Pion-Nucleon Spin-Flip Sum Rule*

I. S. GERSTEIN

Laboratory for Nuclear Science and Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts

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Current algebra and the hypothesis of partially conserved axial-vector current are used to derive a sum rule for the spin-flip pion-nucleon scattering amplitude analogous to the Weisberger-Adler (W-A) sum rule for the non-spin-flip amplitude. We find that such a sum rule requires information about the weak amplitude for axial-vector–nucleon scattering, in contrast to the W-A relation, which requires no such information. As a byproduct we give a decomposition of this weak amplitude into tensor covariants which is more useful than those previously appearing in the literature. This expansion applies to polar-vector–nucleon scattering as well, and hence is relevant to Compton scattering.

I. INTRODUCTION

SUM rules for pion-nucleon scattering which follow from the equal-time commutation relation

$$\delta(x^0)[A_\mu{}^a(x),A_0{}^b(0)]=\delta^4(x)i\epsilon^{abc}V_\mu{}^c(x)+\cdots, \quad (1)$$

together with some form of the PCAC (partially conserved axial-vector current) hypothesis

$$\partial^\mu A_\mu{}^a(x)=c_\pi\Pi^a(x), \quad (2)$$

have been intensively investigated recently.[1] The most celebrated of these is the Weisberger[2]-Adler[3] (W-A) sum rule which relates the forward (or non-spin-flip) π-N amplitude and the charge and axial vector coupling constants of the nucleon. It is natural to expect that if the spins of the nucleons are not summed than a second W-A type relation should emerge, one which would relate the spin-flip π-N amplitude (at zero momentum transfer) to the nucleon form factors and it is this question which we study in the following. Such a sum rule has, in fact, been proposed by Fuchs[4] and by Bouchiat, Flamand, and Meyer,[5] however, the results of these authors are not in agreement with each other.[6]

In a recent paper Schnitzer[7] has given convincing arguments which suggest strongly that no spin-flip sum rule of the W-A type should exist.[8] He studied the problem of computing the p-wave scattering lengths and s-wave effective ranges from Eqs. (1) and (2) and found he had to introduce a model for the weak axial-vector–nucleon scattering amplitude to do so. It is known[8a] that the non-spin-flip W-A relation is equivalent to a prediction of the s-wave π-N scattering lengths, and it is not difficult to see that a calculation of Schnitzer's type involves some knowledge of the forward spin-flip amplitude. If this contention is correct, then neither of the sum rules given in Refs. 4 and 5 can be completely right since both of these are model independent. In any event, we believe the situation warrants a careful investigation.

Our derivation of pion-nucleon sum rules will proceed in two steps. First we derive sum rules for the invariant amplitudes appearing in a decomposition of the weak axial-vector–nucleon scattering amplitude,[9] using the method introduced by Fubini,[10] then we use the PCAC condition to combine certain of these sum rules into sum rules for π-N scattering. This procedure has the advantage that the use of PCAC is clearly separated from the derivation of the sum rule and that no singular limits involving Born terms, of the type familiar in early derivations of the W-A relation, appear. It is, perhaps, the complexities of this last which lead to the differences between Refs. 4, 5, and our result. We proceed in a covariant manner throughout and in this respect our derivation may be more transparent than that of Ref. (7). Our conclusion is that there is no model-independent spin-flip sum rule.

In Sec. II, we choose a set of tensor covariants to expand the weak amplitude. This choice is much more subtle than has been previously recognized in the literature and is the crux of this paper. In fact, this decomposition has been made differently by several authors[11–13] but we find all of their choices to be deficient in some respect. We shall mention the points of disagreement in the appropriate place. We derive sum

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[1] The currents $A_\mu{}^a(x)$, $V_\mu{}^a(x)$ are the isotopic vector, axial and polar vector weak hadron currents. The omitted terms in Eq. (1) are proportional to gradients of δ functions and will be ignored in the following. In Eq. (2) $\pi^a(x)$ is the pion field operator.

[2] W. I. Weisberger, Phys. Rev. Letters **14**, 1047 (1965); Phys. Rev. **143**, 1302 (1966).

[3] S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965).

[4] N. H. Fuchs, Phys. Rev. **150**, 1241 (1966).

[5] C. Bouchiat, G. Flamand, Ph. Meyer, Orsay Report No. th/187, 1967 (unpublished).

[6] The sum rules in question are Eq. (18) of Ref. 4 and Eq. (11) of Ref. (5).

[7] H. J. Schnitzer, Phys. Rev. **158**, 1471 (1967).

[8] By "sum rule of the W-A type" we mean a relation which follows from Eqs. (1) and (2) alone without any further model-dependent assumptions.

[8a] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

[9] We shall refer to this amplitude, or its vector counterpart, or any of the invariant amplitudes as a weak amplitude. It should be clear from the context which function is meant.

[10] S. Fubini, Nuovo Cimento **43A**, 475 (1966).

[11] M. Gourdin, Lectures perpared for 1966 Cargese Summer School, Orsay Report No. th/161, 1966 (unpublished); (also Ref. 15).

[12] J. W. Meyer, Phys. Rev. **153**, 1653 (1967).

[13] D. Amati, R. Jengo, and E. Remiddi, Nuovo Cimento (to be published).

rules for the weak amplitude in Sec. III and for the π-N amplitude in Sec. IV.

II. DECOMPOSITION OF THE WEAK AMPLITUDE

We define the weak axial-vector–nucleon amplitude to be

$$T_{\mu\nu}{}^{ab}=-i\int d^4x\, e^{iq_2\cdot x}\theta(x_0)\langle p_2|[A_\mu{}^a(x),A_\nu{}^b(0)]|p_1\rangle\,, \quad (3)$$

with imaginary part

$$A_{\mu\nu}{}^{ab}=-\frac{1}{2}\int d^4x\, e^{iq_2\cdot x}\langle p_2|[A_\mu{}^a(x),A_\nu{}^b(0)]|p_1\rangle\,. \quad (4)$$

In Eqs. (3) and (4), $|p_1\rangle$ and $\langle p_2|$ are nucleon states, we do not specify their helicity or isospin label explicitly. We also define

$$\begin{aligned} q_2&=p_1+q_1-p_2\,,\\ P&=\tfrac{1}{2}(p_1+p_2)\,,\\ Q&=\tfrac{1}{2}(q_1+q_2)\,,\\ \Delta&=p_2-p_1\,,\\ \nu&=P\cdot Q=P\cdot q_1=P\cdot q_2\,,\\ t&=\Delta^2\,. \end{aligned}$$

Invariant functions will be considered to be functions of ν, t, $q_2{}^2$, and $q_1{}^2$.

In order to apply the technique of Ref. 10, we have to expand $T_{\mu\nu}$ and $A_{\mu\nu}$ in a complete set of independent covariants. Since we will have to assume unsubtracted dispersion relations in the variable ν for the invariant amplitudes appearing in this decomposition, it is necessary that these functions have no kinematical singularities in this variable. If we think of $T_{\mu\nu}$ as a real scattering amplitude for axial-vector and pseudoscalar particles from nucleons, then an elementary helicity counting yields the result that there are 32 independent amplitudes.[14] If we have equal current masses, $q_1{}^2=q_2{}^2$, then time reversal invariance reduces[15] the number of amplitudes to 20.

The difficulty which arises in choosing an appropriate set of 32 tensor covariants comes from the fact that at first sight we can write down 34 apparently independent tensors, namely,

$P_\mu P_\nu$,	$P_\mu P_\nu\gamma\cdot Q$,	$P_\mu Q_\nu$,	$P_\mu Q_\nu\gamma\cdot Q$,
$P_\mu\Delta_\nu$,	$P_\mu\Delta_\nu\gamma\cdot Q$,	$Q_\mu P_\nu$,	$Q_\mu P_\nu\gamma\cdot Q$,
$Q_\mu Q_\nu$,	$Q_\mu Q_\nu\gamma\cdot Q$,	$Q_\mu\Delta_\nu$,	$Q_\mu\Delta_\nu\gamma\cdot Q$,
$\Delta_\mu P_\nu$,	$\Delta_\mu P_\nu\gamma\cdot Q$,	$\Delta_\mu Q_\nu$,	$\Delta_\mu Q_\nu\gamma\cdot Q$,
$\Delta_\mu\Delta_\nu$,	$\Delta_\mu\Delta_\nu\gamma\cdot Q$,	$g_{\mu\nu}$,	$g_{\mu\nu}\gamma\cdot Q$,
$P_\mu i\sigma_{\nu\lambda}Q^\lambda$,	$P_\nu i\sigma_{\mu\lambda}Q^\lambda$,	$Q_\mu i\sigma_{\nu\lambda}Q^\lambda$,	$Q_\nu i\sigma_{\mu\lambda}Q^\lambda$,
$\Delta_\mu i\sigma_{\nu\lambda}Q^\lambda$,	$\Delta_\nu i\sigma_{\mu\lambda}Q^\lambda$,	$P_\mu\gamma_\nu$,	$P_\nu\gamma_\mu$,
$Q_\mu\gamma_\nu$,	$Q_\nu\gamma_\mu$,	$\Delta_\mu\gamma_\nu$,	$\Delta_\nu\gamma_\mu$,
	$\tfrac{1}{2}(\gamma_\mu\gamma\cdot Q\gamma_\nu-\gamma_\nu\gamma\cdot Q\gamma_\mu)$,	$i\sigma_{\mu\nu}$,	

all of which are understood to appear between Dirac spinors $\bar{u}(p_2)$ and $u(p_1)$.[16] Of course, only 32 of these are independent and a straightforward but tedious algebraic manipulation yields the identities

$$\begin{aligned} 0={}&P^2\tfrac{1}{2}(\gamma_\mu\gamma\cdot Q\gamma_\nu-\gamma_\nu\gamma\cdot Q\gamma_\mu)-m\nu i\sigma_{\mu\nu}\\ &-\tfrac{1}{2}(P_\mu\Delta_\nu-P_\nu\Delta_\mu)\gamma\cdot Q+\tfrac{1}{2}Q\cdot\Delta(P_\mu\gamma_\nu-P_\nu\gamma_\mu)\\ &-\tfrac{1}{2}\nu(\Delta_\mu\gamma_\nu-\Delta_\nu\gamma_\mu)-m(P_\mu i\sigma_{\nu\lambda}Q^\lambda-P_\nu i\sigma_{\mu\nu}Q^\lambda)\,, \end{aligned} \quad (5)$$

$$\begin{aligned} 0={}&\Delta^2K^2i\sigma_{\mu\nu}+2K^2(P_\mu\Delta_\nu-P_\nu\Delta_\mu)-2Q\cdot\Delta(P_\mu K_\nu-P_\nu K_\mu)\\ &+2\nu(\Delta_\mu K_\nu-\Delta_\nu K_\mu)-2m(\Delta_\mu K_\nu-\Delta_\nu K_\mu)\gamma\cdot Q\\ &+2mK^2(\Delta_\mu\gamma_\nu-\Delta_\nu\gamma_\mu)-2mQ\cdot\Delta(K_\mu\gamma_\nu-K_\nu\gamma_\mu)\\ &+\Delta^2(K_\mu i\sigma_{\nu\lambda}Q^\lambda-K_\nu i\sigma_{\mu\lambda}Q^\lambda)\,, \end{aligned} \quad (6)$$

where we have defined

$$K_\mu=Q_\mu-(\nu/P^2)P_\mu-(Q\cdot\Delta/\Delta^2)\Delta_\mu\,,$$

and find

$$K^2=Q^2-(\nu^2/P^2)-[(Q\cdot\Delta)^2/\Delta^2]\,.$$

Now when we choose to eliminate two covariants we must ensure that the use of Eqs. (5) and (6) will not introduce any kinematical singularities in ν when these invariants are projected onto the remaining 32. That is, we should not eliminate any covariant whose coefficient is K^2 or ν. We should further not discard covariants whose coefficient is $Q\cdot\Delta=\frac{1}{2}(q_1{}^2-q_2{}^2)$ since this factor vanishes when the current masses are equal. Put another way we would be eliminating covariants whose coefficients vanished by time reversal invariance when $q_1{}^2=q_2{}^2$ anyway so that in this limit we would have to use Eqs. (5) and (6) again to remove two other covariants.

To be specific it is useful to look at the one-nucleon intermediate-state contribution to the absorptive part. This is

$$\begin{aligned} (A_{\mu\nu})_{\text{pole}}={}&-\pi\delta(s-m^2)\bar{u}(p_2)i\gamma_5[\gamma_\mu G_A(q_2{}^2)-q_{2\mu}F_P(q_2{}^2)]\\ &\times[\gamma\cdot(P+Q)+m]i\gamma_5[\gamma_\nu G_A(q_1{}^2)\\ &+q_{1\nu}F_P(q_1{}^2)]u(p_1)+\text{crossed term} \end{aligned} \quad (7)$$

for axial-vector currents, and

$$\begin{aligned} (A_{\mu\nu})_{\text{pole}}={}&-\pi\delta(s-m^2)\bar{u}(p_2)[\gamma_\mu F_1(q_2{}^2)-(i/2m)\\ &\times\sigma_{\mu\lambda}q_2{}^\lambda F_2(q_2{}^2)][\gamma\cdot(P+Q)+m]\\ &\times[\gamma_\nu F_1(q_1{}^2)+(i/2m)\sigma_{\nu\lambda}q_1{}^\lambda F_2(q_1{}^2)]u(p_1)\\ &+\text{crossed term} \end{aligned} \quad (8)$$

for the polar vector currents. We have used the usual definition for the vector vertex function of the nucleon and have omitted isospin factors. We now see, for example, that the covariant $\frac{1}{2}[\gamma_\mu\gamma\cdot Q\gamma_\nu-\gamma_\nu\gamma\cdot Q\gamma_\mu]$ appears in both of these so that according to Eqs. (5) and (6) unless the covariant $(i\sigma_{\mu\nu})$ is included in our set of 32, we will already have kinematical singularities in the Born approximation.

We consider briefly the choices made by other workers. Gourdin[11,15] uses the first 32 of the covariants. Thus, in omitting $(i\sigma_{\mu\nu})$ his amplitudes have kinemat-

[14] This result holds equally well if the axial-vector currents in Eq. (3) are replaced by vector currents and, in fact, all the results of this section and the following section hold for the latter case also.

[15] Cf. M. Gourdin, Nuovo Cimento 47A, 145 (1967), Appendix I.

[16] We shall sometimes omit these spinors in the rest of this paper. It is understood that any Dirac matrix appears between them.

ical singularities. Meyer[12] omits $(P_\mu\gamma_\nu - P_\nu\gamma_\mu)$ and $(Q_\mu P_\nu - P_\mu Q_\nu)\gamma Q$. But the first appears multiplied by $Q\cdot\Delta$, while the second does not appear at all in Eqs. (5) and (6), and so is actually independent. This is reflected in the fact that he finds 22 nonvanishing amplitudes after using time reversal rather than 20 which is the correct number.[15] Amati, Jengo, and Remiddi[13] use all 34 covariants and so have a redundant set.

We finally give an expansion of $A_{\mu\nu}$ which we believe meets all the requirements for the correct derivation of sum rules[16a]

$$\begin{aligned}A_{\mu\nu}=&P_\mu P_\nu[a_1+\bar{a}_1\gamma\cdot Q]+P_\mu q_{2\nu}[a_2+\bar{a}_2\gamma\cdot Q]+P_\mu q_{1\nu}[a_3+\bar{a}_3\gamma\cdot Q]+q_{2\mu}P_\nu[a_4+\bar{a}_4\gamma\cdot Q]+q_{2\mu}q_{2\nu}[a_5+\bar{a}_5\gamma\cdot Q]\\&+q_{2\mu}q_{1\nu}[a_6+\bar{a}_6\gamma\cdot Q]+q_{1\mu}P_\nu[a_7+\bar{a}_7\gamma\cdot Q]+q_{1\mu}q_{2\nu}[a_8+\bar{a}_8\gamma\cdot Q]+q_{1\mu}q_{1\nu}[a_9+\bar{a}_9\gamma\cdot Q]+g_{\mu\nu}[a_{10}+\bar{a}_{10}\gamma\cdot Q]\\&+i\sigma_{\mu\nu}d_1+\tfrac{1}{2}[\gamma_\mu\gamma\cdot Q\gamma_\nu-\gamma_\nu\gamma\cdot Q\gamma_\mu]d_2+P_\mu\gamma_\nu b_1+P_\nu\gamma_\mu b_2+q_{2\mu}\gamma_\nu b_3+q_{2\nu}\gamma_\mu b_4+q_{1\mu}\gamma_\nu b_5+q_{1\nu}\gamma_\mu b_6\\&+[P_\mu i\sigma_{\nu\lambda}Q^\lambda+P_\nu i\sigma_{\mu\lambda}Q^\lambda]c_1+[Q_\mu i\sigma_{\nu\lambda}Q^\lambda+Q_\nu i\sigma_{\mu\lambda}Q^\lambda]c_2+[\Delta_\mu i\sigma_{\nu\lambda}Q^\lambda+\Delta_\nu i\sigma_{\mu\lambda}Q^\lambda]c_3+[\Delta_\mu i\sigma_{\nu\lambda}Q^\lambda-\Delta_\nu i\sigma_{\mu\lambda}Q^\lambda]c_4.\end{aligned}\quad(9)$$

We have not specified isospin labels since all our results will refer to isospin one in the t ($N\bar{N}$ annihilation) channel. Specifically, we decompose

$$A_{\mu\nu}{}^{ab}=A_{\mu\nu}{}^{(+)}\delta_{ab}+A_{\mu\nu}{}^{(-)}\tfrac{1}{2}[\tau_a,\tau_b],$$

and we work throughout with $A_{\mu\nu}{}^{(-)}$. With this understood, we find that the invariant functions satisfy

$$a(\nu,t,q_1^2,q_2^2)=\pm a(-\nu,t,q_1^2,q_2^2),$$

the even amplitudes (plus sign) being

$$a_1,\bar{a}_2,\bar{a}_3,\bar{a}_4,a_5,a_6,\bar{a}_7,a_8,a_9,a_{10},b_1,b_2,c_1,d_2,\quad(10)$$

and the remaining ones being odd. For completeness, we give the properties of these amplitudes which follow from time-reversal invariance[17]:

$$\begin{aligned}a_2(\nu,t,q_1^2,q_2^2)&=a_7(\nu,t,q_2^2,q_1^2),\\ \bar{a}_2(\cdots)&=\bar{a}_7(\cdots),\\ a_3(\cdots)&=a_4(\cdots),\\ \bar{a}_3(\cdots)&=\bar{a}_4(\cdots),\\ a_5(\cdots)&=a_9(\cdots),\\ \bar{a}_5(\cdots)&=\bar{a}_9(\cdots),\\ b_1(\cdots)&=b_2(\cdots),\\ b_3(\cdots)&=b_6(\cdots),\\ b_4(\cdots)&=b_5(\cdots),\\ c_i(\cdots)&=-c_i(\cdots),\quad i=1,2,4.\end{aligned}$$

The remaining amplitudes go into themselves upon interchange of q_1^2 and q_2^2.

III. SUM RULES FOR WEAK AMPLITUDES

The method of Fubini for deriving sum rules from commutation relations is by now well known, and we refer the reader to the original paper,[10] here we merely outline the important steps. We compute the quantities $q_{2\mu}A^{\mu\nu}$, $q_{2\mu}T^{\mu\nu}$, $A^{\mu\nu}q_{1\nu}$, and $T^{\mu\nu}q_{1\nu}$ and project these onto the set of eight independent vectors

$$\begin{matrix}P_\mu, & P_\mu\gamma\cdot Q, & q_{2\mu}, & q_{2\mu}\gamma Q,\\ q_{1\mu}, & q_{1\mu}\gamma\cdot Q, & \gamma_\mu, & i\sigma_{\mu\lambda}Q^\lambda.\end{matrix}\quad(11)$$

We assume unsubtracted dispersion relations for the invariant amplitudes and obtain a sum rule for those amplitudes whose coefficient is the variable ν. The lack of conservation of the axial-vector current is no handicap here if we also assume unsubtracted dispersion relations for the invariant amplitudes appearing in the decomposition of

$$D_\mu=-\frac{1}{2}\int d^4x\,e^{iq_2\cdot x}\langle p_2|[A_\mu(x),\partial^\nu A_\nu(0)|p_1\rangle,\quad(12)$$

on the vector covariants (11). We obtain 16 sum rules this way, six of them being identically satisfied because of the crossing relations (10). The remaining ten are,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu(a_1-c_1)=\frac{F_2(t)}{4m},\quad(13a)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu(a_1+c_1)=\frac{F_2(t)}{4m},\quad(13b)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu\,\bar{a}_i=0,\quad i=2,3,4,7\quad(13c)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu(b_i+d_2)=-\tfrac{1}{4}[F_1(t)+F_2(t)],\quad i=1,2\quad(13d)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu\,c_i=0.\quad i=1,2\quad(13e)$$

In deriving these sum rules we have expanded

$$\langle p_2|V_\mu{}^a(0)|p_1\rangle=\tfrac{1}{2}\tau^a[\gamma_\mu(F_1(t)+F_2(t))-(P_\mu/m)F_2(t)].\quad(14)$$

We obtain two additional sum rules by considering the coefficients of $P_\mu\gamma Q$ and $P_\nu\gamma Q$ in the decomposition

[16a] Our choice of eliminating $Q_\mu i\sigma_{\nu\lambda}Q^\lambda - Q_\nu i\sigma_{\mu\lambda}Q^\lambda$ using Eq (6) suffers from the defect that a pole at $t=0$ is thereby introduced. In fact, for the purpose of this paper, this choice is correct because the pole cancels in all amplitudes which are continued to $t=0$. In general, however, $(\Delta_\mu Q_\nu - \Delta_\nu Q_\mu)\gamma\cdot Q$ should be eliminated and then no such problems arise. I would like to thank M. A. B. Bég for a conversation regarding this point.

[17] These hold for both the isospin-even and isospin-odd amplitudes.

of $q_{2\mu}A^{\mu\nu}$ and $A^{\mu\nu}q_{1\nu}$, respectively. That is, we define

$$D_\mu=-\frac{1}{2}\int d^4x\, e^{iq_2\cdot x}\langle p_2|[A_\mu(x),\partial^\nu A_\nu(0)]|p_1\rangle$$
$$=\alpha_1P_\mu+\bar{\alpha}_1P_\mu\gamma Q+\cdots+\alpha_5 i\sigma_{\mu\lambda}Q^\lambda\,, \quad (15a)$$

and

$$\bar{D}_\nu=-\frac{1}{2}\int d^4x\, e^{iq_2\cdot x}\langle p_2|[\partial^\mu A_\mu(x),A_\nu(0)]|p_1\rangle$$
$$=\beta_1P_\nu+\bar{\beta}_1P_\nu\gamma Q+\cdots+\beta_5 i\sigma_{\nu\lambda}Q^\lambda\,, \quad (15b)$$

and we have

$$\nu\bar{a}_1+q_2^2\bar{a}_4+q_1\cdot q_2\bar{a}_7+b_2+mc_1-d_2=i\bar{\beta}_1\,, \quad (16a)$$

$$\nu\bar{a}_1+q_1\cdot q_2\bar{a}_2+q_1^2\bar{a}_3-mc_1+b_1-d_2=-i\bar{\alpha}_1\,. \quad (16b)$$

Integrating Eqs. (16a) and (16b) over ν and using Eqs. (15) yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu(\nu\bar{a}_1+b_2-d_2)=i\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu\bar{\beta}_1\,, \quad (17a)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu(\nu\bar{a}_1+b_1-d_2)=-i\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu\bar{\alpha}_1. \quad (17b)$$

We could now derive sum rules for $\bar{\alpha}_1$ and $\bar{\beta}_1$ by applying the Fubini technique to D_μ and $\bar{D}_\nu$ if we knew the equal-time commutator

$$\delta(x^0)[\partial_\mu A^\mu(x),A_0(0)]\,.$$

In fact, it is sufficient to assume that this commutator is local to obtain such sum rules. This is because

$$q_{2\mu}D^\mu=(\nu\alpha_1+\cdots)+\gamma Q(\nu\bar{\alpha}_1+\cdots)\,,$$

but if

$$\delta(x^0)[\partial_\mu A^\mu(x),A_0(0)]=\delta^4(x)G(x)\,,$$

then the expansion analogous to Eq. (14) is

$$\langle p_2|G(0)|p_1\rangle=g(t)\,.$$

Since there is no term proportional to $\gamma\cdot Q$ here, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu\,\bar{\beta}_1=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu\,\bar{\alpha}_1=0\,,$$

and the additional sum rules

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu(\nu\bar{a}_1+b_i-d_2)=0\,,\quad i=1,2. \quad (13f)$$

We get some estimate of the validity of these sum rules if we can find out some information about the asymptotic behavior of the integrands. It has been repeatedly emphasized by Fubini and his co-workers that the Regge-pole model provides just this information[18] and it has been made plausible[19] that the Regge asymptotic behavior controlled by t-channel exchanges will apply to the absorptive parts of weak amplitudes. Taking into account the complications due to spin, we find that all of the integrands in Eqs. (13) behave asymptotically $\sim\nu^{\alpha(t)-2}$, where the leading Regge trajectory is that of the ρ meson. Thus, according to the arguments of Ref. 19 we may have confidence in the convergence of the sum rules for $t<m_\rho^2$.

IV. SUM RULES FOR PION-NUCLEON SCATTERING

In this section we combine the sum rules of Eq. (13), using the PCAC hypothesis Eq. (2), to obtain sum rules for the pion-nucleon scattering amplitude which we define as

$$T^{ab}=-i\int d^4xe^{iq_2\cdot x}\theta(x_0)\langle p_2|[j_\pi{}^a(x),j_\pi{}^b(0)]|P_1\rangle \quad (18)$$

$$=A^{ab}(\nu,\tau,q_1^2,q_2^2)+\gamma QB^{ab}(\nu,\tau,q_1^2,q_2^2)\,. \quad (19)$$

Applying PCAC to Eq. (4) we have

$$q_2{}^\mu A_{\mu\nu}{}^{ab}q_1{}^\nu=c_\pi{}^2(q_1^2-m_\pi{}^2)^{-1}(q_2^2-m_\pi{}^2)^{-1}\times[\mathrm{Im}A^{ab}+\gamma Q\,\mathrm{Im}B^{ab}]\,. \quad (20)$$

We restrict ourselves to the antisymmetric isospin amplitude and find

$$\begin{aligned}\mathrm{Im}A(\nu,t,q_1^2,q_2^2)&\\=[(q_1^2-m_\pi^2)(q_2^2-m_\pi^2)/c_\pi^2][&\nu^2a_1+\nu q_1\cdot q_2a_2+\nu q_1^2a_3\\&+\nu q_2^2a_4+q_1\cdot q_2q_2^2a_5+q_1^2q_2^2a_6+\nu q_1\cdot q_2a_7+(q_1\cdot q_2)^2a_8\\&+q_1\cdot q_2q_1^2a_9+q_1\cdot q_2a_{10}+\tfrac{1}{2}\nu(q_2^2-q_1^2)c_2-\nu tc_3\\&-\nu(q_2^2-q_1^2)c_4-2\nu d_1]\,,\end{aligned} \quad (21)$$

$$\begin{aligned}\mathrm{Im}B(\nu,t,q_1^2,q_2^2)&\\=[(q_1^2-m_\pi^2)(q_2^2-m_\pi^2)/c_\pi^2][&\nu^2\bar{a}_1+\nu q_1\cdot q_2\bar{a}_2+\nu q_1^2\bar{a}_3\\&+\nu q_2^2\bar{a}_4+q_1\cdot q_2q_2^2\bar{a}_5+q_1^2q_2^2\bar{a}_6+\nu q_1\cdot q_2\bar{a}_7+(q_1\cdot q_2)^2\bar{a}_8\\&+q_1\cdot q_2q_1^2\bar{a}_9+q_1\cdot q_2\bar{a}_{10}+\nu b_1+\nu b_2+q_2^2b_3+q_1\cdot q_2b_4\\&+q_1\cdot q_2b_5+q_1^2b_6-\tfrac{1}{2}m(q_2^2-q_1^2)c_2+mtc_3\\&+m(q_2^2-q_1^2)c_4+2md_1]\,,\end{aligned} \quad (22)$$

and so

$$(1/\nu^2)\,\mathrm{Im}[A(\nu,0,0,0)+(\nu/m)B(\nu,0,0,0)]=(2/f_\pi{}^2)[(\nu/m)\bar{a}_1+(b_1+b_2)/m+a_1]\,, \quad (23)$$

$$(1/\nu)\,\mathrm{Im}B(\nu,0,0,0)=(2/f_\pi{}^2)[\nu\bar{a}_1+b_1+b_2+(2m/\nu)d_1]\,, \quad (24)$$

where we have used the charged-pion decay constant f_π, related to c_π by

$$m_\pi{}^4/c_\pi{}^2=2/f_\pi{}^2\,.$$

[18] S. Fubini and G. Segrè, Nuovo Cimento 45A, 641 (1966); V. de Alfaro, S. Fubini, G. Furlan, and C. Rossetti, University di Torino Report, 1966 (unpublished).

[19] J. B. Bronzan, I. S. Gerstein, B. W. Lee, and F. E. Low, Phys. Rev. Letters 18, 32 (1967).

We can now write sum rules for the pion-nucleon amplitudes if we have sum rules for the weak amplitudes appearing on the right-hand side of Eqs. (23) and (24). As a technical convenience, we remove the one-nucleon intermediate-state contributions to the sum rules (13) using Eq. (7) and apply PCAC only to the integrands above the physical threshold. We obtain[20]

$$[G_A(0)]^2-2f_\pi^2\frac{1}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu^2}\times\mathrm{Im}[A^{(-)}(\nu,0,0,0)+\nu B^{(-)}(\nu,0,0,0)]=F_1(0)\,, \quad (25a)$$

$$[G_A(0)]^2-2f_\pi^2\frac{1}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu}\mathrm{Im}B^{(-)}(\nu,0,0,0)=F_1(0)+F_2(0)-\frac{4}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu}2md_1^{(-)}(\nu,0,0,0)\,. \quad (25b)$$

Our normalization, Eq. (14), implies

$$F_1(0)=1\,,$$
$$F_1(0)+F_2(0)=\mu_p-\mu_n=4.7\,,$$

and Eq. (25a) is immediately recognized as the Weisberger-Adler relation. Equation (25b) is the spin-flip sum rule we hoped to derive. The appearance of d_1, a weak amplitude, on the right-hand side of Eq. (25b) occurs because no sum rule for d_1 is contained among Eqs. (13). This is the covariant analog of Schnitzer's[7] statement that to go beyond the s-wave scattering lengths [Eq. (25a)] some model must be assumed for the weak amplitude.

V. CONCLUSIONS

It is interesting to observe that the weak amplitude which survives in Eq. (25b) is precisely the coefficient of the covariant ($i\sigma_{\mu\nu}$) which we have argued in Sec. II must be included in the expansion of $A_{\mu\nu}$ if kinematical singularities are to be avoided. It is easy to see that no matter how we choose the remaining 31 covariants we can never obtain a model-independent spin-flip sum rule. This is because

$$q_2^\mu i\sigma_{\mu\nu}=-i\sigma_{\nu\lambda}Q^\lambda+m\gamma_\nu-P_\nu\,,$$

so that the Fubini procedure cannot yield a sum rule for d_1 and

$$q_2^\mu i\sigma_{\mu\nu}q_1^\nu=-\tfrac{1}{2}[\gamma\cdot q_2,\gamma\cdot q_1]\,,$$

which is precisely the spin-flip amplitude covariant so that clearly d_1 survives in the spin-flip invariant even in the limit $q_1^2=q_2^2=t=0$. It is possible that, in fact, a general statement may be made that for any pionic process the only model-independent sum rule which can be obtained by this method is a non-spin-flip one although this is only a speculation at this point.

Fuchs'[4] sum rule omits both the $(G_A)^2$ and the weak amplitude in Eq. (25b) and his numerical evaluation of the resulting sum rule was in satisfactory agreement with experiment. Bouchiat, Flamand, and Meyer[5] obtained the $(G_A)^2$ term and only neglected the weak amplitude and, of course, obtained poorer results. Thus, if the sum rule is correct, the weak term presumably is the same order of magnitude as $(G_A)^2\sim1.4$. This conclusion is strengthened by Schnitzer's[7] calculation. He is able to show that for the p-wave scattering lengths and s-wave effective ranges only intermediate states with $J=\frac{3}{2}^+$ in the direct channel contribute to the weak amplitude and he obtains reasonable agreement with this approximation.

We could attempt an evaluation of Eq. (25b) by carrying out a procedure analogous to Schnitzer's for evaluating the weak contribution and evaluating the integral over $B^{(-)}(\nu,0,0,0)$ using phase-shift analysis as was done in Refs. 4 and 5. This involves having some reasonable information about the weak amplitude, at least a good knowledge of the axial form factors for N^* production. We hope to return to this question in the future. At present, it is probably best to use Eq. (25b) as a low-energy theorem, in which case our conclusions would be identical to those of Ref. 8.

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[20] We have changed variables to the more conventional $\nu=P\cdot Q/m$ and have used the crossing relations (10). The threshold is $\nu_0=m_\pi(1+m_\pi/2m)$.