Low-Energy Theorem for Pion Photoproduction from the Hypothesis of Partially Conserved Axial-Vector Current*

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By using gauge invariance and the hypothesis of partially conserved axial-vector current, the pion-photoproduction amplitude at threshold is calculated to all orders in strong interactions neglecting terms of order $(m_\pi/M_N)^2$ but including terms of order m_π/M_N. The calculated cross section is consistent with experimental results for charged-pion photoproduction near threshold.

I. INTRODUCTION

As was first shown by Kroll and Ruderman,[1] the pion-photoproduction amplitude at threshold is given, to all orders in the pion-nucleon coupling constant g, simply by the Born-approximation amplitude in the limit as the pion-nucleon mass ratio m_π/M_N approaches zero. For positive-pion production from protons, $\gamma+p\rightarrow n+\pi^+$, the calculated cross section in the c.m. system gives at threshold

$$\frac{|\mathbf{k}|}{|\mathbf{q}|}\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}}(\gamma+p\rightarrow n+\pi^+)=\frac{e^2}{4\pi}\frac{g^2}{4\pi}\frac{1}{2M_N{}^2}=23.1\ \mu\text{b/sr},$$

where $|\mathbf{k}|$ and $|\mathbf{q}|$ are the photon and pion c.m. momenta, respectively, $e^2/4\pi\approx 1/137$ is the fine-structure constant, and $g^2/4\pi\approx 14.4$.

However, the experimental result[2] is

$$\frac{|\mathbf{k}|}{|\mathbf{q}|}\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}}(\gamma+p\rightarrow n+\pi^+)=(15.6\pm0.5)\ \mu\text{b/sr}$$

at threshold, which suggests that corrections to the Kroll-Ruderman theorem of order m_π/M_N may not be neglected.

The proof of Kroll and Ruderman is based essentially on the gauge invariance of the photoproduction amplitude. Their result can also be obtained by relating the pion field to the divergence of the weak axial current through the hypothesis of partially conserved axial current[3] (PCAC). We wish to point out here that by using both gauge invariance and PCAC, the first-order terms in an expansion of the threshold amplitude in powers of m_π/M_N may also be calculated. The agreement with the experimental results is then considerably improved.

II. THE LOW-ENERGY THEOREM

Consider the process $\gamma+p\rightarrow n+\pi^+$. The S-matrix amplitude has the form

$$\langle\pi_q{}^+n_{p'}\text{ out}|\gamma_kP_p\text{ in}\rangle=i(2\pi)^4\delta^{(4)}(p+k-p'-q)\mathfrak{M}_+,$$

where $\mathfrak{M}_+\equiv\langle n_{p'}|j_\pi{}^+(0)|\gamma_kP_p\text{ in}\rangle$ and $j_\pi{}^+(x)$ is the source of the pion field $\phi_\pi{}^+(x)$, i.e., $(\square+m_\pi{}^2)\phi_\pi{}^+(x)=j_\pi{}^+(x)$. According to the PCAC hypothesis, including electromagnetic interactions to first order in e,

$$\partial_\mu A_+{}^\mu(x)+ie\mathfrak{A}_\mu(x)A_+{}^\mu(x)=ia\phi_\pi{}^+(x)\,, \quad (1)$$

where $A_+{}^\mu(x)$ is the positive-charged component of the weak axial current, $\mathfrak{A}_\mu(x)$ is the electromagnetic potential, and

$$a=\sqrt{2}M_Nm_\pi{}^2F_A(0)/g(0)\,.$$

$F_A(0)\approx 1.18$ is the weak axial coupling constant and $g(0)$ is the off-mass-shell pion-nucleon coupling constant $[g^2(m_\pi{}^2)/4\pi\approx 14.4]$.

Taking matrix elements of this expression between states $\langle n_{p'}|$ and $|\gamma_kP_p\text{ in}\rangle$, we have

$$a\frac{1}{m_\pi{}^2-q^2}\langle n_{p'}|j_\pi{}^+(0)|\gamma_kP_p\text{ in}\rangle=-q^\mu\langle n_{p'}|A_\mu{}^+(0)|\gamma_kP_p\text{ in}\rangle+e\langle n_{p'}|\mathfrak{A}_\mu(0)A_+{}^\mu(0)|\gamma_kP_p\text{ in}\rangle. \quad (2)$$

In the first term on the right side of Eq. (2), we separate out the pion pole contribution to the axial-current matrix element

$$-q^\mu\langle n_{p'}|A_\mu{}^+(0)|\gamma_kP_p\text{ in}\rangle=\frac{q^2}{m_\pi{}^2-q^2}\frac{\sqrt{2}MF_A(0)}{g(0)}\times\langle n_{p'}|j_\pi{}^+(0)|\gamma_kP_p\text{ in}\rangle-q^\mu\langle n_{p'}|A_\mu{}^+(0)|\gamma_kP_p\text{ in}\rangle',$$

where the prime on the second term indicates that the pion pole term has been subtracted. Inserting this in Eq. (2), we then obtain

$$\langle n_{k'}|j_\pi{}^+(0)|\gamma_kP_p\text{ in}\rangle\frac{a}{m_\pi{}^2}=-q^\mu\langle n_{p'}|A_\mu{}^+(0)|\gamma_kP_p\text{ in}\rangle'+e\langle n_{p'}|\mathfrak{A}_\mu(0)A_+{}^\mu(0)|\gamma_kP_p\text{ in}\rangle. \quad (3)$$

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To lowest order in e, $\mathcal{A}_\mu(x)=\mathcal{A}_\mu^{\text{in}}(x)$, so that

$$e\langle n_{p'}|\mathcal{A}_\mu(0)A_+^\mu(0)|\gamma_k P_p \text{ in}\rangle = e\epsilon_\mu(k)\langle n_{p'}|A_+^\mu(0)|P_p\rangle + O(e^2) = e\epsilon_\mu(k)\bar{U}(p')\bigg[\gamma^\mu\gamma_5 F_A((q-k)^2)+(q-k)^\mu$$

$$\times\frac{2M_N F_A(0)}{(q-k)^2-m_\pi^2}\gamma_5+O\left(\frac{(q-k)^2}{M_N^2}\right)\bigg]U(p)+O(e^2)=eF_A(0)\bar{U}(p')\left[\gamma\cdot\epsilon\gamma_5+\frac{2M_N q\cdot\epsilon}{-2q\cdot k}\gamma_5\right]U(p)+O\left(\frac{(q-k)^2}{M_N^2}\right)+O(e^2), \quad (4)$$

assuming $F_A((q-k)^2)=F_A(0)+O((q-k)^2/M_N^2)$. Here $\epsilon_\mu(k)$ is the polarization vector of the photon ($k\cdot\epsilon=0$).

Also, by isolating the Born contribution[4] to the first term on the right in Eq. (3), we may write

$$q^\mu\langle n_p'|A^+_\mu(0)|\gamma_k P_p \text{ in}\rangle' = e\bar{U}(p')\left[q\gamma_5\frac{p+k+M_N}{(p+k)^2-M_N^2}\left(\gamma\cdot\epsilon-\frac{\kappa_p}{2M_N}\gamma\cdot\epsilon\right)-\frac{\kappa_n}{2M_N}\gamma\cdot\epsilon\frac{p'-k+M_N}{(p'-k)^2-M_N^2}q\gamma_5\right]U(p)F_A(q^2)$$

$$+e\frac{q\cdot\epsilon}{m_\pi^2-(q-k)^2}\frac{a}{m_\pi^2}\sqrt{2}g[(q-k)^2]\bar{U}(p')\gamma_5 U(p)+q^\mu\epsilon^\nu N_{\mu\nu}, \quad (5)$$

where κ_p and κ_n are the proton and neutron anomalous moments, respectively. The non-Born amplitude $N_{\mu\nu}$ is finite as q, $k\to 0$ (with $m_\pi^2=q^2\to 0$), so we have

$$q^\mu N_{\mu\nu}=q^\mu N_{\mu\nu}(q^2=m_\pi^2=0, q^\mu=0, k^\mu=0)+O(q^2/M_N^2, q\cdot k/M_N^2).$$

Combining Eqs. (3), (4), and (5), we obtain

$$\mathfrak{M}_+\equiv\langle n_{p'}|j_\pi^+(0)|\gamma_k P_p \text{ in}\rangle = e\frac{g(0)}{\sqrt{2}M_N}\bar{U}(p')\bigg[\gamma\cdot\epsilon\gamma_5-\frac{2M_N q\cdot\epsilon}{q\cdot k}\gamma_5-q\gamma_5\frac{p+k+M_N}{2p\cdot k}\left(\gamma\cdot\epsilon-\frac{\kappa_p}{2M_N}\gamma\cdot\epsilon k\right)$$

$$-\frac{\kappa_n}{2M_N}\gamma\cdot\epsilon k\frac{p'-k+M_N}{-2p'\cdot k}q\gamma_5\bigg]U(p)+q^\mu\epsilon^\nu N_{\mu\nu}(q^2=0, q^\mu=k^\mu=0)+O(q^2/M_N^2, q\cdot k/M_N^2)+O(e^2). \quad (6)$$

Now, writing $\mathfrak{M}_+=\epsilon_\mu M_+^\mu$, gauge invariance of the S-matrix amplitude requires that $k^\mu M_\mu^+=0$. Since the first term in Eq. (6) is separately gauge invariant, we must have

$$k^\nu N_{\mu\nu}(q^2=0, q^\mu=k^\mu=0)=0,$$

which implies $N_{\mu\nu}(q^2=0, q^\mu=k^\mu=0)=0$.

Thus, we have shown that neglecting terms of order q^2/M_N^2 and $q\cdot k/M_N^2$, the S-matrix amplitude $\mathfrak{M}_+$, for $\gamma+p\to n+\pi^+$, is given by the first term in Eq. (6), which can be rewritten in the form

$$\mathfrak{M}_+=-\sqrt{2}ge\bar{U}(p')\left\{\left[\frac{\gamma\cdot\epsilon k}{2p\cdot k}-\left(\frac{p\cdot\epsilon}{p\cdot k}-\frac{q\cdot\epsilon}{q\cdot k}\right)-\frac{\kappa_p}{2M_N}\frac{q\gamma\cdot\epsilon k}{2p\cdot k}+\frac{\kappa_n}{2M_N}\frac{\gamma\cdot\epsilon kq}{2p'\cdot k}\right]+\frac{\kappa_p+\kappa_n}{4M_N^2}\gamma\cdot\epsilon k\right\}\gamma_5 U(p), \quad (7)$$

where we have assumed that $g(0)=g(m_\pi^2)+O(m_\pi^2/M_N^2)$. The factor in square brackets is just the usual Born amplitude. The additional term is of order $(\kappa_p+\kappa_n)m_\pi/M_N$, which may be neglected since $(\kappa_p+\kappa_n)m_\pi/M_N<(m_\pi/M_N)^2$. The anomalous moment terms in the Born amplitude also contribute to the cross section a term of order $(\kappa_p+\kappa_n)m_\pi/M_N$, and hence they may be ignored.

[4] The last of the Born terms in Eq. (5) involves the π-γ axial-vector vertex. This is evaluated by using PCAC.

III. COMPARISON WITH EXPERIMENTAL RESULTS

A. π^+ Production

The differential cross section in the c.m. system obtained from Eq. (7) gives

$$\frac{|\mathbf{k}|}{|\mathbf{q}|}\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}}(\gamma+p\to n+\pi^+)=\frac{e^2}{4\pi}\frac{g^2}{4\pi}\frac{1}{2(p_0+k_0)^2}$$

$$\times\left[\frac{p'\cdot k}{p\cdot k}+\tfrac{1}{2}q^2\left(\frac{p^\mu}{p\cdot k}-\frac{q^\mu}{q\cdot k}\right)^2\right]. \quad (8)$$

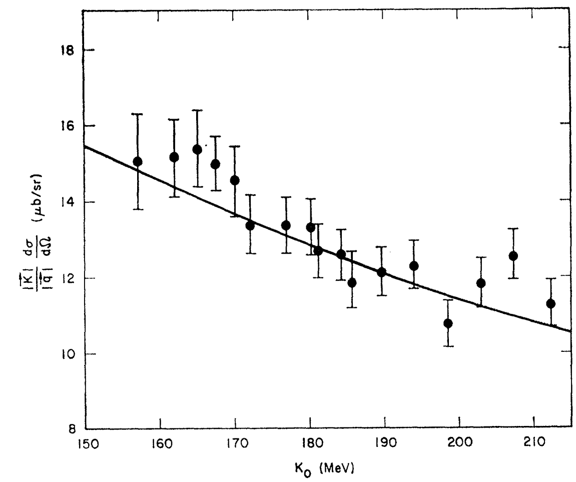

FIG. 1. The differential cross section in the c.m. system (times a kinematical factor $|\mathbf{k}|/|\mathbf{q}|$) for photoproduction of π^+ mesons from protons near threshold. The momentum transfer is held fixed at its value at threshold as the photon energy is varied. The experimental points are taken from Ref. 2.

Figure 1 shows the experimental data[2] for $(|\mathbf{k}|/|\mathbf{q}|)$-$(d\sigma/d\Omega)_{\text{c.m.}}$ near threshold for the momentum transfer fixed at its value at threshold, together with the theoretical curve predicted by Eq. (8). At threshold ($|\mathbf{q}|=0$), we find from Eq. (8)

$$\frac{|\mathbf{k}|}{|\mathbf{q}|}\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}}(\gamma+p\rightarrow n+\pi^+)=15.5\ \mu\text{b/sr}.$$

This value is consistent with the experimental result[2] of $(15.6\pm0.5)\ \mu$b/sr.

We see from Fig. 1 that Eq. (8) correctly predicts the slope of the cross section near threshold. The angular distribution has been observed experimentally[5] in the region just above threshold and it does not agree with Eq. (8). However, since the angular variations are small, this discrepancy is not surprising, because of the approximate nature of the PCAC relation. The observed distribution is presumably due to the tail of the $N^*(1236)$ resonance.

B. π^- Production

A calculation for π^- photoproduction from neutrons similar to the one in Sec. II gives the result

$$R\equiv\frac{(d\sigma/d\Omega)_{\text{c.m.}}(\gamma+n\rightarrow p+\pi^-)}{(d\sigma/d\Omega)_{\text{c.m.}}(\gamma+p\rightarrow n+\pi^+)}$$
$$=(p\cdot k/p'\cdot k)^2\approx1.3 \text{ at threshold}.$$

A recent experimental value[2] is

$$R=1.265\pm0.075,$$

which agrees with our result, whereas the Kroll-Ruderman limit gives $R=1$.

C. π^0 Production

For π^0 photoproduction the amplitude vanishes in zeroth order (the Kroll-Ruderman limit). Calculation of the first-order terms gives

$$\frac{|\mathbf{k}|}{|\mathbf{q}|}\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}}(\gamma+p\rightarrow\pi^0+p)=\frac{e^2}{4\pi}\frac{g^2}{4\pi}\frac{1}{4M_N^2}\times\left(\frac{m_\pi}{M_N}\right)^2=0.24\ \mu\text{b/sr} \quad (9)$$

at threshold. Furthermore $(|\mathbf{k}|/|\mathbf{q}|)(d\sigma/d\Omega)_{\text{c.m.}}$ should be approximately constant as a function of photon energy just above threshold. Experimentally[6] this is not the case. $(|\mathbf{k}|/|\mathbf{q}|)(d\sigma/d\Omega)_{\text{c.m.}}$ increases quadratically with $|\mathbf{q}|$, and at 160 MeV is still over twice as large as Eq. (9). Also the angular distribution disagrees with the calculated result. Clearly then, for π^0 production near threshold, the N^* resonance may not be ignored because of the vanishing of the Born amplitude in the limit as $m_\pi/M_N\rightarrow0$.

IV. CONCLUSION

We have shown[7] that, by using gauge invariance and the PCAC hypothesis, one is justified in using the Born approximation for pion photoproduction near threshold if we neglect terms of order m_π^2/M_N^2 and $(\kappa_p+\kappa_n)$-m_π/M_N in the amplitude, and if the $N^*(1236)$ resonance can be ignored. For charged-pion production the agreement with experiment is good, showing our assumptions are justified. For neutral pions, due to the smallness of the Born amplitude, the N^* resonance apparently dominates near threshold.

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[7] After this work had been completed, our attention was called to a paper by S. Ragusa [Enrico Fermi Institute for Nuclear Study EFINS Report No. 67-9 (unpublished)] in which essentially the same results were obtained. Our derivation of the low-energy theorem is somewhat simpler and avoids taking the pion off the mass shell.