Connection between Regge-Pole and Single-Particle Exchange Models for High-Energy Reactions*

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It is shown that the $s^{\alpha(0)}$ energy dependence of high-energy transition amplitudes characteristic of the Regge-pole model can be obtained directly in the s channel by summing the most divergent terms resulting from an infinite set of single-particle exchanges in the t channel.

It has become evident in the past two years that the Regge-pole model provides a remarkably successful description of particle reactions at high energies and small momentum transfers, including, in particular, the energy dependence of the transition amplitudes. On the other hand, single-particle exchange models have been applied with considerable success to the description of two-body reactions at lower energies. The connection between the two models has been obscure. It has, of course, long been known that a single Regge-pole term in a transition amplitude is equivalent, through its partial-wave expansion, to the sum of an infinite set of resonance terms in the channel in which the pole appears[1] (we shall take this as the t channel). Conversely, as re-emphasized recently by van Hove,[2] the sum of an infinite set of single-particle exchange amplitudes, with masses and couplings which satisfy appropriate analyticity conditions in the complex j plane, is equivalent through the Watson transform to a set of Regge-pole amplitudes. This equivalence is not obvious in the crossed or s channel. If we attempt to write the transition amplitude as the sum of an infinite set of single-particle exchange amplitudes, we encounter the well-known difficulty that the individual terms in the (divergent) series increase as s^j for large s and increasing spin j of the particle exchanged. It is only by performing the Watson transform in the t channel before continuing to the s channel, hence making explicit use of the j plane analyticity as well as crossing, that one avoids this problem and obtains a well-behaved Regge-pole amplitude. Moreover, the characteristic $s^{\alpha(0)}$ energy dependence of this amplitude has no obvious connection with the energy dependence of the individual single-particle exchange amplitudes.

In the present paper, we wish to examine the possibility, that by appropriate summation of the most divergent terms in the single-particle-exchange series one can recover the Regge-pole amplitude, and in particular, its energy dependence, without the direct use of crossing and j-plane analyticity.[3] The conjecture is correct. The method, while less precise mathematically than the conventional methods based on the Watson transform, provides interesting insights, first, into the connection between the Regge-pole and particle exchange models, and second, into the possible behavior of particle couplings for large j.

For simplicity, we will consider only the scattering of spinless particles of equal mass; these restrictions are not essential. The s-channel scattering amplitude arising from the exchange of a particle of spin j and mass μ_j in the t channel is readily calculated using a result for the spin-j projection operator $P_{\mu\cdots\nu;\,\mu'\cdots\nu'}{}^j$ given elsewhere.[4] The coupling at each vertex involves a form factor $g_j(t)$, and j factors of the sum of the external momenta at that vertex. Aside from irrelevant factors,

$$\begin{aligned}F_j(s,t)&=g_j^2(p+p')_\mu\cdots(p+p')_\nu P_{\mu\cdots\nu;\,\mu'\cdots\nu'}{}^j(q+q')_{\mu'}\cdots\\&\qquad\times(q+q')_{\nu'}/(\mu_j^2-t)\\&=g_j^2(-1)^j[\pi^{1/2}\Gamma(j+1)/2^{j+1}\Gamma(j+\tfrac{3}{2})]\\&\qquad\times(2j+1)(4p_t^2)^jP_j(\cos\theta_t)/(\mu_j^2-t)\\&\qquad\to g_j^2(-2s)^j/(\mu_j^2-t)+\cdots,\quad s\to\infty\,.\end{aligned}\tag{1}$$

In these expressions, p_t and $\cos\theta_t$ are the 3-momentum of any of the particles in the t-channel center-of-mass system, and the cosine of the t-channel scattering angle. We have suppressed a dimensional factor m^{-2j} for simplicity. The forward-scattering amplitude in the s channel is thus given for large s by

$$A^{\pm}(s,0)=\tfrac{1}{2}\sum_{j=0}^{\infty}(g_j/\mu_j)^2[(-1)^j\pm1](2s)^j,\tag{2}$$

where we have distinguished even and odd values of j to allow for possible effects of the signature of the particles exchanged. Note that we have dropped the less singular terms in the high-energy limit of the F_j, Eq. (1).

We will suppose initially that the g_j are all equal, and that the masses μ_j^2 increase linearly with j as suggested by the mass-angular-momentum relation for the ob-

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[1] For example, the Khuri form of the single Regge-pole amplitude gives

$$A(s,t)=\sum_{j=0}^{\infty}(2j+1)e^{-(j-\alpha)\xi}P_j(\cos\theta_t)/(j-\alpha).$$

[N. N. Khuri, Phys. Rev. 130, 429 (1963).]

[2] L. van Hove, Phys. Letters 24B, 183 (1967).

[3] The author would like to thank Professor R. Oehme and Dr. Y. T. Chiu for an interesting discussion of a related conjecture of R. P. Feynman's.

[4] L. Durand, Phys. Rev. Letters 18, 58 (1967); and to be published.

served Regge trajectories[5]:

$$\mu_j^2=\mu^2(j-\alpha). \tag{3}$$

These conditions will be relaxed below. Note that $j=\alpha$ would be the $t=0$ intercept for a linear Regge trajectory containing the set of particles considered. However, we assume nothing at this point about the existence of Regge poles, or about the behavior of the masses away from $t=0$. Clearly, the smallest value of j which appears in Eq. (2) is $j=0$ if $\alpha<0$, and $j=\nu$, $\nu>\alpha>\nu-1$ if $\alpha>0$. We are thus led to consider the (divergent) series

$$\begin{aligned}\bar{A}(s,0)&=\sum_{j=\nu}^{\infty}(2s)^j/(j-\alpha)\\&=(2s)^\nu\sum_{n=0}^{\infty}(2s)^n/(n+\nu-\alpha),\quad \alpha>0.\end{aligned} \tag{4}$$

The series can be summed for $|2s|<1$; we define $\bar{A}(s,0)$ for $|2s|>1$ as the analytic continuation of this function. An elementary calculation yields the desired result,

$$\begin{aligned}\bar{A}(s,0)&=(2s)^\nu\int_1^\infty t^{\alpha-\nu}(t-2s)^{-1}dt\\&=(2s)^\nu\,{}_2F_1(1,\nu-\alpha;1+\nu-\alpha;2s)/(\nu-\alpha),\\&\qquad 0<\arg s<2\pi.\end{aligned} \tag{5}$$

For $s\to\infty$,

$$\bar{A}(s,0)\to-(\pi/\sin\pi\alpha)(-1)^\nu e^{-i\pi\alpha}(2s)^\alpha,\quad 0<\arg s<2\pi \tag{6}$$

provided that $\alpha>-1$. For $\alpha<-1$, the lower limit of summation in Eq. (4) is $j=0$, and $|\bar{A}(s,0)|\propto|s|^{-1}$ for $|s|\to\infty$.

The asymptotic behavior of the amplitudes $A^{\pm}(s,0)$ is now readily determined:

$$A^{\pm}(s,0)\to-\frac{g^2}{2\mu^2}\frac{\pi}{\sin\pi\alpha}[1\pm e^{-i\pi\alpha}](2s)^\alpha,\quad |s|\to\infty,\quad \alpha>-1. \tag{7}$$

[5] See, for example, V. Barger and D. Cline, Phys. Rev. Letters **16**, 913 (1966). It has been suggested that the Regge sequences continue indefinitely. From a theoretical point of view, we expect the partial-wave amplitudes in a theory with Regge behavior to vary as indicated in Ref. 1, whether or not $\alpha(t)$ reaches physical values of j.

For $\alpha<-1$, $|A^{\pm}(s,0)|\propto|s|^{-1}$, $|s|\to\infty$. *The asymptotic behavior of the amplitudes $A^{\pm}(s,0)$ given in Eq. (7) is precisely that obtained for a single t-channel Regge pole with a $t=0$ intercept $\alpha(0)=\alpha$.* The lower powers of s in the Regge-pole amplitude may be obtained by retaining lower powers of s in Eq. (1), and summing the appropriate series.

The foregoing result is readily generalized. First, the method is clearly not restricted to $t=0$: if we retain t in the denominator in Eq. (1), we obtain the single-Regge-pole amplitude for a linear trajectory $\alpha(t)=\alpha-t/\mu^2$. Second, the j dependence of the factors g_j^2/μ_j^2, or more generally, of the complete single-particle exchange amplitudes at fixed t, need not be as simple as was assumed above. Similar results, generally involving several Regge-like terms, are obtained if this factor is replaced by a rational function of j multiplied by an exponential e^{-cj}, $c=\text{const}$. However, these factors cannot decrease too rapidly with increasing j if Regge behavior is to be obtained: If $|g_j^2/\mu_j^2|\lesssim e^{-jf(j)}$, where $f(j)\to\infty$ for $j\to\infty$, the functions defined by the series in Eq. (2) are entire, and the resulting amplitudes are real for all real s. In this situation, the Watson transform may be expected to fail. It is only because we encounter a branch point of the amplitude $\bar{A}$ in continuing this function to large s that the amplitudes in Eq. (7) are complex. The possible growth of the factors g_j^2/μ_j^2 for $j\to\infty$ is correspondingly limited by the requirement that the partial-wave series for $A(s,t)$ converge in t channel. Moreover, if the functional dependence of g_j^2/μ_j^2 on j involves branch-point singularities, non-Regge behavior is obtained. For example, if $\mu_j^2=\mu^2(j-\alpha)^\sigma$, the amplitudes $\bar{A}(s,0)$ vary at high energies as $s^\alpha(\ln s)^{\sigma-1}$. It is clear that our smoothness assumptions for the particle masses and couplings are substantially equivalent to the usual assumptions of j-plane analyticity for the t-channel scattering amplitude, though the method can be used even when those simpler assumptions fail. Finally, it is only because we sum an *infinite* set of particle-exchange amplitudes that the $s^{j_{\max}}$ behavior of the finite series is converted to the s^α behavior of the Regge-pole amplitudes. However, it should be emphasized that the "particles" which are exchanged need not actually appear as physical resonances, since μ_j^2 may well become negative for large j.