the tremendous suppression of the $\Delta I = \frac{3}{2}$, $K^+ \to \pi^+\pi^0$ decay by some dynamical mechanism. An alternative approach is to postulate that the nonleptonic Hamiltonian only contains a $\Delta I = \frac{1}{2}$ part and to assume that the small $K^+ \to \pi^+\pi^0$ amplitude comes from electromagnetic violation of isotopic spin symmetry. A current-current Hamiltonian density of this type is

$$H_2 = G_2\{[(V_c{}^3+P_c{}^3)_\mu,(V_2{}^c+P_2{}^c)_\mu]_+ + (2\leftrightarrow 3)\}\,, \quad (2)$$

where G_2 is not *a priori* known. Another Hamiltonian density which has the $\Delta I = \frac{1}{2}$ property is most conveniently written in the framework of the quark model as

$$H_3{}^{\rm NL} = G_3(\bar{q}_3 q_2 + \bar{q}_2 q_3) + G_3'(\bar{q}_3\gamma_5 q_2 - \bar{q}_2\gamma_5 q_3)\,, \quad (3)$$

where the q_a are the quark fields and G_3 and G_3' are also not *a priori* known.

In order to investigate the consequences of these weak interactions, both the group theory[1–7] and current-algebra[8–14] approaches have been used. Of these two, the latter is clearly the more powerful since it does not necessarily introduce unknown parameters into the theory and is even expected to hold in the case when the exact symmetry group invariance is violated. Furthermore, it may also give some insight into the construction of dynamical models.[15,16] On the other hand, the interpretation of the current-algebra results is not unambiguous since what is actually calculated are amplitudes with one or more meson 4-momenta extrapolated to zero. In addition, it is not always clear what part of the current-algebra results comes from dynamics and what part comes from the symmetry structure of the theory. Therefore, as a supplement to understanding this new method, it seems desirable to investigate the restrictions on the $K \to 2\pi$ amplitudes which follow from symmetry considerations alone. An easy way to do this is to construct an *effective* interaction which gives the decay in first order and which has the appropriate symmetry property.

Many authors[1–7] have shown that an effective nonderivative interaction having the $SU(3)$ transformation property of either Eq. (1) or Eq. (2) vanishes identically. On the other hand, no useful restriction is obtained in the case of Eq. (3), which has a different transformation property ($T_3{}^2 - T_2{}^3$ instead of $T_3{}^2 + T_2{}^3$).

In Sec. II we shall consider the general effective interaction, allowing derivatives, which transforms as Eq. (2). By allowing derivatives, we obtain momentum dependences for the amplitudes which reproduce uniquely the current-algebra results in the appropriate extrapolations to zero meson 4-momenta. We note that this method takes some account of both medium strong and electromagnetic violation of $SU(3)$ sym-

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metry if the amplitudes are evaluated at the invariant masses of the *physical* particles.

In Sec. III the general effective interaction which transforms as Eq. (1) will be studied and the possible relevance of these results to current-algebra calculations briefly considered. Section IV will contain a discussion of the additional restrictions imposed by chiral $SU(3)\otimes SU(3)$ invariance. This leads to one way of obtaining octet dominance. Finally Sec. V is a short resumé.

II. OCTET CURRENT-CURRENT HAMILTONIAN

The $SU(3)$ transformation property of Eq. (2) is most conveniently described[3] by the "spurion" matrix

$$S=\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}. \tag{4}$$

We define π to be the 3×3 matrix of pseudoscalar meson fields. Then the general effective interaction for $K\to 2\pi$ which transforms as Eq. (4) is given by

$$\begin{aligned}H_{\text{eff}}=&f_1(\pi\pi\pi S)+f_2(\pi\pi)(\pi S)\\&+g_1(\Box\pi\pi\pi S)+g_2(\pi\Box\pi\pi S)+g_3(\pi\pi\Box\pi S)\\&+g_4(\pi\Box\pi)(\pi S)+g_5(\pi\pi)(\Box\pi S)+\cdots,\end{aligned} \tag{5}$$

where the bracket denotes the trace operation and the $f_i, g_i, \cdots$ are some arbitrary constants. To see that our enumeration of terms in Eq. (5) is complete, we note the existence in this context of relations like

$$(\partial_\mu\pi\partial_\mu\pi\pi S)=\tfrac{1}{2}[(\pi\pi\Box\pi S)-(\Box\pi\pi\pi S)-(\pi\Box\pi\pi S)].$$

Next we impose CP invariance on Eq. (5). Under CP, $\pi(\mathbf{x},t)\to -\pi^T(-\mathbf{x},t)$. This immediately leads to

$$f_1=f_2=0, \tag{6a}$$

$$g_2=g_4=g_5=0, \tag{6b}$$

$$g_1=-g_3. \tag{6c}$$

Equation (6a) is the well-known result[1–3] that the nonderivative effective interaction vanishes. There is only one nonzero term to first order in the squared momenta. Carrying through the above analysis also shows that the general result to all orders in momenta is

$$H_{\text{eff}}=\sum_{a=1}^{\infty}\sum_{\substack{b=0\\a>c}}^{\infty}\sum_{c=0}^{\infty}g_{abc}\times[(\Box^a\pi\Box^b\pi\Box^c\pi s)-(\Box^c\pi\Box^b\pi\Box^a\pi s)], \tag{7}$$

where the g_{abc} are arbitrary constants and $\Box^a$ denotes the d'Alembertian raised to the ath power. The expressions for the three decay amplitudes in momentum space which follow from Eq. (7) are

$$\begin{aligned}A_{+-}(K^2,P_+^2,P_-^2)=\sum\frac{(-1)^{a+b+c}}{\sqrt{2}}g_{abc}&\times[K^{2c}(P_+^{2a}P_-^{2b}+P_-^{2a}P_+^{2b})\\&-K^{2a}(P_+^{2b}P_-^{2c}+P_-^{2b}P_+^{2c})],\end{aligned} \tag{8a}$$

$$\begin{aligned}A_{00}(K^2,P_1^2,P_2^2)=\sum\frac{(-1)^{a+b+c}}{\sqrt{2}}g_{abc}&\times[K^{2c}(P_1^{2a}P_2^{2b}+P_2^{2a}P_1^{2b})\\&-K^{2a}(P_1^{2b}P_2^{2c}+P_2^{2b}P_1^{2c})],\end{aligned} \tag{8b}$$

$$\begin{aligned}A_{+0}(K^2,P_+^2,P_0^2)=\sum\frac{(-1)^{a+b+c}}{\sqrt{2}}g_{abc}&\times[K^{2a}(P_0^{2b}P_+^{2c}-P_+^{2b}P_0^{2c})\\&+K^{2c}(P_+^{2b}P_0^{2a}-P_0^{2b}P_+^{2a})].\end{aligned} \tag{8c}$$

In each amplitude of Eq. (8), K stands for the K-meson four-momentum and the subscripted P's for the final pion-momenta as indicated. This approach requires the conservation of four-momentum in each case. Nevertheless, the extrapolation to the (unphysical) $SU(3)$ limit gives

$$A_{ij}(P^2,P^2,P^2)=0, \tag{9}$$

where i and j stand for $+$, $-$, and 0. The evaluation of the A_{ij} in the limit of exact isotopic spin symmetry simply gives the $\Delta I=\frac{1}{2}$ relations,

$$A_{+0}(K^2,P^2,P^2)=0, \tag{10a}$$

$$A_{+-}(K^2,P^2,P^2)=A_{00}(K^2,P^2,P^2). \tag{10b}$$

We also easily verify the following zero-energy relations:

$$A_{+0}(0,P^2,P^2)=0, \tag{11a}$$

$$\begin{aligned}A_{+0}(P^2,0,P^2)&=-A_{+0}(P^2,P^2,0)\\&=-A_{+-}(P^2,0,P^2)=-A_{+-}(P^2,P^2,0)\\&=\tfrac{1}{2}A_{+-}(0,P^2,P^2)=\tfrac{1}{2}A_{00}(0,P^2,P^2)\\&=-A_{00}(P^2,0,P^2)=-A_{00}(P^2,P^2,0).\end{aligned} \tag{11b}$$

These are the equations obtained by Hara and Nambu[11] using current algebra and $SU(3)$ with the Hamiltonian of Eq. (2). From our derivation, it is seen that these are to be expected from any dynamical model with $SU(3)$ symmetry provided the results are suitably analytic in momentum space.

It is interesting to write out the amplitudes of Eqs. (4) explicitly up to second order in the squared momenta

$$A_{+-}(K^2,P_+^2,P_-^2)=(-g_{100}/\sqrt{2})(P_+^2+P_-^2-2K^2)+(g_{200}/\sqrt{2})(P_+^4+P_-^4-2K^4)+(g_{110}/\sqrt{2})[-K^2(P_-^2+P_+^2)+2P_+^2P_-^2]+\cdots, \quad (12a)$$

$$A_{00}(K^2,P_1^2,P_2^2)=(-g_{100}/\sqrt{2})(P_1^2+P_2^2-2K^2)+(g_{200}/\sqrt{2})(P_1^4+P_2^4-2K^4)+(g_{110}/\sqrt{2})[-K^2(P_1^2+P_2^2)+2P_1^2P_2^2]+\cdots, \quad (12b)$$

$$A_{+0}(K^2,P_+^2,P_0^2)=(-g_{100}/\sqrt{2})(P_0^2-P_+^2)+(g_{200}/\sqrt{2})(P_0^4-P_+^4)+(g_{110}/\sqrt{2})(P_0^2-P_+^2)K^2+\cdots. \quad (12c)$$

If only first-order terms are retained, we have the case considered by Hara and Nambu[11] and also shown by Sakurai[16] to follow from a vector-meson-pole model. This case evidently leads to a nonzero value for A_{+0} if the *physical* pion masses are used. This effect represents the contribution to electromagnetic violation of isospin symmetry resulting from the electromagnetic mass splitting of the (external) mesons. In general, contributions are also to be expected from other types of diagrams. Neglecting these, we would have in this scheme

$$\left|\frac{A_{+-}}{A_{+0}}\right|=2\frac{M^2(K^0)-M^2(\pi^+)}{M^2(\pi^+)-M^2(\pi^0)}\simeq 370,$$

which is much larger than the experimental value[17] of 21. Also

$$\left|\frac{A_{+-}}{A_{00}}\right|=\frac{M^2(K^0)-M^2(\pi^+)}{M^2(K^0)-M^2(\pi^0)}\simeq 0.995.$$

Unfortunately the experimental value of $(1-|A_{+-}/A_{00}|)$ is not well determined.[18]

Another possibility is to assume that, for some reason, $g_{100}=g_{200}=0$ in Eqs. (12). Then to second order we have only the g_{110} terms and the results[15]

$$\left|\frac{A_{+-}}{A_{+0}}\right|=2\frac{M^2(\pi^+)}{M^2(K^+)}\frac{M^2(K^0)-M^2(\pi^+)}{M^2(\pi^+)-M^2(\pi^0)}\simeq 30,$$

$$\left|\frac{A_{+-}}{A_{00}}\right|=\frac{M^2(\pi^+)}{M^2(\pi^0)}\frac{M^2(K^0)-M^2(\pi^+)}{M^2(K^0)-M^2(\pi^0)}\simeq 1.07.$$

A model leading to this numerically interesting case has been proposed by Clavelli.[19]

To end this section, we note that, if the universal current-current Hamiltonian of Eq. (1) is adopted, it is not necessary to explain A_{+0} as an electromagnetic correction.

III. UNIVERSAL CURRENT-CURRENT HAMILTONIAN

The effective Hamiltonian in this case consists of Eq. (7) plus a corresponding term which transforms like $(T_{12}{}^{13}+T_{13}{}^{12})$, where $T_{cd}{}^{ab}$ is the symmetric traceless $SU(3)\{27\}$ tensor. Following a similar procedure to the last section, we find the required additions to the amplitudes of Eq. (8) to be

$$\delta A_{+-}(K^2,P_+^2,P_-^2)=\sum_{b>c}\frac{(-1)^{a+b+c}}{\sqrt{2}}h_{abc}\times[K^{2c}(P_+^{2a}P_-^{2b}+P_+^{2b}P_-^{2a})-K^{2b}(P_+^{2a}P_-^{2c}+P_+^{2c}P_-^{2a})], \quad (13a)$$

$$\delta A_{00}(K^2,P_1^2,P_2^2)=\sum_{b>c}\frac{(-1)^{a+b+c}}{\sqrt{2}}h_{abc}\times[K^{2b}(P_1^{2a}P_2^{2c}+P_1^{2c}P_2^{2a})-K^{2c}(P_1^{2a}P_2^{2b}+P_1^{2b}P_2^{2a})], \quad (13b)$$

$$\delta A_{+0}(K^2,P_+^2,P_0^2)=\sum_{b>c}\frac{(-1)^{a+b+c}}{\sqrt{2}}h_{abc}\times[2K^{2a}(P_0^{2b}P_+^{2c}-P_0^{2c}P_+^{2b})+K^{2b}(P_0^{2a}P_+^{2c}+P_+^{2a}P_0^{2c})-K^{2c}(P_0^{2a}P_+^{2b}+P_0^{2b}P_+^{2a})], \quad (13c)$$

where the h_{abc} are some new arbitrary constants. From now on let us, for convenience, set A_{ij} to be the sum of terms from Eqs. (8) and (13). We see, first of all, that the new A_{ij} still do not contain any momentum-independent terms and that Eq. (9) still holds. This is the familiar result extended to include $\{27\}$ contributions.[4–7]

To proceed we note that in the isotopic-spin-symmetry limit the $\Delta I=\frac{1}{2}$ rules of Eqs. (10) are to be replaced by the $\Delta I=\frac{3}{2}$ rule[20]:

$$A_{00}(K^2,P^2,P^2)-A_{+-}(K^2,P^2,P^2)=2A_{+0}(K^2,P^2,P^2). \quad (14)$$

In comparing this relation with experiment, it is important to take account of final-state interaction effects.[18] Unfortunately, as remarked before, the experimental value of $|A_{+-}/A_{00}|$ is not yet reliably known to the required accuracy.

In this case we find the following seven relations

[17] See Ref. 14, for example, for the calculation of the amplitudes from experiments.

[18] A discussion of the difficulties is given by T. D. Lee and C. S. Wu, Ann. Rev. Nucl. Sci. **16** (1966).

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among the nine zero-energy limits:

$$A_{ij}(P^2,0,P^2)+A_{ij}(P^2,P^2,0)=-A_{ij}(0,P^2,P^2)\,, \quad (15a)$$

$$A_{00}(0,P^2,P^2)-A_{+-}(0,P^2,P^2)=2A_{+0}(0,P^2,P^2)\,, \quad (15b)$$

$$A_{+-}(P^2,0,P^2)=A_{+-}(P^2,P^2,0)\,, \quad (15c)$$

$$A_{00}(P^2,0,P^2)=A_{00}(P^2,P^2,0)\,, \quad (15d)$$

$$A_{+0}(P^2,0,P^2)-A_{+0}(P^2,P^2,0)=\tfrac{3}{2}A_{+-}(0,P^2,P^2)-\tfrac{1}{2}A_{00}(0,P^2,P^2)\,. \quad (15e)$$

With Eqs. (15), we are in a position to discuss the best way of getting results from the current-algebra treatment of these decays. In this method only the zero-energy limits are known and the problem[11] is how to relate these to the physical amplitudes. First, it is natural to ask if there is any rigorous way to do this in the most general case when all terms of Eqs. (8) and (13) are allowed. It is trivial to see that this is impossible since the terms with all of (a,b,c) nonzero will contribute to the physical amplitudes but not to the zero-energy limits. What we can say is that, by Eq. (15b), the results obtained by letting the K-meson momenta go to zero will still satisfy the $\Delta I=\frac{3}{2}$ rule and so will not be manifestly unreasonable. From Eq. (15a) we see that (apart from an over-all factor) this is equivalent to symmetrizing each amplitude in the two outgoing pions, and letting each pion-momentum go to zero independently. Various workers have used these methods.[8–14] If only terms up to first order in squared momenta are kept, the K-meson zero-energy limits are seen from Eq. (12) and the expansion of Eq. (13) to be actually proportional to the physical amplitudes. Finally, we stress that the considerations of this paragraph are only relevant for the current-current Hamiltonian.

IV. RESTRICTIONS FROM CHIRAL $SU(3)\times SU(3)$

Although the chiral $SU(3)\otimes SU(3)$ group[21–24] does not appear to be a very good symmetry of the strong interactions, there are some indications that it may be useful in treating low-energy pion phenomena.[25]

Under this group the universal current-current Hamiltonian of Eq. (1) transforms as $(8,1)+(27,1)$, where the terminology is standard.[26] The quark-model Hamiltonian of Eq. (3), on the other hand, transforms as $(3,3^*)-(3^*,3)$. The representation assignment for the pseudoscalar mesons has generally been taken to be either $(8,1)-(1,8)$ or $(3,3^*)-(3^*,3)$. Nonlinear representations have also been considered.[25]

If we assume that the mesons belong to $(8,1)-(1,8)$, we find no restrictions beyond $SU(3)$ for Eqs. (1) and (2) but that all $K\to 2\pi$ decays vanish with Eq. (3).

The meson representation $(3,3^*)-(3^*,3)$ does not lead to any restrictions for the quark-model Hamiltonian. However, it leads to the complete suppression of the $(27,1)$ part of the universal current-current Hamiltonian (octet dominance) without giving any further restrictions to the $(8,1)$ part. To see this, we merely note that the decomposition of $[(3,3^*)-(3^*,3)]^3$ does not contain $(27,1)$. This appealing feature holds if any number of derivatives are allowed in the effective interaction.

V. DISCUSSION

In this article we have only considered those restrictions on the amplitudes which were independent of dynamics. We cannot say anything, by this method, about $K\to 3\pi$ decays or about $K\to 2\pi$ decays with the quark-model Hamiltonian of Eq. (3). However, the very interesting case of $K\to 2\pi$ resulting from the current-current Hamiltonian turned out to be highly constrained from the requirement of $SU(3)$ symmetry. If the universal Hamiltonian, Eq. (1) is correct we must imagine a dynamical mechanism or a higher symmetry like in Sec. IV suppressing the $K^+\to\pi^+\pi^0$ rate. Then, if the electromagnetic corrections to this rate are assumed negligible, the careful comparison of Eq. (14) with experiment can be considered as an excellent test of whether or not Eq. (1) is correct.

It should be stressed that we have *assumed* that the effects of medium strong and electromagnetic breaking of $SU(3)$ symmetry arise only as a result of giving the external mesons their physical [as opposed to $SU(3)$ degenerate] invariant masses. This is somewhat different from the usual group-theory approach in which symmetry-breaking perturbations are added while the degeneracy of the masses is maintained. As support for our alternative approach, we may mention that such a situation explicitly holds for the medium-strong breaking in Sakurai's vector-meson model[16] (see footnote 28 of this reference). In the more general case, if the internal particle masses are large compared with the pseudoscalar meson masses, we may expect the resulting $SU(3)$ breaking effects on internal masses and couplings to be of the order of 10%. On the other hand, the percentage changes in the external masses due to $SU(3)$ symmetry breaking are enormous (especially if the degenerate pseudoscalar multiplet is considered to have zero "bare" mass). Finally since this approach reproduces the current-algebra limits, it may be possible to show that it has a certain type of "exact" validity.

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