Once the analyticity of the matrix elements has been established, the analyticity of the roots of Eq. (5.1), $l=\alpha(k^2)$, should follow.

As another incidental point, we tried to apply the modified Cheng representation[10,11] to compute the S matrix from the leading trajectory of the BBS equation and obtained poor results. If one believes that this representation is applicable to realistic problems such as considered by Abbe *et al.*,[11] then one should expect that this representation should also hold for the BBS equation. Our negative result in obtaining a converging solution suggests that in a realistic problem the Cheng representation should perhaps be generalized to include relativistic kinematics.

Figures 3, 4, and 5 show the result of numerical computations of the S matrix with the BBS equation. In both the Noyes-Kowalski method and the singular-equation method, about twenty mesh points were sufficient to give 1% accuracy. To estimate this accuracy we have solved the nonrelativistic Lippmann-Schwinger equation and compared the results with solutions from the Schrödinger equation.[3] We conclude that with present computer facilities (eg., CDC 3600), direct solutions of singular integral equations seems to be a feasible method for studying models of scattering and Regge behavior, and we expect that the methods studied here will be applicable to the solution of the Bethe-Salpeter equation, even above the inelastic threshold.

[10] Hung Cheng, Phys. Rev. **144**, 1237 (1966); W. J. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, *ibid.* **140**, B1595 (1965).

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Sum Rules for Almost Backward Pion-Nucleon Elastic Scattering*

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Sum rules are derived, assuming Regge asymptotic behavior, for the πN elastic scattering amplitudes at $u=0$. The four sum rules obtained are found to be consistent with each other and with the forward-scattering sum rule of Sakita and Wali if ρ and $f^0(1250)$ are included in the t channel. Without the f^0, the sum rules are inconsistent. Predictions are made for the nucleon–ρ–meson coupling constants.

I. INTRODUCTION

IN this paper we investigate the consequences of the following assumptions:

(a) The asymptotic behavior of the πN elastic-scattering amplitudes at zero cross-momentum transfer ($u=0$) is given by the Reggeized u-channel trajectories of the nucleon and N^* (1236).

(b) The superconvergence relations implied by this high-energy behavior can be well approximated by the contributions of a finite number of single-particle bound or resonant states [in our case the N and N^* in the s channel, the ρ and $f(1250)$ in the t channel].

Several recent papers have explored similar assumptions[1] and obtained results consistent with experiment. In at least one case,[2] inconsistencies have been found, and it has been suggested by Fubini[3] that an infinite number of particles are needed to satisfy the superconvergence relations. One of the most interesting of the sum rules is the Sakita-Wali[4] result for forward πN elastic scattering. On the assumption that the contribution of the higher spin particles is small, Sakita and Wali relate the πNN coupling to the πNN^* coupling and obtain remarkable agreement with experiment. This leads us to hope that the finite-pole approximation may also be valid for backward πN elastic scattering. In what follows we shall rederive the Sakita-Wali sum rule to show its close connection with our sum rule.

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[1] V. De Alfaro, S. Fubini, G. Rossetti, and G. Furlan, Phys. Letters **21**, 576 (1966).

[2] F. E. Low, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, California, 1966* (University of California Press, Berkeley, 1967). Low points out that the authors of Ref. 1 have not considered all the sum rules implied by their model, and that a sum rule can be obtained from the model which is inconsistent with the published sum rules. It is unclear whether this indicates a failure of the superconvergence relation itself or of the finite-pole approximation used to evaluate it.

[3] Sergio Fubini, in Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energies, 1967 (unpublished).

[4] Bunji Sakita and Kameshwar C. Wali, Phys. Rev. Letters **18**, 29 (1967).

II. KINEMATICS AND REGGE BEHAVIOR

(i) We use the standard notation for πN scattering with (q_1,α) the 4-momentum and isospin of the initial pion and (p_1,i) that of the initial nucleon, in the s channel (see Fig. 1). We define

$$s=-(p_1+q_1)^2,\quad Q=\tfrac{1}{2}(q_1+q_2),$$
$$t=-(p_1-p_2)^2,\quad P=\tfrac{1}{2}(p_1+p_2),$$
$$u=-(p_1-q_1)^2,\quad \nu=P\cdot Q.$$

The πN scattering amplitude is, suppressing isospin dependence,

$$M=\bar{u}(p_2)[-A+i\gamma\cdot QB]u(p_1)$$
$$=\bar{u}(p_2)Tu(p_1).$$

(ii) Since we shall be interested in amplitudes whose asymptotic behavior is dominated by specific isospin exchange in the u channel, it is appropriate to decompose the amplitudes in terms of isospin eigenstates of the u-channel system. The projection operators for πN states are given by

$$P^{1/2}=\tfrac{1}{3}[1-2\mathbf{I}_\pi\cdot\mathbf{I}_N],$$
$$P^{3/2}=\tfrac{2}{3}[1+\mathbf{I}_\pi\cdot\mathbf{I}_N].$$

The matrix elements of these operators in the s channel are given by

$$\langle\beta j|P^{1/2}|\alpha i\rangle=\tfrac{1}{3}[\delta_{\beta\alpha}\delta_{ji}+i\tau_{ji}{}^{\gamma}e_{\beta\alpha\gamma}],$$
$$\langle\beta j|P^{3/2}|\alpha i\rangle=\tfrac{2}{3}[\delta_{\beta\alpha}\delta_{ji}-\tfrac{1}{2}i\tau_{ji}{}^{\gamma}e_{\beta\alpha\gamma}],$$

and in the u channel by permitting α and β in the above expressions.

(iii) The exchange of a u-channel pole gives rise, as discussed by Frautschi *et al.*,[5] to the asymptotic behavior $[s\to\infty]$

$$A^{(I)}\to s^{[\alpha_I(u)-1/2]},$$
$$B^{(I)}\to s^{[\alpha_I(u)-1/2]}. \tag{1}$$

At one time there was some doubt as to the validity of the Regge asymptotic formula for $u\geq 0$,[6] but recently Freedman and Wang[7] have solved this problem by introducing "daughter" Regge trajectories. Thus, when $\alpha_I(u)<-\frac{1}{2}$ we expect the superconvergence relations

$$\int_{-\infty}^{\infty}\mathrm{Im}A^{(I)}(s,u)ds=0,$$
$$\int_{-\infty}^{\infty}\mathrm{Im}B^{(I)}(s,u)ds=0, \tag{2}$$

where (I) is the isospin in the u channel.

[5] S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2204 (1962).
[6] D. A. Atkinson and V. Barger, Nuovo Cimento **38**, 634 (1965).
[7] Daniel Z. Freedman and Jiunn-Ming Wang, Phys. Rev. **153**, 1596 (1967).

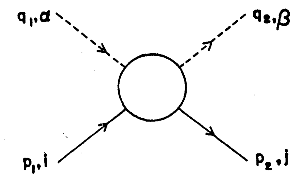

FIG. 1. πN elastic scattering kinematical notation.

The Regge trajectories appear to be more a matter of conjecture than experiment. For instance, only one particle is known on the ρ trajectory, so that $\alpha_\rho(0)$ can hardly be thought of as well established. Therefore, we feel free to make the conjecture most useful to us, namely, $\alpha_{1/2}(0)$ and $\alpha_{3/2}(0)\leq-\frac{1}{2}$. Barger and Cline[8] have recently sought to establish these trajectories from a study of backward πN scattering, and their results suggest that $\alpha_{1/2}(0)$, $\alpha_{3/2}(0)$ do not reach $-\frac{1}{2}$ until u becomes negative. However, it is not clear to us how much difference the daughter hypothesis would make to their results, and we believe the intercepts at $u=0$ still remain in doubt. In any case, $\alpha_{3/2}(0)<-\frac{1}{2}$ and $\alpha_{1/2}(0)<-\frac{1}{2}$ constitute an interesting and not implausible working hypothesis. (We can always go to negative momentum transfer to take advantage of the more sharply damped Regge behavior; in fact, we find our predictions for the vector-meson–nucleon coupling constants relatively insensitive to this change.)

III. SUM RULES

Before writing the sum rules at $u=0$, let us briefly rederive the Sakita-Wali result for forward scattering. In this case the amplitudes are dominated at high energy by meson exchange trajectories, with the behavior $[s\to\infty]$

$$A^{(I)}\to s^{\alpha_I(t)},$$
$$B^{(I)}\to s^{[\alpha_I(t)-1]},$$

where (I) denotes the isospin in the t channel. Assuming that $\alpha_I(0)<0$ for $I=0$ and 1, we have

$$\int_{-\infty}^{\infty}\mathrm{Im}B^{(I)}(s,t)ds=0. \tag{3}$$

For the time being we avoid the question of momentum-transfer dependence and evaluate the sum rule at $t=0$. Since the amplitudes obey the crossing relations

$$A^{(I)}(s,u)=(-1)^I A^{(I)}(u,s),$$
$$B^{(I)}(s,u)=(-1)^{I+1}B^{(I)}(u,s),$$

we have

$$\mathrm{Im}B^{(I)}(\nu)=(-1)^I\,\mathrm{Im}B^{(I)}(-\nu),$$

with $\nu=\frac{1}{4}(u-s)$. The superconvergence relation (3) is therefore automatically satisfied (the result of crossing symmetry) for $I=1$, leaving the nontrivial relation

$$\int_0^{\infty}\mathrm{Im}B^{(0)}(\nu)\,d\nu=0. \tag{4}$$

[8] V. Barger and D. Cline, Phys. Rev. Letters **16**, 913 (1966).

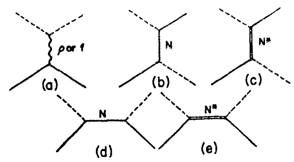

FIG. 2. Single-particle contributions to the πN elastic scattering amplitudes. (a) ρ- or f-meson exchange (t channel); (b) nucleon exchange (u channel); (c) N^* exchange (u channel); (d) direct nucleon pole (s channel); (e) direct N^* pole (s channel).

Following Sakita and Wali, we approximate the superconvergence relation by including only the N and N^* pole contributions, and thus obtain (at $t=0$)

$$g_{NN\pi}^2=\tfrac{2}{3}(0.755)(M^*/\mu)^2 g_{N^*N\pi}^2\,, \qquad (5)$$

where M^* is the N^* mass, μ is the pion mass, and the coupling constants are defined by the phenomenological Lagrangians

$$\begin{aligned} L(NN\pi)&=i(4\pi)^{1/2}g_{NN\pi}\bar{\Psi}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}\Psi\,,\\ L(N^*N\pi)&=(4\pi)^{1/2}(\tfrac{2}{3})(g_{N^*N\pi}/\mu)N_{\beta j}^{*\mu}[\delta_{\beta\alpha}\delta_{jn} \\ &\qquad -\tfrac{1}{2}i\tau_{jn}{}^{\gamma}e_{\beta\alpha\gamma}]\partial_\mu N_n\pi_\alpha+\text{H.c.} \end{aligned} \qquad (6)$$

The N^* width is related to $g_{N^*N\pi}$ by the expression

$$(g_{N^*N\pi})^2=6[M_\mu]^2\Gamma_{N^*}/P^3[(M^*+M)^2-\mu^2]\,, \qquad (7)$$

where $2P$ is the relative πN momentum and M is the nucleon mass. With $\Gamma_{N^*}=120$ MeV,

$$(g_{N^*N\pi})^2=0.36$$

and using this result in (5) we have

$$g_{NN\pi}^2=14.6\,. \qquad (8)$$

The close agreement of this result and experiment implies that either higher spin members of the trajectories contribute small residues or that the sum rule is valid, term by term, for corresponding members of the N and N^* trajectories. As we shall see, this sum rule, the result of a superconvergence relation at $t=0$, is very similar to the sum rules we derive at $u=0$ provided that the contribution of the f^0 meson's residue to the B amplitude is small.

Let us now develop the sum rules at $u=0$ in some detail. The single-particle contributions to the πN amplitudes are given by the diagrams of Fig. 2. At fixed u, only the ρ and f exchange and the N and N^* direct diagrams contribute to the discontinuities in s. With the phenomenological Lagrangians

$$\begin{aligned} L(\rho\pi\pi)&=-(4\pi)^{1/2}g_{\rho\pi\pi}e_{\alpha\beta\gamma}\pi_\alpha(\partial_\mu\pi_\beta)\rho_\gamma{}^{\mu}\,,\\ L(f\pi\pi)&=(4\pi)^{1/2}(g_{f\pi\pi}/M_f)(\partial_\mu\boldsymbol{\pi})\cdot(\partial_\nu\boldsymbol{\pi})f_{\mu\nu}\,,\\ L(\rho NN)&=i(4\pi)^{1/2}g_v\bar{\Psi}\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\varrho}_\mu\Psi \\ &\qquad +(4\pi)^{1/2}g_t\bar{\Psi}(\sigma_{\mu\nu}/2M)(\boldsymbol{\tau}\cdot\partial_\nu\boldsymbol{\varrho}_\mu)\Psi\,, \qquad (9)\\ L(fNN)&=(4\pi)^{1/2}(g_{fNN}{}^{(1)}/2M)[(\partial_\nu\bar{\Psi})\gamma_\mu f_{\mu\nu}\Psi \\ &\qquad -\bar{\Psi}\gamma_\mu f_{\mu\nu}(\partial_\nu\Psi)]+(4\pi)^{1/2}(g_{fNN}{}^{(2)}/M^2) \\ &\qquad \times(\partial_\nu\bar{\Psi})(\partial_\nu\Psi)f_{\mu\nu}\,, \end{aligned}$$

and with the $NN\pi$ and $N^*N\pi$ Lagrangian defined in (6), we have

$$\rho\text{ exchange: } T_{\alpha\beta}{}^{ij}=-i\frac{e_{\alpha\beta\gamma}}{M_\rho{}^2-t}\tau_{ji}{}^{\gamma}\left[(g_v+g_t)2i\gamma\cdot Q-\frac{2g_t}{M}P\cdot Q\right]g_{\rho\pi\pi}\,, \qquad (10\text{a})$$

$$\begin{aligned} f\text{ exchange: } & g_{f\pi\pi}\frac{\delta_{ji}\delta_{\beta\alpha}}{M_f{}^2-t}\bigg(-\frac{g_{fNN}{}^{(1)}}{M_fM}[2i\gamma\cdot Q(P\cdot Q) \\ &+\tfrac{1}{6}M_f{}^2]+(g_{fNN}{}^{(2)}/2M^2)[4(P\cdot Q)^2 \\ &+\tfrac{1}{3}M_f{}^2(p_1\cdot p_2)+\tfrac{1}{2}M_f{}^4]\bigg) \\ &\equiv[\delta_{ji}\delta_{\beta\alpha}/(M_f{}^2-t)](-f_A+i\gamma\cdot Qf_B)\,, \end{aligned} \qquad (10\text{b})$$

$$N\text{ direct: } \tau_{jk}{}^{\beta}\tau_{ki}{}^{\alpha}g_{NN\pi}{}^2 i\gamma\cdot Q/(M^2-s)\,, \qquad (10\text{c})$$

$$N^*\text{ direct: } \frac{(g_{N^*N\pi})^2}{\mu^2}\tfrac{2}{3}[\delta_{\beta\alpha}\delta_{ji}-\tfrac{1}{2}i\tau_{ji}{}^{\gamma}e_{\beta\alpha\gamma}]\times\frac{(-A^*+i\gamma\cdot QB^*)}{(M^{*2}-s)} \qquad (10\text{d})$$

where, at the pole,

$$\begin{aligned} A^*&=\tfrac{1}{3}(M+\epsilon)[3(\epsilon-M)(M+M^*)\cos\theta_s \\ &\qquad +(M^*-M)(\epsilon+M)]\approx 0.406M^{*3}\,, \quad u=0\,. \\ B^*&=\tfrac{1}{3}(M+\epsilon)[3(\epsilon-M)\cos\theta_s-(\epsilon+M)] \\ &\qquad \approx -0.667M^{*2}\,, \end{aligned}$$

with $\epsilon=(M^{*2}+M^2-\mu^2)/2M^*$.

Projecting out u-channel isospin contributions to the A and B amplitudes, and evaluating the residues, we obtain the finite-pole approximation to the superconvergence relations (2) (evaluated at $u=0$):

$$\begin{aligned} A^{(1/2)}&:\ \left(\frac{2M^2+2\mu^2-M_\rho{}^2}{M^2}\right)g_tg_{\rho\pi\pi}-\frac{4}{3}\frac{g_{N^*N\pi}{}^2A^*}{\mu^2M}+\frac{f_A}{M}=0\,,\\ A^{(3/2)}&:\ -\frac{(2M^2+2\mu^2-M_\rho{}^2)}{2M^2}g_tg_{\rho\pi\pi}-\frac{1}{3}\frac{g_{N^*N\pi}{}^2A^*}{\mu^2M}+\frac{f_A}{M}=0\,,\\ B^{(1/2)}&:\ -4g_{\rho\pi\pi}(g_v+g_t)+g_{NN\pi}{}^2-\tfrac{4}{3}B^*g_{N^*N\pi}{}^2/\mu^2+f_B=0\,,\\ B^{(3/2)}&:\ 2g_{\rho\pi\pi}(g_v+g_t)-2g_{NN\pi}{}^2-\tfrac{1}{3}B^*g_{N^*N\pi}{}^2/\mu^2+f_B=0\,. \end{aligned} \qquad (11)$$

If we solve the two relations for the B amplitudes by eliminating the ρ-meson contributions, we have

$$g_{NN\pi}^2=\tfrac{2}{3}(0.667)g_{N^*N\pi}^2(M^*/\mu)^2+3f_B\,, \tag{12}$$

which is consistent with the forward scattering relation (5) provided that $f_B\approx 0.037g_{NN\pi}^2$. Using this value for f_B, and again using the forward scattering relation, we can solve for the vector-meson coupling term, with the result

$$g_{\rho\pi\pi}(g_v+g_t)=0.74g_{NN\pi}^2\,. \tag{13}$$

If we eliminate f_A from the two equations for the A amplitude, we have [with $(2M^2+2\mu^2-M_\rho^2)/M^2\approx\frac{4}{3}$]

$$g_t g_{\rho\pi\pi}=0.53g_{NN\pi}^2\,. \tag{14}$$

Thus we obtain the relations

$$g_v=\frac{0.21g_{NN\pi}^2}{g_{\rho\pi\pi}}\,,\quad g_t=\frac{0.53g_{NN\pi}^2}{g_{\rho\pi\pi}}\,,\quad g_t/g_v=2.52\,. \tag{15}$$

With $g_{\rho\pi\pi}\sim 1.7$ (for a width $\Gamma_\rho=150$ MeV), we have $g_v=1.4$, which is in agreement with current estimates. A tensor coupling 2 to 4 times larger than the vector coupling is in agreement with a bootstrap model.[9]

We are encouraged, therefore, by the plausible results of the superconvergence model at $u=0$; at least there is no obvious inconsistency. Such an inconsistency might appear, for example, as an unphysically large prediction for the product $g_v g_{\rho\pi\pi}$ (say, $g_v g_{\rho\pi\pi}=10g_{NN\pi}^2$). We note, however, that if we had neglected the f-meson contributions there would have been an immediate contradiction of the sum rules for $A^{1/2}$ and $A^{3/2}$, although the sum rules for $B^{1/2}$ and $B^{3/2}$ would still have been essentially consistent with the t-channel results of Sakita and Wali. The identity of the t- and u-channel results for $g_{N^*N\pi}^2/g_{NN\pi}^2$, assuming $f_B\approx 0$, depends on the relation between the isospin projection operators (in the corresponding channels) which are closely related to the crossing matrix. The consistency is reminiscent of static-model bootstrap results.

We have, in (11), evaluated the sum rules at $u=0$. If we take the Regge predictions for asymptotic behavior to be valid at other values of u (say, $u<0$), then we have a one-parameter set of sum rules, all of which cannot be valid. The sensitivity to momentum transfer, however, is surprisingly small. For example, if we evaluate our sum rules at $u=-M_\rho^2$, Eq. (12) becomes

$$g_{NN\pi}^2=\tfrac{2}{3}(0.467)g_{N^*N\pi}^2(M^*/\mu)^2+3f_B'\,, \tag{12$'$}$$

which is consistent with the forward scattering relation provided that $f_B'\approx 0.126g_{NN\pi}^2$. As before, we can solve for the vector-meson coupling term, with the result

$$g_{\rho\pi\pi}(g_v+g_t)=0.72g_{NN\pi}^2\,. \tag{13$'$}$$

Similarly, Eq. (14) becomes, when the sum rules are evaluated at $u=-M_\rho^2$,

$$g_t g_{\rho\pi\pi}=0.58g_{NN\pi}^2\,. \tag{14$'$}$$

Varying u has therefore little effect on the ρ-meson coupling-constant relations, although the residues change appreciably. However, since we still do not know exactly where to evaluate the sum rules, this remains a fundamental difficulty.

IV. CONCLUSIONS

The two assumptions we have explored in this paper, that of superconvergent behavior based on a Regge hypothesis and the finite-pole approximation, give rise to highly suggestive results. We believe that they merit further study. It would especially be interesting to have more definite knowledge of the behavior of the Regge trajectories for small and negative momentum transfer. We believe that this behavior is possibly related to the fundamental problem of the momentum-transfer dependence of the sum rules. The contribution of higher spin resonances should also be investigated. This work is in progress.

Note added in proof. After our article had been accepted we learned of two other manuscripts on the $U=0$ sum rules [Douglas S. Beder and Jerome Finkelstein, Lawrence Radiation Laboratory Report No. UCRL-17407, 1967 (unpublished); R. Ramachandran, International Centre for Theoretical Physics Report, 1967 (unpublished)]. Beder and Finkelstein investigate the $I_u=\frac{1}{2}$ sum rules and Ramachandran the $I_u=\frac{3}{2}$ sum rules. Our results are in agreement with the conclusions of these authors, in particular that the $B^{(1/2)}$ and $B^{(3/2)}$ sum rules are well satisfied by ρ, N and N^* alone, while the $A^{(3/2)}$ and $A^{(1/2)}$ sum rules, if valid, require additional positive terms. These authors note that a sigma meson will contribute a positive term to the A amplitudes while the B amplitudes are unaffected. Thus the sigma, if it exists, can play essentially the same role as the $f(1250)$ in our analysis.

ACKNOWLEDGMENTS

We are grateful for illuminating conversations with Professor G. Feldman, Professor T. Fulton, and Professor G. Kane.

[9] G. Kane and W. F. Palmer (unpublished).