exists. Taking this limit, we have

$$\bar{X}\int dt\,\tilde{f}(t)e^{itM^2}b=\sum_{s=0}^{s_0}\int dt\,f(t)\frac{(-it)^s}{s!}e^{itM^2}b_s. \quad (25)$$

On using the spectral decomposition of M^2, this becomes

$$\bar{X}\int f(\lambda)dE(\lambda)b=\sum_{s=0}^{s_0}\int\frac{f^{(s)}(\lambda)}{s!}dE(\lambda)b_s, \quad (26)$$

or

$$\bar{X}Qb=Q\sum_{s=0}^{s_0}\frac{f^{(s)}(m^2)}{s!}b_s=Qb_0=Q\bar{X}b. \quad (27)$$

Now let f be any vector in $\Delta(\bar{X})$. Since, from lemma 2, $\bar{X}$ is the closure of $X/\mathcal{B}$, there exists a sequence $b_n\in\mathcal{B}$ such that $b_n\rightarrow f$, $\bar{X}b_n\rightarrow\bar{X}f$, and hence, since Q is bounded, $Qb_n\rightarrow Qf$, $\bar{X}Qb_n=Q\bar{X}b_n\rightarrow Q\bar{X}f$. Thus $Qf\in\Delta(\bar{X})$ and $\bar{X}Qf=Q\bar{X}f$. This establishes the lemma.

Since $\bar{X}$ is also the closure of $X/\mathfrak{D}$ and $\mathfrak{D}$ is analytic with respect to the connected group $R(g)$, it follows immediately that Q reduces $R(g)$. This is the theorem stated in the Introduction.

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Analytic Continuation of the Froissart-Gribov Partial-Wave Amplitude to the Left-Half l Plane*

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A continuation of the Froissart-Gribov definition of the partial-wave amplitude to the left of the poles in the l plane is obtained under the assumption of power-law behaviors of the Mandelstam weight functions at high energy. Discrete and continuous powers in these weight functions are seen to yield, respectively, poles and cuts in the continued partial-wave amplitude. This continuation is then used to prove that in the presence of cuts a generalized form of the Mandelstam symmetry relation for the partial-wave amplitudes about $l=-\frac{1}{2}$ for the half-odd-integral values of l holds at energies where there are no Regge poles passing through half-odd integers. The discontinuity across the cut at a half-odd integer is always equal to discontinuity across the cut at the half odd integer obtained by reflection about $l=-\frac{1}{2}$. The case of Regge poles passing through half-odd integers is considered in detail, and the results derived by Mandelstam for potential scattering are shown to follow from our continuation in a straightforward manner. The continued partial-wave amplitude has the desirable feature that every term in it has the correct threshold behavior, $(q^2)^l$.

I. INTRODUCTION

SINCE the time Froissart[1] and Gribov[2] independently proposed the formula for the partial-wave amplitude for complex angular momentum l valid to the right of the leading Regge pole α in the complex l plane, there have been several attempts[3] to obtain continuations to the left of the line $\mathrm{Re}l=\mathrm{Re}\alpha$. However, to the author's knowledge, most of these have been confined to cases of nonrelativistic potential scattering. One interesting result of such investigations is that a symmetry relation for the partial-wave amplitudes about $l=-\frac{1}{2}$ for the half-odd-integral values of l, proposed by Mandelstam, is true for a large class of potentials. We examine here (Sec. II) the problem in the relativistic case under the assumption that the high-energy behavior of the Mandelstam weight functions in a fixed-t dispersion relation (t is the Regge-pole channel) consists of powers of s and u. Discrete and continuous powers yield, respectively, poles and cuts in the l plane, and these are explicitly exhibited in our formula for the partial-wave amplitude. We find that the continuation

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[1] M. Froissart, Report at La Jolla Conference, 1961 (unpublished).

[2] V. N. Gribov, Zh. Eksperim. i Teor. Fiz. 42, 1260 (1961) [English transl.: Soviet Phys.—JETP 15, 873 (1962)].

[3] S. Mandelstam, Ann. Phys. (N.Y.) 19, 254 (1962); M. Froissart, J. Math. Phys. 3, 922 (1962); R. Blankenbecler and M. L. Goldberger, Phys. Rev. 126, 766 (1962). See for other references, E. J. Squires, *Complex Angular Momenta and Particle Physics* (W. A. Benjamin, Inc., New York, 1964).

we obtain satisfies in the presence of cuts a generalized form of the Mandelstam symmetry relation, at energies where there are no Regge poles passing through half-odd integers. We also find that the discontinuity across the cut at a half odd integer is always equal to the discontinuity across the cut at the half-odd integer obtained by reflection about $l=-\frac{1}{2}$. In Sec. III we study the formula for the Regge residue with particular attention to the case of Regge poles passing through half-odd integers. It follows in a straightforward manner that the results proved by Mandelstam for potential scattering[3] are true in this case also. Thus, at energies where a Regge pole passes through a half-odd integer, either the residue of the Regge pole must vanish or there must exist another Regge pole of equal residue at the half odd integer obtained by reflection about $l=-\frac{1}{2}$. In Sec. IV we show that the continued partial-wave amplitude is a good threshold representation in the sense that every term in it has the correct threshold behavior, $(q^2)^l$. This is to be contrasted with an analogous representation guessed by Gribov and Pomeranchuk,[4] in which the Regge-pole terms do not have this threshold behavior.

II. ANALYTIC CONTINUATION OF THE PARTIAL-WAVE AMPLITUDE TO THE LEFT OF THE LEADING SINGULARITIES IN THE COMPLEX l PLANE

Let the invariant scattering amplitude for the process $p_1+p_2\rightarrow p_3+p_4$ of scattering of spinless particles of masses m_1, m_2, m_3, and m_4 be assumed to satisfy a fixed-t dispersion relation with weight functions $A_s(t,s')$ and $A_u(t,u')$, where we have defined $s=-(p_1+p_2)^2$, $t=-(p_1-p_3)^2$, $u=-(p_1-p_4)^2$. In the t channel, the signatured Froissart-Gribov partial-wave amplitudes are given by

$$a^{\pm}(l,t)=\frac{1}{2\pi q_{13}(t)q_{24}(t)}\left[\int_{s_0}^{\infty}ds'\,A_s(t,s')Q_l(z_1(t,s'))\pm\int_{u_0}^{\infty}du'\,A_u(t,u')Q_l(-z_2(t,u'))\right],\tag{2.1}$$

where

$$z_1(t,s')=\frac{s'+\frac{1}{2}t-\frac{1}{2}(m_1^2+m_2^2+m_3^2+m_4^2)+(m_1^2-m_3^2)(m_2^2-m_4^2)/2t}{2q_{13}(t)q_{24}(t)},\tag{2.2}$$

$$z_2(t,u')=-\frac{u'+\frac{1}{2}t-\frac{1}{2}(m_1^2+m_2^2+m_3^2+m_4^2)-(m_1^2-m_3^2)(m_2^2-m_4^2)/2t}{2q_{13}(t)q_{24}(t)},\tag{2.3}$$

and $q_{13}(t)$ and $q_{24}(t)$ denote the magnitudes of the t-channel center-of-mass momenta of the two pairs of particles. $q_{13}(t)q_{24}(t)$ has a simple pole at $t=0$, if $m_1\neq m_3$ and $m_2\neq m_4$. The weight functions $A_s(t,s')$ and $A_u(t,u')$ are analytic at $t=0$. We assume that we can write

$$A_s(t,s')=\sum_i c_s^{(i)}(t)[z_1(t,s')]^{\alpha_s^{(i)}(t)}+\int_{\alpha_{s1}(t)}^{\alpha_{s2}(t)}d\alpha'\,g_s(t,\alpha')[z_1(t,s')]^{\alpha'}+A_s^0(t,s'),\tag{2.4}$$

and

$$A_u(t,u')=\sum_i c_u^{(i)}(t)[-z_2(t,u')]^{\alpha_u^{(i)}(t)}+\int_{\alpha_{u1}(t)}^{\alpha_{u2}(t)}d\alpha'\,g_u(t,\alpha')[-z_2(t,u')]^{\alpha'}+A_u^0(t,u'),\tag{2.5}$$

where

$$A_s^0(t,s')=O((s')^{-L}),\quad s'\rightarrow\infty,\tag{2.6}$$

and

$$A_u^0(t,u')=O((u')^{-L}),\quad u'\rightarrow\infty,\tag{2.7}$$

where L is a real number satisfying

$$\mathrm{Re}\alpha(t)>-L,\tag{2.8}$$

$\mathrm{Re}\alpha(t)$ being the largest real part of the powers appearing in Eqs. (2.4) and (2.5). The integrals over α' in (2.4) and (2.5) are, in general, line integrals in the complex α' plane. The essential assumption embodied in Eqs. (2.4)–(2.8) is that the high-energy behavior of the Mandelstam weight functions consists of discrete and continuous powers of s' and u' down to a certain power $-L$.

For the sake of clarity, we may mention that although such asymptotic behaviors result when the partial-wave amplitude is assumed to be analytic except for poles and branch points in the l plane and to have asymptotic behavior in the l plane suitable for making a Mandelstam-Sommerfeld-Watson transformation[3] with background integral at $\mathrm{Re}l=-L$, we are *not* making such *a priori* hypotheses for the partial-wave amplitude. The assumptions (2.4)–(2.8) will be regarded as fundmental (*not* derived) assumptions in relativistic theory analogous to assumptions about classes of potentials in nonrelativistic theory, and consequences for the partial-wave amplitude will then be derived.

[4] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters **9**, 238 (1962), Eq. (4).

For $-L<\mathrm{Re}l<\mathrm{Re}\alpha(t)$, the divergence of the Froissart-Gribov integrals (2.1) comes entirely from the terms under the summation and integral signs in (2.4) and (2.5); however, the point is that analytic expressions can be found for the integrals of these terms for $\mathrm{Re}l>\mathrm{Re}\alpha(t)$, and these expressions can then be continued down to $\mathrm{Re}l=-L$. Our assumptions yield

$$a^{\pm}(l,t)=\frac{1}{2\pi q_{13}(t)q_{24}(t)}\left[\int_{s_0}^{\infty}ds'\,A_s{}^0(t,s')Q_l(z_1(t,s'))\pm\int_{u_0}^{\infty}du'\,A_u{}^0(t,u')Q_l(-z_2(t,u'))\right]$$
$$+\frac{1}{\pi}\sum_i\left[c_s{}^{(i)}(t)I(l,\alpha_s{}^{(i)}(t),(z_1)_0)\pm c_u{}^{(i)}(t)I(l,\alpha_u{}^{(i)}(t),(-z_2)_0)\right]$$
$$+\frac{1}{\pi}\left[\int_{\alpha_{s1}(t)}^{\alpha_{s2}(t)}d\alpha'\,g_s(t,\alpha')I(l,\alpha',(z_1)_0)\pm\int_{\alpha_{u1}(t)}^{\alpha_{u2}(t)}d\alpha'\,g_u(t,\alpha')I(l,\alpha',(-z_2)_0)\right],\quad \mathrm{Re}l>-L,\qquad(2.9)$$

where $(z_1)_0$ and $(-z_2)_0$ are the values of z_1 and $-z_2$ defined by (2.2) and (2.3) at $s'=s_0$ and $u'=u_0$, respectively, and we have denoted

$$I(l,\alpha,x_0)\equiv\int_{x_0}^{\infty}dx\,x^{\alpha}Q_l(x),\quad \mathrm{Re}l>\mathrm{Re}\alpha.\qquad(2.10)$$

Expressing the Legendre function in Eq. (2.10) as a hypergeometric function[5] we can perform the integral to obtain

$$I(l,\alpha,x_0)=(\sqrt{\pi})2^{-l-1}\frac{\Gamma(l+1)}{\Gamma(1+\frac{1}{2}l)\Gamma(\frac{1}{2}+\frac{1}{2}l)}\times\sum_{r=0}^{\infty}\frac{\Gamma(1+\frac{1}{2}l+r)\Gamma(\frac{1}{2}+\frac{1}{2}l+r)}{\Gamma(l+\frac{3}{2}+r)\Gamma(r+1)}\frac{(x_0)^{\alpha-l-2r}}{(l-\alpha+2r)}.\qquad(2.11)$$

Equation (2.11) gives the continuation of the function defined in (2.10) to $\mathrm{Re}l<\mathrm{Re}\alpha$. The series in (2.11) is seen to be absolutely convergent for $x_0^2>1$ by noting that without the denominator $(l-\alpha+2r)$ we would obtain a hypergeometric series which is absolutely convergent, and the denominator only improves the convergence. Thus, at least for values of t such that $(z_1)_0^2$ and $(-z_2)_0^2$ are greater than unity, Eq. (2.9) together with (2.11) yields a continuation of the partial-wave amplitude to the left half l plane down to $\mathrm{Re}l=-L$, where L is the largest number for which our assumptions (2.4)–(2.8) are valid. This continuation exhibits the singularities of $a^{\pm}(l,t)$ in the l plane down to $\mathrm{Re}l=-L$. The terms in (2.9) involving $A_s{}^0$ and $A_u{}^0$ are analytic in the half-plane $\mathrm{Re}l>-L$ except possibly at the negative integers where the Q_l functions have poles. It is enough to consider the integrals over α' in Eqs. (2.4), (2.5), and (2.9) to run over finite regions ($\mathrm{Re}\alpha'\geq-L$), and since the sums (2.11) are clearly uniformly convergent with respect to α' for all such values of α', the integration over α' in (2.9) and the summation over r resulting after the substitution of (2.11) in (2.9) can be interchanged if both the results exist. It then follows that each term in the sum over r contributes fixed poles in the l plane at the negative integers $l=-1,-2,\cdots$, moving poles at $\alpha_s{}^{(i)}(t)-2r$, $\alpha_u{}^{(i)}(t)-2r$, and cuts in the l plane from $\alpha_{s1}(t)-2r$ to $\alpha_{s2}(t)-2r$ and from $\alpha_{u1}(t)-2r$ to $\alpha_{u2}(t)-2r$ to the partial-wave amplitude in (2.9). Not all of these singularities need actually occur in $a^{\pm}(l,t)$. From (2.9) and (2.11) the relations between the $c_s{}^{(i)}(t)$, $\alpha_s{}^{(i)}(t)$, $g_s(t,\alpha')$, $\alpha_{s1}(t)$, $\alpha_{s2}(t)$ and the $c_u{}^{(i)}(t)$, $\alpha_u{}^{(i)}(t)$, $g_u(t,\alpha')$, $\alpha_{u1}(t)$, $\alpha_{u2}(t)$ in order that certain Regge poles or cuts occur only in certain signature amplitudes can be easily found. It has been shown[6] that the residues of the fixed poles in $a^{\pm}(l,t)$ at the negative integers cannot vanish in a relativistic theory with a third double-spectral function (except for negative integers of the correct signature for identical particle scattering) and that such fixed poles for l on one side of the cut are consistent with unitarity[7] in the presence of cuts in the l plane.

We shall now use (2.9) and (2.11) to prove that in the presence of cuts a generalized form of the Mandelstam-symmetry relation for the partial-wave amplitudes about $l=-\frac{1}{2}$ for the half-odd-integral values of l holds at values of t where there are no discrete powers $\alpha_s{}^{(i)}(t)$, $\alpha_u{}^{(i)}(t)$ passing through half-odd integers. For a given pair of half-odd integers l, and $(-l-1)$ the symmetry relation will still hold if the half-odd-integer powers are less than both l and $(-l-1)$.

For proof, we examine first Eq. (2.11) at half-odd-integral values of l, $l=\frac{1}{2}n$, and $l=-\frac{1}{2}n-1$, where n is a positive odd integer. We assume that if α is a half-odd integer, it is less than $\frac{1}{2}n$ and $(-\frac{1}{2}n-1)$, so that the denominators $(l-\alpha+2r)$ in Eq. (2.11) never vanish. We then obtain

$$I(-\tfrac{1}{2}n-1,\alpha,x_0)=(\sqrt{\pi})2^{n/2}\frac{\Gamma(-\frac{1}{2}n)}{\Gamma(\frac{1}{2}-\frac{1}{4}n)\Gamma(-\frac{1}{4}n)}\times\sum_{r=0}^{\infty}\frac{\Gamma(\frac{1}{2}-\frac{1}{4}n+r)\Gamma(-\frac{1}{4}n+r)}{\Gamma(-\frac{1}{2}n+\frac{1}{2}+r)\Gamma(r+1)}\frac{(x_0)^{\alpha+\frac{1}{2}n+1-2r}}{(2r-\alpha-\frac{1}{2}n-1)}.\qquad(2.12)$$

[5] *Bateman Manuscript Project, Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, p. 134, (41)

[6] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Letters **2**, 239 (1962).

[7] C. Edward Jones and Vigdor L. Teplitz, Phys. Rev. **159**, 1271 (1967); Stanley Mandelstam and Ling-Lie Wang, *ibid.* **160**, 1490 (1967).

We notice that the first few terms of the summation on the right-hand side corresponding to $r\leq(n-1)/2$ vanish on account of the Γ function $\Gamma(-\frac{1}{2}n+\frac{1}{2}+r)$ in the denominator. The remaining terms can be written by translating the summation index $(r\rightarrow r+\frac{1}{2}n+\frac{1}{2})$ into the form

$$I(-\tfrac{1}{2}n-1,\,\alpha,x_0)=(\surd\pi)2^{n/2}\frac{\Gamma(-\frac{1}{2}n)}{\Gamma(\frac{1}{2}-\frac{1}{4}n)\Gamma(-\frac{1}{4}n)}\times\sum_{r=0}^{\infty}\frac{\Gamma(1+\frac{1}{4}n+r)\Gamma(\frac{1}{2}+\frac{1}{4}n+r)}{\Gamma(r+1)\Gamma(r+\frac{1}{2}n+\frac{3}{2})}\frac{(x_0)^{\alpha-\frac{1}{2}n-2r}}{(\frac{1}{2}n-\alpha+2r)}.\quad(2.13)$$

The coefficient in front of the summation symbol in Eq. (2.13) can be proved to be equal to the coefficient in front of the summation symbol in Eq. (2.11) with $l=\frac{1}{2}n$, by using elementary properties of Γ functions. The terms inside the summation in (2.13) are the same as those in (2.11) with $l=\frac{1}{2}n$. We thus obtain

$$I(l,\alpha,x_0)=I(-l-1,\,\alpha,x_0),\quad l=\text{half odd integer},\quad(2.14)$$

with the previously stated restriction on α. We also know that[8]

$$Q_l(z)=Q_{-l-1}(z),\quad l=\text{half-odd integer.}\quad(2.15)$$

If we write Eq. (2.9) in the form

$$a^{\pm}(l,t)=a_R^{\pm}(l,t)+a_C^{\pm}(l,t)\,,\quad(2.16)$$

where $a_C^{\pm}(l,t)$ denotes the terms involving integrals over α' and hence containing the cuts in the l plane, and $a_R^{\pm}(l,t)$ denotes the remaining terms, then (2.14) and (2.15) show that

$$a_R^{\pm}(l,t)=a_R^{\pm}(-l-1,\,t),\quad l=\text{half-odd integer},\quad(2.17)$$

provided that discrete half-odd-integer powers, if any, in Eqs. (2.4) and (2.5) are smaller than l and $(-l-1)$. The term $a_C^{\pm}(l,t)$ could be shown to satisfy a relation similar to (2.17) by direct use of (2.14) if the cuts were not along the real axis. For the case where the cuts are along the real axis, we obtain for values of l just above the cut

$$a_C^{\pm}(l+i\epsilon,\,t)=\sum_{r=0}^{\infty}\left[\frac{P}{\pi}\int_{\alpha_{s1}(t)}^{\alpha_{s2}(t)}d\alpha'\,g_s(t,\alpha')\frac{I(r,l)}{(l-\alpha'+2r)}(z_1)_0{}^{\alpha'-l-2r}\pm\frac{P}{\pi}\int_{\alpha_{u1}(t)}^{\alpha_{u2}(t)}d\alpha'\,g_u(t,\alpha')\frac{I(r,l)}{(l-\alpha'+2r)}(-z_2)_0{}^{\alpha'-l-2r}\right]$$
$$-i\sum_{r=0}^{\infty}I(r,l)[g_s(t,\,l+2r)\theta(\alpha_{s1}(t),\,l+2r,\,\alpha_{s2}(t))\pm g_u(t,\,l+2r)\theta(\alpha_{u1}(t),\,l+2r,\,\alpha_{u2}(t))],\quad(2.18)$$

where we have denoted

$$I(r,l)=(\surd\pi)2^{-l-1}\frac{\Gamma(l+1)}{\Gamma(1+\frac{1}{2}l)\Gamma(\frac{1}{2}+\frac{1}{2}l)}\frac{\Gamma(1+\frac{1}{2}l+r)\Gamma(\frac{1}{2}+\frac{1}{2}l+r)}{\Gamma(l+\frac{3}{2}+r)\Gamma(r+1)},\quad(2.19)$$

and

$$\theta(a,b,c)\equiv1,\quad\text{if}\quad a\leq b\leq c,\qquad \equiv0,\ \text{otherwise.}\quad(2.20)$$

In the terms involving principal-value integrations in (2.18) the denominators $(l-\alpha'+2r)$ never vanish, and hence (2.14) shows that they are the same for $l=\frac{1}{2}n$ and $l=-\frac{1}{2}n-1$. The remaining terms in (2.18) are also shown to be equal for $l=\frac{1}{2}n$ and $l=-\frac{1}{2}n-1$ by noting that the denominator $\Gamma(l+\frac{3}{2}+r)$ in (2.19) becomes infinite for $r\leq-l-\frac{3}{2}$ when l is a negative half-odd integer, and using a procedure identical to that used in deriving (2.14). We thus have

$$a_C^{\pm}(l+i\epsilon,\,t)=a_C^{\pm}(-l-1+i\epsilon,\,t),\qquad l=\text{half-odd integer.}\quad(2.21)$$

An analogous relation below the cut $(i\epsilon\rightarrow-i\epsilon)$ is obtained exactly similarly. Hence, by subtracting from (2.21) the analogous relation below the cut we conclude the discontinuity across the cut at a half-odd integer is always equal to the discontinuity across the cut at the half-odd integer obtained by reflection about $l=-\frac{1}{2}$. Combining (2.16), (2.17), and (2.21) we finally have

$$a^{\pm}(l+i\epsilon,\,t)=a^{\pm}(-l-1+i\epsilon,\,t),\qquad l=\text{half-odd integer},\quad(2.22)$$
$$\text{Re}l>-L,\quad\text{Re}(-l-1)>-L,$$

provided that discrete half-odd-integer powers, if any, in Eqs. (2.4) and (2.5) are smaller than l and $(-l-1)$. The $i\epsilon$ instruction in (2.22) is relevant only when there are cuts in the l plane along the real axis. The analog of (2.22) below the cut is also valid. Equation (2.22) for the case of no cuts is the symmetry relation proposed by Mandelstam.[3] He showed that when this relation is true, the high-energy behavior of the total scattering amplitude is dominated by Regge-pole terms even when the background integral is shifted to Rel $=-L$ (assuming meromorphy in the l plane down to Re$l=-L$). Since Regge-pole dominance at high energies leads to discrete powers in the asymptotic behavior of $A_s(t,s')$, $A_u(t,u')$ for high s', u', and hence satisfies our assumptions (2.4)–(2.8) (which postulate discrete and continuous powers in the asymptotic behavior), our result will not be considered surprising. It is of interest, however, to compare the assumptions about the potential in Mandelstam's derivation of (2.22) for potential scattering with the assumptions in the present derivation. The detailed discussion in the following section of

[8] Reference 5, p. 140 (9).

the case of discrete powers passing through half-odd integers reveals further analogies with potential scattering results.

III. REGGE POLES PASSING THROUGH HALF-ODD INTEGERS AND NEGATIVE INTEGERS

We have seen that possible poles in the partial-wave amplitude arise from discrete powers in the asymptotic behavior of the Mandelstam weight functions. The partial-wave amplitude $a^{\pm}(l,t)$ can have a pole at $l=\alpha(t)$ if there are powers $\alpha_s^{(i)}(t)$, $\alpha_u^{(i)}(t)$ in Eqs. (2.4) and (2.5) such that $\alpha(t)=\alpha^{(i)}(t)-2r^{(i)}$, where $r^{(i)}$ is equal to zero or a positive integer. The residue at such a pole is given by ($\mathrm{Re}\alpha>-L$),

$$\beta^{\pm}(\alpha(t),t)=\lim_{l\to\alpha(t)}[l-\alpha(t)]a^{\pm}(l,t)=\frac{1}{\pi}[\sum_i c_s^{(i)}(t)I(r^{(i)},\alpha_s^{(i)}(t)-2r^{(i)})\pm\sum_i c_u^{(i)}(t)I(r^{(i)},\alpha_u^{(i)}(t)-2r^{(i)})], \quad (3.1)$$

where the first sum runs over those $\alpha_s^{(i)}(t)$, $r^{(i)}$ for which $\alpha_s^{(i)}(t)-2r^{(i)}=\alpha(t)$, and the second sum runs over those $\alpha_u^{(i)}(t)$, $r^{(i)}$ for which $\alpha_u^{(i)}(t)-2r^{(i)}=\alpha(t)$, and $I(r^{(i)},\alpha^{(i)}-2r^{(i)})$ defined as in (2.19) is given by

$$I(r^{(i)},\alpha^{(i)}-2r^{(i)})=\lim_{l\to(\alpha^{(i)}-2r^{(i)})}(l-\alpha^{(i)}+2r^{(i)})I(l,\alpha^{(i)},x_0)$$
$$=(\sqrt{\pi})2^{-\alpha^{(i)}+2r^{(i)}-1}\frac{\Gamma(\alpha^{(i)}-2r^{(i)}+1)\Gamma(1+\frac{1}{2}\alpha^{(i)})\Gamma(\frac{1}{2}+\frac{1}{2}\alpha^{(i)})}{\Gamma(1+\frac{1}{2}\alpha^{(i)}-r^{(i)})\Gamma(\frac{1}{2}+\frac{1}{2}\alpha^{(i)}-r^{(i)})\Gamma(\alpha^{(i)}+\frac{3}{2}-r^{(i)})\Gamma(r^{(i)}+1)}. \quad (3.2)$$

We shall first study (3.2) at a fixed half-odd-integral value of $\alpha^{(i)}(t)$. Equation (3.2) shows that if $\alpha^{(i)}\leq-(\frac{3}{2})$, $I(r^{(i)},\alpha^{(i)}-2r^{(i)})$ vanishes for all $r^{(i)}$ because $r^{(i)}$ is constrained to be zero or a positive integer and then the Γ function in the denominator with argument $(\alpha^{(i)}+\frac{3}{2}-r^{(i)})$ becomes infinite. If $\alpha^{(i)}\geq-(\frac{1}{2})$, $I(r^{(i)},\alpha^{(i)}-2r^{(i)})$ is nonzero and finite for $r^{(i)}\leq\alpha^{(i)}+\frac{1}{2}$, i.e., for $\alpha^{(i)}-2r^{(i)}=\alpha^{(i)},\alpha^{(i)}-2,\cdots,-\alpha^{(i)}-1$. We notice that whenever $I(l,\alpha^{(i)},x_0)$ has a pole with nonvanishing residue at a half-odd integer $l=\alpha^{(i)}-2r^{(i)}$, it also has a pole with nonvanishing residue at the half-odd integer $l=-(\alpha^{(i)}-2r^{(i)})-1$, obtained by reflection about $l=-\frac{1}{2}$. Further it is a simple matter to prove, using (3.2) and elementary relations between Γ functions, that the residues at these poles are in fact equal. Now, any half-odd-integer power $\alpha_s^{(i)}(t)$ or $\alpha_u^{(i)}(t)$ contributes at most one term to the Regge residue at $l=\alpha(t)$ given by (3.1). The result obtained above implies that every power that makes a nonzero contribution to $\beta^{\pm}(\alpha(t),t)$ makes an equal contribution to $\beta^{\pm}\times(-\alpha(t)-1,t)$. Thus, whenever $a^{\pm}(l,t)$ has a pole in the l plane with nonvanishing residue at a half-odd integer, it also has a pole with equal residue at the half-odd integer obtained by reflection about $l=-\frac{1}{2}$. As noted by Mandelstam,[3] this fact ensures the finiteness of the contributions of these Regge poles to the Mandelstam-Sommerfeld-Watson formula for the total amplitude.

Consider next the case where the residue of the pole at the half-odd integer $l=\alpha-2r$ in $I(l,\alpha,x_0)$ vanishes, i.e., $\alpha+\frac{3}{2}-r\leq0$. Such a value of l is necessarily a negative half-odd integer. A procedure similar to that used in deriving (2.14) now shows that there is an extra term on the right-hand side of (2.13), because, of the first few terms in (2.12) which have an exploding Γ function in the denominator, the term in which the summation index in (2.12) equals the value r above also has a vanishing denominator. Thus the relation (2.14) is not satisfied at a negative half-odd-integer value of l at a value of t where a pole of $I(l,\alpha,x_0)$ passes through l with vanishing residue. Hence, in general, the partial-wave amplitude will not satisfy the symmetry relation (2.22) at negative half-odd-integral values of l at values of t where there are Regge poles passing through l with vanishing residue. This result and the result of the preceding paragraph are in agreement with the potential scattering results of Mandelstam.[3]

We note now some further simple consequences of the residue formula (3.1). At a certain value of t, let $\alpha(t)$ be a discrete power occurring in Eqs. (2.4) and (2.5) with nonvanishing coefficients $c_s(t)$ and $c_u(t)$, respectively, such that $\mathrm{Re}\alpha(t)$ is greater than the real part of other discrete powers, if any, which are separated by even integers from it. Then $a^{\pm}(l,t)$ has a pole at $l=\alpha(t)$ with residue given by

$$\beta^{\pm}(\alpha(t),t)=\frac{1}{\sqrt{\pi}}2^{-\alpha(t)-1}\frac{\Gamma(\alpha(t)+1)}{\Gamma(\alpha(t)+\frac{3}{2})}[c_s(t)\pm c_u(t)]. \quad (3.3)$$

A pure signatured pole results if $c_s(t)=\pm c_u(t)$. At a value of t where $\alpha(t)$ passes through a negative half-odd integer $\leq-\frac{3}{2}$, with no half-odd-integer powers larger than $\alpha(t)$ in (2.4) and (2.5), the residue vanishes since $c_s(t)$ and $c_u(t)$ are in general finite at such a point. Further, if $c_s(t)$ and $c_u(t)$ are nonzero at a value of t where the trajectory $\alpha(t)$ goes through a negative integer, the residue will have a pole as a function of t at the value in question.

IV. THRESHOLD REPRESENTATION OF THE CONTINUED PARTIAL-WAVE AMPLITUDE

The continuation provided by (2.9) and (2.11) is particularly convenient for a threshold representation of the partial-wave amplitude because, when t is close to t_0, the threshold value, $(z_1)_0$ and $(-z_2)_0$ are very large and the series (2.11) converges rapidly for large x_0. We shall assume that the threshold behaviors of the coefficients $c_s^{(i)}(t)$ and $c_u^{(i)}(t)$ in (2.4) and (2.5) are $[q_{13}(t)q_{24}(t)]^{\alpha_s^{(i)}(t)+2r_s^{(i)}}$ and $[q_{13}(t)q_{24}(t)]^{\alpha_u^{(i)}(t)+2r_u^{(i)}}$, respectively, where $r_s^{(i)}$ and $r_u^{(i)}$ are positive integers or zero. This corresponds to the usual assumption about the threshold behavior of the Regge residue. We shall also assume that the quantities $g_s(t,\alpha')$ and $g_u(t,\alpha')$ have the threshold behavior $[(q_{13}(t)q_{24}(t)]^{\alpha'}$. Substituting (2.11) and the hypergeometric series for the Legendre function in Ref. 5 into Eq. (2.9), we obtain

$$a^{\pm}(l,t)=\sum_{r=0}^{\infty}\frac{1}{\pi}I(r,l)[2q_{13}(t)q_{24}(t)]^{l+2r}\Bigg[d_s(l,r,t)\pm d_u(l,r,t)+\sum_i\Bigg\{\bar{c}_s^{(i)}(t)[2q_{13}(t)q_{24}(t)]^{2r_s^{(i)}}\frac{[s_0+f_1(t)]^{\alpha_s^{(i)}(t)-l-2r}}{l-\alpha_s^{(i)}(t)+2r}$$
$$\pm\bar{c}_u^{(i)}(t)[2q_{13}(t)q_{24}(t)]^{2r_u^{(i)}}\frac{[u_0+f_2(t)]^{\alpha_u^{(i)}(t)-l-2r}}{l-\alpha_u^{(i)}(t)+2r}\Bigg\}+h_s(l,r,t)\pm h_u(l,r,t)\Bigg],\quad(4.1)$$

where we have denoted

$$c_s^{(i)}(t)=\bar{c}_s^{(i)}(t)[2q_{13}(t)q_{24}(t)]^{\alpha_s^{(i)}(t)+2r_s^{(i)}},\quad(4.2)$$

$$c_u^{(i)}(t)=\bar{c}_u^{(i)}(t)[2q_{13}(t)q_{24}(t)]^{\alpha_u^{(i)}(t)+2r_u^{(i)}},\quad(4.3)$$

$$g_s(t,\alpha')=\bar{g}_s(t,\alpha')[2q_{13}(t)q_{24}(t)]^{\alpha'},\quad(4.4)$$

$$g_u(t,\alpha')=\bar{g}_u(t,\alpha')[2q_{13}(t)q_{24}(t)]^{\alpha'},\quad(4.5)$$

$$f_1(t)=\tfrac{1}{2}t-\tfrac{1}{2}(m_1^2+m_2^2+m_3^2+m_4^2)+(m_1^2-m_3^2)(m_2^2-m_4^2)/2t,\quad(4.6)$$

$$f_2(t)=\tfrac{1}{2}t-\tfrac{1}{2}(m_1^2+m_2^2+m_3^2+m_4^2)-(m_1^2-m_3^2)(m_2^2-m_4^2)/2t,\quad(4.7)$$

$$d_s(l,r,t)=\int_{s_0}^{\infty}ds'\,A_s^0(t,s')[s'+f_1(t)]^{-(l+1+2r)},\quad(4.8)$$

$$d_u(l,r,t)=\int_{u_0}^{\infty}du'\,A_u^0(t,u')[u'+f_2(t)]^{-(l+1+2r)},\quad(4.9)$$

$$h_s(l,r,t)=\int_{\alpha_{s1}(t)}^{\alpha_{s2}(t)}d\alpha'\,\bar{g}_s(t,\alpha')\frac{[s_0+f_1(t)]^{\alpha'-l-2r}}{(l-\alpha'+2r)},\quad(4.10)$$

and

$$h_u(l,r,t)=\int_{\alpha_{u1}(t)}^{\alpha_{u2}(t)}d\alpha'\,\bar{g}_u(t,\alpha')\frac{[u_0+f_2(t)]^{\alpha'-l-2r}}{(l-\alpha'+2r)}.\quad(4.11)$$

It is seen that the dominant threshold behavior corresponding to each type of term in (4.1) is $[q_{13}(t)q_{24}(t)]^l$. This formula should be contrasted with the form guessed by Gribov and Pomeranchuk.[4] Apart from the fact that our formula exhibits the contribution of cuts in the l plane, there is the important difference that in their formula the Regge-pole terms have the threshold behavior $[q_{13}(t)q_{24}(t)]^{\alpha(t)}$. This difference has interesting consequences for the discussion of the unitarity condition continued below Re$l=-\frac{1}{2}$, and will be considered in detail elsewhere. For the present we mention that in the presence of cuts in the l plane the threshold behavior of $a^{\pm}(l,t)$ will be $[q_{13}(t)q_{24}(t)]^l$ at least on one side of the cut, because the discontinuity across the cut has this threshold behavior.

One drawback of the continuation provided by (2.9) and (2.11) is that they are not suited for a direct study of the behavior of $a^{\pm}(l,t)$ at $t=0$ in the unequal-mass case. This is so because power behaviors of $A_s(t,s')$, $A_u(t,u')$ at large s', u' are not conveniently expressed by the power behaviors in $z_1(t,s')$ and $-z_2(t,u')$ for small values of t. Thus, the sum of finite numbers of discrete powers exhibited in (2.4) and (2.5) contain in the coefficients of powers of s' and u' lower than the powers of z_1 and $(-z_2)$ occurring in the sums, singularities at $t=0$ which are not present in $A_s(t,s')$, $A_u(t,u')$. One needs, instead of (2.4) and (2.5), rearranged series involving the powers of s' and u' directly. The result of using such a series is to find that the partial-wave amplitude has the behavior at $t=0$ found by Freedman and Wang[9] for all values of l to which such a continuation of the Froissart-Gribov formula is possible. The procedure is similar to that in Ref. 9 and will not be repeated here.

Added note. An earlier version of this paper had an error on account of using formula (26) [*Bateman Manuscript Project*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2, p. 325], which is in error.

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[9] D. Z. Freedman and J. M. Wang, Phys. Rev. **153**, 1596 (1967).