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Equilibrium and Linear Response of a Classical Scalar Plasma

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The properties of a classical many-particle system interacting through a scalar field (a scalar plasma) are studied. A relativistic Hartree approximation is applied to equilibrium to evaluate thermodynamic characteristics such as the energy, entropy, and equation of state. A gas-liquid phase transition appears at moderate densities. There is no collapse for an attractive interaction: The system can accomodate any density. Stability against small perturbation and the behavior of collective excitations are analyzed through the linear-response function.

I. INTRODUCTION

A RELATIVISTIC many-particle system of identical particles, in an energy state sufficiently high to eliminate all bound states composed of a small number of particles, and interacting through a scalar field, will be termed in the present paper as a *scalar plasma.* Such a system can be regarded as a crude model for a neutron gas,[1] and it will reflect some properties of the nuclear matter.[2] The primary purpose of the present paper is, however, to explore the properties of such a system, relegating to Sec. V the question of applicability of the model to real physical systems.

The peculiarities of a scalar plasma, as contrasted with those of an electromagnetic plasma, stem (a) from the special behavior of the equation of motion[3,4] relative to a particle in a scalar field and of the scalar field equation under relativistic conditions; (b) from the short-range character of the interaction; and (c) from the fact that like particles are supposed to attract rather than repel each other.

In this paper the system will be treated classically; the essential features brought about by the characteristics (a), (b), and (c) are expected to be well described by a classical approach. In a subsequent paper we will provide a more complete treatment by analyzing the Green's function of the system, and we will also consider the consequence of the replacement of the scalar interaction by a pseudoscalar one.

The approximation to be used throughout this paper is based on representing the interparticle interaction by an average field. In equilibrium this amounts to a Hartree approximation, which in its nonrelativistic, or in an inconsistent relativistic form,[2,5] has extensively been used to determine properties of a neutron gas or of nuclear matter. In nonequilibrium situations the method is tantamount to using the Vlasov equation to evaluate a response function or transport properties.

In Sec. II we review the basic equation pertaining to the motion in a scalar field; we demonstrate an energy-conservation theorem and evaluate the potential energy which is highly dependent on velocity. In Sec. III we consider the equilibrium state and calculate the value of the constant average potential as a function of density and temperature, we derive expressions for the characteristic thermodynamic quantities, and we demonstrate the existence of a gas-liquid phase transition in the system. Section IV is devoted to the determination of the frequency and wavelength-dependent linear-response function, and the behavior of the collective excitations is analyzed; we concentrate in particular on the evaluation of stability criteria. Quite independently of its physical implications, the model has some interesting features from the purely statistical-mechanical viewpoint. Both the gas-liquid phase transition and the behavior of the system in the entire density range before and after the phase transition can be described in terms of a simple analytic interaction; the conditions leading to the phase transition can be studied both by following the equilibrium characteristics of the system and by analyzing the behavior of the linear-response function.

II. GENERAL RELATIONS

The equation of motion of a particle of free rest mass m, and of coupling constant g in a scalar field characterized by its invariant potential Φ is[4]

$$\dot{p}_\mu = -g\partial\Phi/\partial x_\mu,$$
$$p_\mu \equiv M\dot{x}_\mu, \qquad (1)$$
$$M \equiv m + g\Phi(x).$$

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In three-dimensional notation this takes the form[6]

$$dp_m/dt=f_m,$$
$$f_m=-g(1-v^2)^{1/2}\partial\Phi/\partial x_m, \qquad (2)$$
$$p_m\equiv[M/(1-v^2)^{1/2}]v_m.$$

The energy

$$E=M/(1-v^2)^{1/2}=(p^2+M^2)^{1/2} \qquad (3)$$

is a constant of the motion, provided that Φ is explicitly time-independent $[(\partial\Phi/\partial t)=0]$. E, moreover, has the property that $E(\mathbf{p},\mathbf{x})$ can be regarded as the three-dimensional Hamiltonian function generating the equations of motion (2) as canonical equations. It follows from (1) that M may become negative if $\Phi<0$ and $|g\Phi|>m$. It follows, on the other hand, from the constancy of E that this should have no bearing on the positiveness of E, even for negative mass states. Thus $E=|M|$ (and not $E=M$) should be understood when $p\rightarrow 0$.

$\Phi(x)$ is determined from the field equation

$$\Delta\Phi(x)=-4\pi\eta g\nu(x),$$
$$\Delta(x)=\partial^2/\partial x_\mu\partial x_\mu-\kappa^2. \qquad (4)$$

$\eta=\pm 1$ stands for repulsive field, $\eta=-1$ for attractive field. For a system composed of N particles

$$\nu(x)=\sum_i\int d\tau\,\delta(x-x^{(i)}[\tau])$$
$$=\sum_i(1-v^{(i)2})^{1/2}\delta(\mathbf{x}-\mathbf{x}^{(i)}[t]). \qquad (5)$$

Our main concern in this paper is the behavior of a system interacting through a physically realistic attractive field.[7] Nevertheless, some features of a system with repulsive interaction will be pointed out, mainly to facilitate comparison with electromagnetic plasmas.

We consider now the total energy W of a many-body system interacting through a scalar field. The energy-momentum tensor of the free field

$$T_{\mu\nu}=(1/4\pi)\tfrac{1}{2}\eta\{[\kappa^2\Phi^2+(\partial\Phi/\partial x_\alpha)(\partial\Phi/\partial x_\alpha)]\delta_{\mu\nu}$$
$$-2(\partial\Phi/\partial x_\mu)(\partial\Phi/\partial x_\nu)\} \qquad (6)$$

satisfies the conservation equation[4]

$$-\int_1^2 dx(dT_{\mu\nu}/dx_\mu)=P_\nu(2)-P_\nu(1). \qquad (7)$$

The energy conservation is related to the 44-component of $T_{\mu\nu}$:

$$T_{44}=(1/4\pi)\tfrac{1}{2}\eta\{K^2+L^2+\kappa^2\Phi^2\},$$
$$K_m\equiv-\partial\Phi/\partial x_m,$$
$$L\equiv\partial\Phi/\partial t, \qquad (8)$$

and can be expressed as

$$(1/4\pi)\tfrac{1}{2}\eta\int d\mathbf{x}(K^2+L^2+\kappa^2\Phi^2)$$
$$-\sum_i\frac{m+g\Phi(x^{(i)})}{(1-v^{(i)2})^{1/2}}=\text{const.} \qquad (9)$$

The first term in this expression can be transformed with the aid of (4) and (5) into a potential and a radiation energy:

$$(1/4\pi)\tfrac{1}{2}\eta\int d\mathbf{x}(K^2+L^2+\kappa^2\Phi^2)$$
$$=\tfrac{1}{2}\int d\mathbf{x}\nu\Phi+(1/4\pi)\tfrac{1}{2}\eta\int d\mathbf{x}\{L^2-(\partial L/\partial t)^2\}$$
$$=\tfrac{1}{2}\sum_i V^{(i)}-\eta W_{\text{rad}},$$
$$V^{(i)}=\Phi(x^{(i)})(1-v^{(i)2})^{1/2},$$
$$W_{\text{rad}}=(8\pi)^{-1}\int d\mathbf{x}\{L^2-(\partial L/\partial t)^2\}. \qquad (10)$$

Thus the conserved quantity that can be associated with the energy of the system is

$$W=W_{\text{part}}+W_{\text{rad}},$$
$$W_{\text{part}}=\sum_i(E^{(i)}-\tfrac{1}{2}V^{(i)}). \qquad (11)$$

A more symmetric expression for W_{part} can be derived:

$$W_{\text{part}}=\sum_i\frac{m}{(1-v^{(i)2})^{1/2}}0-\tfrac{1}{2}g^2$$
$$\times\sum_{ij}\frac{1-v^{(i)2}v^{(j)2}}{(1-v^{(i)2})^{1/2}(1-v^{(j)2})^{1/2}}\phi_{ij}, \qquad (12)$$

with ϕ_{ij} representing the velocity-independent part of the potential

$$\Delta_j\phi_{ij}=\delta(\mathbf{x}^{(j)}-\mathbf{x}^{(i)}[t]). \qquad (13)$$

A three-dimensional Hamiltonian for the total system can be constructed on the basis of (9) and (2):

$$H=\sum_i\{p^{(i)2}+[m+g\sum_{\mathbf{k}}\Phi_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{x}^{(i)})]^2\}^{1/2}$$
$$+(\eta/8\pi)\sum_{\mathbf{k}}\{(\kappa^2+k^2)\Phi_{\mathbf{k}}\Phi_{-\mathbf{k}}+\Psi_{\mathbf{k}}\Psi_{-\mathbf{k}}\}, \qquad (14)$$

where $\Phi_{\mathbf{k}}$ and $\Psi_{-\mathbf{k}}$ are the field coordinates and their conjugate momenta.

[6] The notation employed in this paper is: Greek subscripts run from 1 to 4, Latin ones from 1 to 3; no metric is introduced, $x_4=ict$, $c=1$ (except where it is useful to include c explicitly).

[7] The repulsive scalar field can actually be ruled out on the basis that the field energy is not positive definite.

The Hartree–Vlasov approximation amounts to the neglect of correlation effects and to the characterization of the system in terms of one-particle distribution functions. Such a one-particle distribution function $F(\mathbf{xp};\, t)$ obeys the Vlasov equation, which in view of the fact that a three-dimensional description of the system can be formulated in terms of Hamiltonian dynamics, assumes its customary form

$$\partial F/\partial t+v_m\partial F/\partial x_m+f_m\partial F/\partial p_m=0,$$
$$v_m=P_m/E,$$
$$f_m=-g(M/E)\partial\Phi/\partial x_m. \tag{15}$$

Φ in (15) represents the average potential calculated from

$$\Phi(\mathbf{x},t)=-4\pi\eta g\int d\mathbf{p}\,d\mathbf{x}'\,dt'\,\Delta^{-1}(\mathbf{x}t;\mathbf{x}'t')$$
$$\times\frac{m+\Phi(\mathbf{x}'t')}{\{p^2+[m+\Phi(\mathbf{x}'t')]^2\}^{1/2}}F(\mathbf{x}'\mathbf{p};t'). \tag{16}$$

III. EQUILIBRIUM

The equilibrium properties of the system in the Hartree–Vlasov approximation will be considered in this section. In virtue of the approximation stated, the canonical distribution function is decomposed into one-particle equilibrium distributions over single-particle energies

$$F(\mathbf{p})=(4\pi/V)z^{-1}(\beta)\exp[-\beta E(\mathbf{p})],$$
$$E(\mathbf{p})=(p^2+M^2)^{1/2},$$
$$M=m+g\Phi, \tag{17}$$

where Φ is the average potential reigning over the system, and M is the effective rest mass.

$z(\beta)$ is determined from the normalization condition[8]

$$z(\beta)=(4\pi)^{-1}\int d\mathbf{p}e^{-\beta E},$$
$$=\int_M^\infty dEE(E^2-M^2)^{1/2}e^{-\beta E},$$
$$=|M|^3\int_1^\infty d\epsilon\epsilon(\epsilon^2-1)^{1/2}e^{-a\epsilon},$$
$$=(a^2/\beta^3)\mathrm{K}_2(a). \tag{18}$$

We summarize the symbols used here or to be used in the sequel:

$$a=\beta|M|,\qquad \mu=|M|/M,$$
$$b=\beta m.$$

$\mathrm{K}_n(a)$ is the modified Bessel function of order n;

$$G_n(a)=\int_1^\infty d\epsilon\epsilon^n(\epsilon^2-1)^{1/2}e^{-a\epsilon}\Big/\int_1^\infty d\epsilon\epsilon(\epsilon^2-1)^{1/2}e^{-a\epsilon},$$
$$G_0(a)=\mathrm{K}_1(a)/\mathrm{K}_2(a). \tag{19}$$

The distribution function depends implicitly on the value of the potential Φ, which, in turn, is determined from the equilibrium variant of (16):

$$\Phi=\eta\frac{(4\pi)^2g^2\eta}{\kappa^2}z^{-1}(\beta)\int d\mathbf{p}\frac{M}{E}F(\mathbf{p}), \tag{20}$$

n being the density of the system, which can be related to the plasma frequency

$$\omega_0^2=4\pi g^2n/m \tag{21}$$

and to the dimensionless density

$$s=\omega_0^2/\kappa^2. \tag{22}$$

Equation (20) implies that Φ or M as a function of a given density (s) and temperature (b) can be inferred from the implicit relation

$$s=-\eta(\mu-a/b)G_0^{-1}(a), \tag{23}$$

which allows for the possibility of a negative effective mass. We now comment on this result.

The character of the solution of (23) depends on the signature of η and μ. A glance at (23) shows, however, that no solution with $\eta=-1$ $\mu=-1$ exists. The remaining three solutions can be characterized as follows (see Fig. 1).

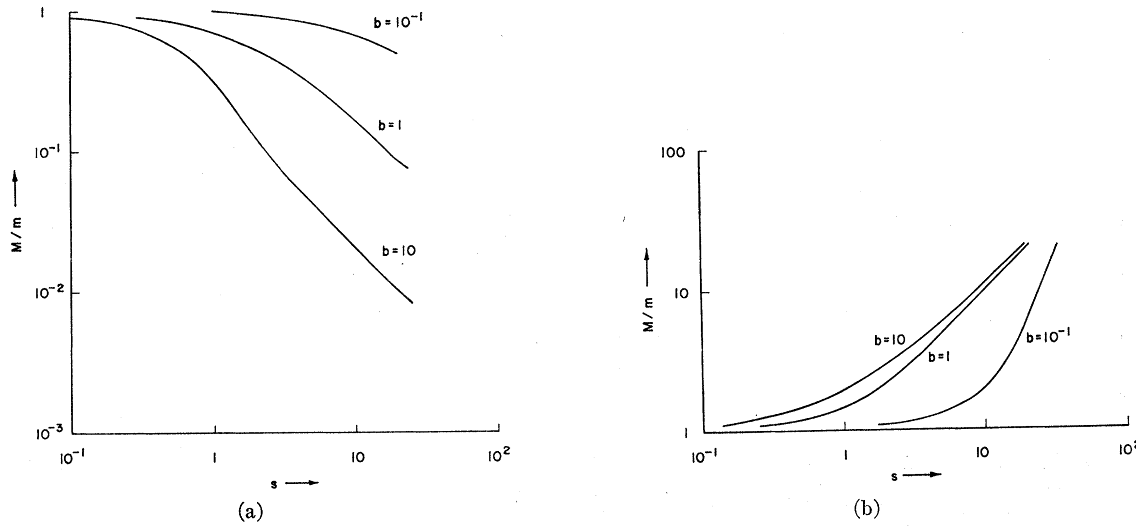
(i) For $\eta=-1$ (attractive interaction), as was just pointed out, M is always positive. This is in contradiction to the naive picture based on a single-particle model[1] which suggests that M becomes negative if $(|g\Phi/m|)\approx(g^2n/mk^2)\approx s>1$. What actually happens is that as the potential well deepens, particles acquire higher velocities; and this, through the $(1-v^2)^{1/2}$ factor in the field equation, softens the interaction in such a way that the $M=0$ limit is never attained. Nevertheless, $M<m$, $(a<b)$ always, and M is a monotonically decreasing function of the density, as expected. The temperature dependence of the effective mass is different in the small density ($s<1$) and in the large density ($s>1$) regions. If $s<1$, then for $b\to\infty$ also $a\to\infty$ and $M\to m(1-s)$; while if $s>1$, then $a\to a_m$, which is the solution of

$$s=G_0^{-1}(a_m), \tag{24}$$

and $M\to 0$. In the very high temperature limit ($b\to 0$) $M\to m$, independently of s, which again reflects the softening of the interaction.

(ii) In the event of a repulsive interaction ($\eta=+1$), a normal ($\mu=+1$) and anomalous ($\mu=-1$) solution can be distinguished. There is no surprising element in the normal solution; it is characterized by $M>m$ ($a>b$), $M\to\infty$ as $s\to\infty$ and again by the weakening of the interaction as $b\to 0$.

[8] J. L. Synge, *The Relativistic Gas* (North-Holland Publishing Company, Amsterdam, 1957).



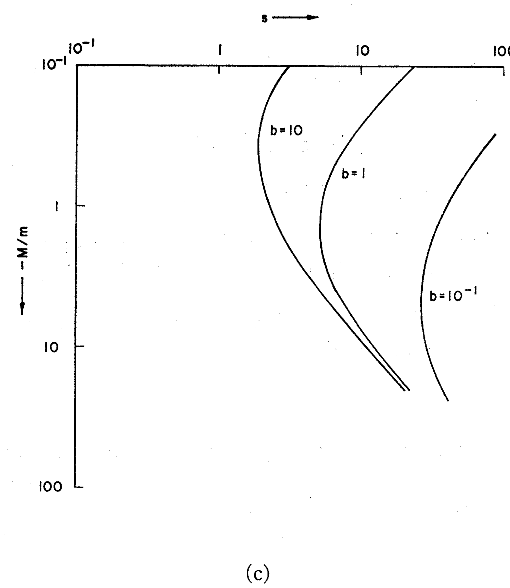


FIG. 1. The ratio of the effective rest mass $M=m+\phi$ to the noninteracting rest mass m versus the dimensionless density $s=4\pi g^2 n/\kappa^2 c^2 m$. The parameter is the dimensionless inverse temperature $b=\beta m$. (a) Attractive interaction. Note that M is finite for any density. (b) Repulsive interaction, normal solution. (c) Repulsive interaction, anomalous solution. M is negative, double valued if the density exceeds a certain limit.

(iii) The anomalous solution is confined to a region $s>s_0>1$. The elucidation of the somewhat bizarre result that the repulsive potential can lead to negative-mass state can be sought in the fact that a change of sign in μ actually entails a change of sign of the interaction (because of the M/E factor); thus in a $\mu=-1$, $\eta=+1$ state the potential is negative. Moreover, if the density is sufficiently high, such a negative potential can sustain in a self-consistent fashion the negative effective mass.

In the remainder of this section we consider case (i) only. The thermodynamic quantities of interest which we are going to evaluate are the internal energy U, the entropy S, the free energy A, and the pressure P.

The total internal energy (per particle) is the sum of the averages of the kinetic and potential energy, as defined in (11):

$$U=\langle E\rangle-\tfrac{1}{2}\langle V\rangle. \tag{25}$$

The former is calculated to yield

$$\langle E\rangle=(1/\beta)\{3+aG_0(a)\}. \tag{26}$$

This is formally identical to the energy expression for the noninteracting relativistic gas[5]—but it should be kept in mind that $a=a(s,b)$ is obtained from (23). For the latter one finds

$$\begin{aligned}\langle V\rangle&=+(1/\beta)(a-b)G_0(a)\\&=-msG_0^2(a).\end{aligned} \tag{27}$$

and thus

$$U=(1/\beta)\{3+\tfrac{1}{2}(a+b)G_0(a)\}. \tag{28}$$

This is depicted in Fig. 2.

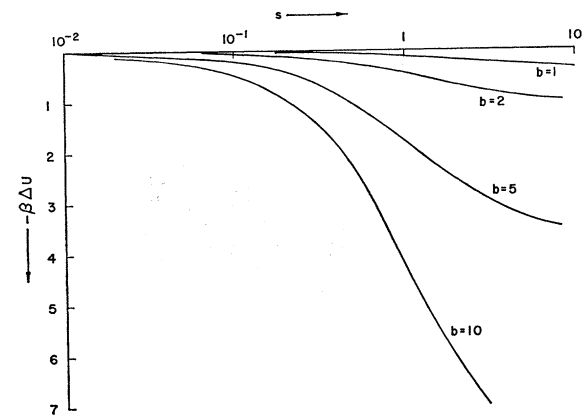


FIG. 2 $\beta\Delta U$, the difference between the energy per particle of the system and the energy per particle of a noninteracting gas with the same temperature and density in units of the temperature versus the dimensionless density $s=4\pi g^2 n/\kappa^2 c^2 m$. The parameter is the dimensionless inverse temperature $b=\beta m$. The curves approach the asymptotic value $-bG(b)$, while the particles in the noninteracting gas have energy $\beta U_0=3+bG(b)$.

The entropy (per particle) of the system can be expressed as

$$S=\beta\langle E\rangle+\ln z(\beta,n)/n+\text{const}$$
$$=aG_0(a)+\ln a^2 K_2(a)/n\beta^3+\text{const}. \tag{29}$$

The characteristics of an adiabatic process may be calculated by requiring the entropy function to be constant.

The free energy (per particle) is constructed from the thermodynamic identity

$$A=U-\beta^{-1}S. \tag{30}$$

Introducing the previously derived results, this yields

$$A=\tfrac{1}{2}msG_0^2(a)-(1/\beta)\ \ln a^2K_2(a)+(1/\beta)\,(\ln s\beta^3+\text{const}). \tag{31}$$

The expression obtained for the free energy leads to an equation of state; its derivation is accomplished through calculating the pressure from

$$P=n^2\partial A/\partial n\,|_\beta,$$
$$=P_0+\Delta P,$$
$$P_0=n/\beta. \tag{32}$$

The relative deviation from the perfect gas pressure $\Delta P/P_0$ is obtained by using (31), (32), and (23):

$$\frac{\Delta P}{P_0}=\tfrac{1}{2}bs\left\{G_0^2(a)+\frac{\partial a}{\partial s}\left[2sG_0(a)G_0'(a)-\frac{2}{b}\frac{d}{da}\ln a^2K_2(a)\right]\right\}, \tag{33}$$

where $\partial a/\partial s$ is to be derived from (23). A somewhat lengthy algebra leads to the expression

$$\Delta P/P_0=-\tfrac{1}{2}(bs)G_0^2(a), \tag{34a}$$

or recalling (27)

$$P=n(1/\beta+\tfrac{1}{2}\langle V\rangle). \tag{34b}$$

Some limiting cases of (34a) are worth mentioning.

(i) *The small temperature* ($b\to\infty$) limit is evaluated, using the asymptotic expansion of the Bessel functions and recalling (24):

$$\Delta P/P_0\to-\tfrac{1}{2}(bs),\qquad s<1,$$
$$\to-b/2s,\qquad s>1. \tag{35}$$

The $s<1$ case is just the nonrelativistic Hartree correction. For $s>1$, $M\to 0$ and the pressure returns to its ideal gas value.

(ii) *In the small density limit* ($s\to 0$), the Hartree result is reobtained with a temperature-dependent correction factor

$$\Delta P/P_0\to-\tfrac{1}{2}(sb)G_0^2(b). \tag{36}$$

(iii) *The high density limit* ($a\to 0$) finally leads again to an expression similar to (35):

$$\Delta P/P_0\to-b/2s. \tag{37}$$

Figure 3 shows the isotherms of the system. The diagram indicates the existence of a gas-liquid phase transition. The physical mechanism entailing such a transition can be understood on referring to (35)–(37): As the density is increased, the attractive Hartree field becomes more and more dominant, approaching an unstable ($\partial P/\partial s<0$) region; for much higher densities, however, the vanishing rest mass results in a "bound" perfect gas behavior again, which is stable. While the first feature—the instability of the system at a certain maximum density—is common to any attractive inter-

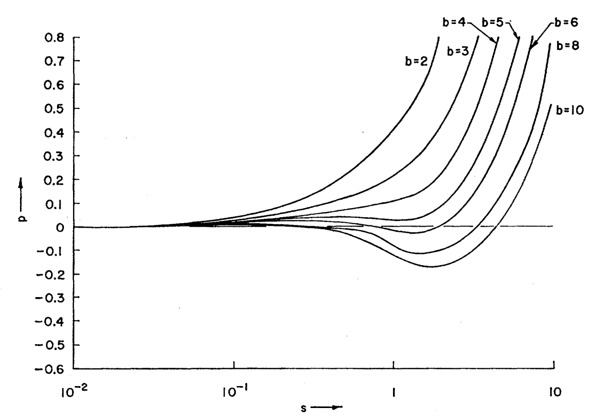


FIG. 3. Isotherms in terms of the dimensionless pressure $p=(P/4\pi)g^2/\kappa^2c^2$ and the dimensionless density $s=4\pi g^2n/\kappa^2c^2m$. The parameter is the dimensionless inverse temperature $b=\beta m$. Note the van der Waals-type behavior for low temperatures in the curves representing $b>3$.

action, the existence of a second stable phase at higher densities is a peculiar feature of the scalar field.

The boundaries of the equilibrium regions in the temperature-density plane (defined by the metastable points where $\partial P/\partial s=0$) are determined by the relation

$$\tfrac{1}{2}[3ba(b,s)-sb]=1 \qquad (38)$$

in conjunction with (23). The critical temperature (b_{c0}^{-1}), the critical density (s_{c0}), and the maximum density (s_{m0}) (beyond which no phase transition occurs) can be derived, with some algebra, from (38): (see also Fig. 4)

$$b_{c0}^{-1}=0.210,$$
$$s_{c0}=0.760,$$
$$s_{m0}=2.252. \qquad (39)$$

The zero subscripts refer to the equilibrium calculation from which these quantities originate; similar critical values will be derived from the evaluation of the response function.

In addition to the equation of state, a quantity of interest that can be derived from equilibrium considerations is the "velocity of sound." The quotation marks refer to the fact that within the Hartree approximation where collisions are inherently ignored, no sound velocity can be defined in a consistent way. In order, however, to compare the behavior of the present model with results pertaining to other model calculations,[9] we consider the sound velocity

$$u=dP/d(nU)|_S. \qquad (40)$$

The reasoning behind this approach is that collisions are sufficiently frequent to maintain local thermodynamic equilibrium, but they affect negligibly the Hartree equation of state. We quote only the interesting limiting results.

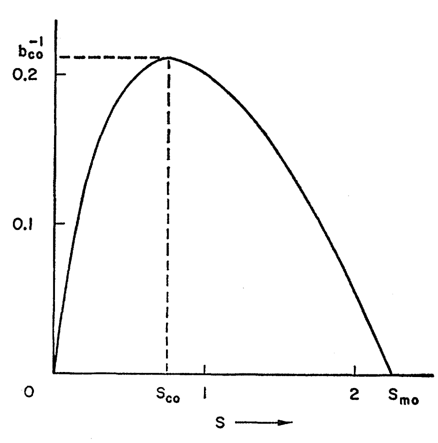


FIG. 4. The critical curve in the dimensionless density $s=4\pi g^2 n/\kappa^2 c^2 m$ and dimensionless temperature $b^{-1}=(\beta m)^{-1}$ plane, as calculated from equilibrium considerations.

[9] Ya. B. Zel'dovich, Zh. Eksperim. i Teor. Fiz. **41**, 1609 (1961) [English transl.: Soviet Phys.—JETP **14**, 1143 (1962)].

(i) In the small temperature ($b\to\infty$), small density ($s\ll 1$) limit the customary perfect gas behavior is recovered:

$$u^2\to(5/3b)c^2 \qquad (41)$$

(ii) In the small temperature, high density limit ($b\to\infty$, $s\gg 1$) one obtains

$$u^2\to\tfrac{1}{3}c^2, \qquad (42)$$

which is a result characteristic to a gas of massless particles.

(iii) Essentially the same result

$$u^2\to\tfrac{1}{3}c^2 \qquad (43)$$

emerges for the high temperature ($b\to 0$) region. These results are not surprising in view of the already discussed perfect gas behavior in the high density and high temperature limits. Since there is no repulsive field here, there is no anomalous behavior, ($u\to c$) like the one demonstrated in Zel'dovich's[9] model.

IV. LINEAR RESPONSE

The effect of a small perturbation on the system is characterized by its linear-response function. This frequency- and wavenumber-dependent response function (its analog in electromagnetic theory is the dielectric tensor) will now be calculated within the framework of the Vlasov-theory, and the behavior of the collective excitations of the system will be inferred from the zeros of the response function on the complex frequency plane.

In order to contemplate the effect of a linear perturbation, we split all the relevant quantities into equilibrium and perturbed parts, such as

$$F=F^0+F',$$
$$f_m=f^0{}_m+\bar{f}'{}_m,$$
$$\bar{f}'{}_m=f'{}_m+\hat{f}'{}_m,$$

where f'_m is the contribution originating from the system itself, while $\hat{f}'_m$ is due to external sources. Thus the linearized Vlasov equation (15) assumes the form

$$\partial F'/\partial t+v_m{}^0\partial F'/\partial x_m+\bar{f}_m{}'\partial F^0/\partial p_m=0. \qquad (44)$$

The explicit expression for the coefficients is

$$v_m{}^0=p_m/E,$$
$$\bar{f}_m{}'=-(M/E)\partial\Phi'/\partial x_m, \qquad (45)$$

where E and M refer to the equilibrium values. Hence, after Fourier transformation,

$$-i(\omega-k_m p_m/E)F'(\mathbf{k}\omega,\mathbf{p})$$
$$-ik_m(M/E)\bar{\Phi}'(\mathbf{k}\omega)\partial F^0(\mathbf{p})/\partial p_m=0 \qquad (46)$$

is obtained.

Φ' is now determined from the linearized field equation (16):

$$\Delta(\mathbf{k}\omega)\Phi'(\mathbf{k}\omega) = -4\pi\eta g\left\{\int \frac{p^2}{E} F^0(\mathbf{p})\,d\mathbf{p}\bar{\Phi}'(\mathbf{k}\omega) + \int \frac{M}{E} F'(\mathbf{k}\omega;\mathbf{p})\,d\mathbf{p}\right\}. \quad (47)$$

The origin of the first term in the right-hand side of (49) is in the M/E factor in the integral, and will be shown to result in the renormalization of the free shielding constant κ^2. The second term may be expressed with the aid of which related F' to $\bar{\Phi}'$:

$$F' = [\omega - (\mathbf{k}\cdot\mathbf{p})/E]^{-1}(M/E)\mathbf{k}\cdot(\partial F/\partial\mathbf{p})\bar{\Phi}'. \quad (48)$$

The definition of the linear-response function is provided by the relation

$$\bar{\Phi}' = \epsilon^{-1}\hat{\Phi}'. \quad (49)$$

Substituting (48) into (47) and separating out terms containing Φ' and $\hat{\Phi}'$ one finds the following expression for $\epsilon(\mathbf{k}\omega)$:

$$\epsilon(\mathbf{k}\omega) = 1 + [\eta\omega_0^2 b/(\omega^2 - k^2\kappa^2)] \times\{(1/a)[G_0(a) - G_{-2}(a)] - G_{-1}(a) - Y(z,a)\}. \quad (50)$$

The notation we have employed is the following:

$$Y(z,a) = -z^2\frac{a}{K_2(a)}\int_1^\infty e^{-a\epsilon}[z^2 - (\epsilon^2-1)/\epsilon^2]^{-1} \times\frac{d\epsilon}{\epsilon^2(\epsilon^2-1)^{1/2}},$$

$$z = \omega/kc. \quad (51)$$

The contour of integration in the vicinity of the singularity is to be taken by adding a small positive imaginary part to z: $z\rightarrow z+i\delta$ as usual. Note that in the nonrelativistic limit $Y(z)\rightarrow(\frac{1}{2}b)^{1/2}zZ[(\frac{1}{2}b)^{1/2}z]$, where Z is the customary plasma dispersion function.[10]

The collective excitations of the system will now be analyzed with the aid of the dispersion relation

$$\epsilon(\mathbf{k}\omega) = 0. \quad (52)$$

This may conveniently be reformulated on noting that

$$Y(\infty, a) = -G_{-1}(a), \quad (53)$$

and by adopting a renormalized shielding coefficient $\bar{\kappa}^2$:

$$\bar{\kappa}^2(s,b) = \kappa^2\{1 - \eta s(b/a)[G_0(a) - G_{-2}(a)]\} \quad (54)$$

to receive

$$\omega^2 - k^2 - \bar{\kappa}^2 = \eta\omega_0^2 b\{Y(z,a) - Y(\infty,a)\}. \quad (55)$$

[10] B. D. Fried and S. D. Conte, *The Plasma Dispersion Function* (Academic Press Inc., New York, 1961).

The following two features of (54) are noteworthy:

(a) $\bar{\kappa}^2 < \kappa^2$ for repulsive interaction ($\eta = +1$)—since $G_0 - G_{-2} > 0$; in this event $\bar{\kappa}^2$ can eventually be zero or negative, leading to an instability of the system;

(b) for attractive interaction ($\eta = -1$) $\bar{\kappa}^2 > \kappa^2$, and even if $\kappa^2\rightarrow 0$, $\bar{\kappa}^2$ remains finite—$\bar{\kappa}^2\rightarrow\frac{1}{2}(\omega_0^2 b)$.

Now we are prepared to obtain a general view of the modes $\omega = \omega(\mathbf{k})$ represented by the dispersion relation (55). If $k=0$, then $z\rightarrow\infty$ and the excitations are "dressed mesons"

$$\omega^2 = \bar{\kappa}^2, \quad (56)$$

the dressing effect being described by (54).

To find the behavior of $\omega(\mathbf{k})$ for finite values of k, the asymptotic development of $Y(z)$ can be employed. In view of the defining Eq. (51),

$$Y(z,a) \sim Y(\infty,a) - (1/z^2)G_{-4}(a)/a. \quad (57)$$

This expansion is valid as long as $z \gg 1$ or, in case $z<1$, if the contribution of the integral in the neighborhood of the point $z = (\epsilon^2-1)/\epsilon^2$ is small (i.e., zc is much larger than the mean thermal velocity).

The a dependence of the second term in (57) can be incorporated in a renormalized plasma frequency:

$$\bar{\omega}_0^2 = \omega_0^2(b/a)G_{-4}(a), \quad (58)$$

which is again temperature- and density-dependent. It is characterized by $\bar{\omega}_0\rightarrow 0$ as $s\rightarrow\infty$. The dispersion relation

$$\omega^2 - \bar{\kappa}^2 - k^2 = \eta\bar{\omega}_0^2 bk^2/\omega^2 \quad (59)$$

is readily solved:

$$\omega^2 = \tfrac{1}{2}\{(\bar{\kappa}^2 + k^2) \pm [(\bar{\kappa}^2+k^2)^2 - 4\eta\bar{\omega}_0^2 k^2]^{1/2}\}. \quad (60)$$

The general behavior of this solution (for $\eta = -1$) is plotted in Fig. 5. The dispersion curve consists of two disjoint modes. The upper "meson" mode is the extension of the "dressed meson" solution (56) approach-

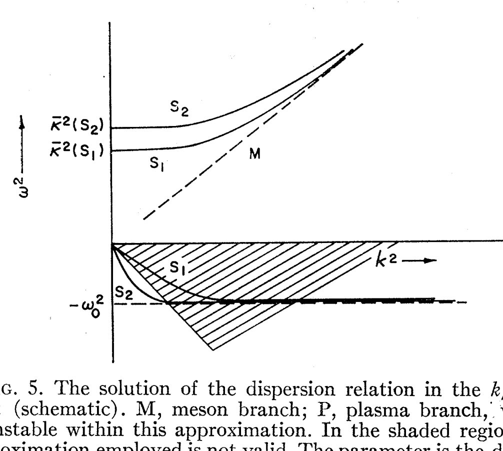


FIG. 5. The solution of the dispersion relation in the $k/\omega\rightarrow 0$ limit (schematic). M, meson branch; P, plasma branch, which is unstable within this approximation. In the shaded region the approximation employed is not valid. The parameter is the dimensionless density s, $s_2 > s_1$.

ing $\omega^2=\bar{\omega}_0^2+\bar{\kappa}^2+k^2$ as $k\rightarrow\infty$. The lower "plasma" mode represents a fully unstable excitation. The rate of instability (Imω) for $k\rightarrow 0$ is $(\bar{s}\bar{k})^{1/2}$ $(\bar{s}=\bar{\omega}_0^2/\bar{\kappa}^2)$ while it evolves toward the constant $\bar{\omega}_0$ value as $k\rightarrow\infty$.

The physical origin of the instability is obviously the mutual attraction between the constituents of the system (manifested by $-\omega_0^2$ replacing ω_0^2 in the dispersion relation). Should relation (60) be exact, the system would be completely unstable and no linear theory would apply. Its validity, however, hinges on the fulfillment of the relation $zc\gg$thermal velocity, or on the more stringent condition $z\gg 1$. If this condition is not satisfied, the expansion leading to (60) is not permissible, and, in particular, Landau-damping effects seriously affect the character of the solution. Thus, without going into the detailed investigation of the exact solution, which would necessitate the evaluation of the relativistic dispersion function $Y(z)$, it can be asserted on physical grounds that the instability will be suppressed completely for $\bar{s}<1$ and will show up in a limited region for $0<k<k_m$ for $\bar{s}>1$. $\bar{s}$ as a function of the density (s) and temperature (b) can be evaluated by virtue of the defining relations (54) and (57) and on referring to (23),

$$\bar{s}=sG_{-4}(a[s,b])/\{1-sG_{-2}(a[s,b])\}. \qquad (61)$$

Although the behavior of this function is rather difficult to visualize, some reflection shows that it is a monotonically increasing function of b, asymptotically approaching the value

$$\bar{s}\rightarrow s/(1-s) \qquad \text{for } s<1$$

$$\rightarrow\frac{sG_{-4}(a_m[s])}{1-sG_{-2}(a_m[s])} \qquad \text{for } s>1, \qquad (62)$$

Instead of discussing, however, the stability problem on the basis of the dynamical behavior of the excitations, it is more practical to analyze the static-response function which itself contains enough information—equivalent, essentially, to that to be inferred from the limiting conditions pertaining to the dynamics of the

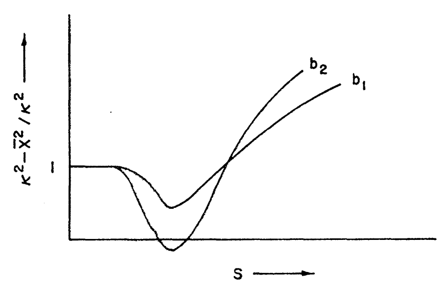


FIG. 6. The variation of the effective shielding range $\kappa^2-\bar{\chi}^2$ versus the dimensionless density (schematic). The parameter is the dimensionless inverse temperature b; $b_2<b_1$. The dip below 0 represents unstable behavior. Note that high density results in enhanced screening instead of leading to antiscreening.

unstable modes—to determine stability criteria.[11] The static-response function will be discussed below; before that, however, a word on the characteristic modes of a repulsive system might be in order. In that case both the "meson" and "plasma" modes are absolutely stable for $\bar{s}<1$. For $\bar{s}>1$ the two might merge, giving rise to a peculiar type of instability.

Turning now to the static-response function, it is evaluated from (50), by setting $\omega=0$. This zero-frequency limit determines the static effective potential around a particle at rest in the system.[12]

$$\epsilon(\mathbf{k}0)=1+\eta[\chi^2/(k^2+\kappa^2)]\{G_{-1}(a) -(1/a)[G_0(a)-G_{-2}(a)]\}$$
$$=1+\eta[\bar{\chi}^2/(k^2+\kappa^2)]. \qquad (63)$$

χ^2 is the Debye wave number

$$\chi^2=\omega_0^2 b, \qquad (64)$$

while $\bar{\chi}^2$ is its renormalized value

$$\bar{\chi}^2=\chi^2\{G_{-1}(a)-(1/a)[G_0(a)-G_{-2}(a)]\}. \qquad (65)$$

The static potential surrounding a perturbation of the invariant density $\nu(\mathbf{k})$ and corresponding to (63) is

$$\Phi(\mathbf{k},0)=4\pi\eta(k^2+\kappa^2+\eta\bar{\chi}^2)^{-1}\nu(\mathbf{k}). \qquad (66)$$

It is obvious that for $\eta=-1$ the Debye wave number represents an antishielding effect and is compensated for by the finite value of κ^2 only. There are two important features to be noted:

(a) $\bar{\chi}^2/\chi^2$ decreases with increasing density; if the temperature does not exceed a certain limit ($b>b_c$), it starts with its normal positive value and switches over to an anomalous negative value for $s>s_c$. If, however, $b<b_c$, then χ^2 is always negative and the effective potential is repulsive for any value of s (see Fig. 6). The physical origin of the strange behavior is that the Debye shielding (i.e., antishielding) effect stems from two competitive sources. The first is the customary plasma behavior, which is represented by the G_{-1} term in (65) and is always positive. The second effect, which is responsible for $\bar{\kappa}^2$, is the modification of the $(1-v^2)^{1/2}$ factor in the field equation because of the external perturbation, as manifested by the first term in the right-hand side of (47) and by the (G_0-G_{-2}) contribution in (65): It can be viewed as an accumulation of particles entailing a deepening of the potential well and hence an increase of particle velocities—which,

[11] N. P. Mermin, Ann. Phys. (N.Y.) **18**, 421 (1962).

[12] It would be tempting to go a step further and to establish the form of the static pair correlation function, evoking the fluctuation-dissipation theorem. Actually, a fluctuation-dissipation theorem can be derived without great difficulty; it, however relates $\epsilon(\mathbf{k},0)$ to $\nu_{\mathbf{k}}\nu_{-\mathbf{k}}$ where $\nu_{\mathbf{k}}=\sum_i(1-v^{(i)2})^{1/2}\exp(-i\mathbf{k}\cdot\mathbf{x}^{(i)})$ is the invariant density—thus it is not immediately amenable to predictions on the pair correlation function.

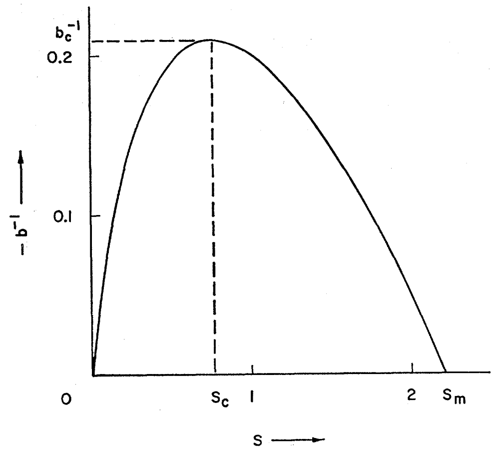


FIG. 7. The critical curve in the dimensionless density $s=4\pi g^2 n/\kappa^2 c^2 m$ and dimensionless temperature $b^{-1}=(\beta m)^{-1}$ plane, as calculated from the static linear-response function.

in turn, tends to decrease the total potential and thus leads to a genuine shielding.

(b) Whether the total shielding constant $\kappa^2-\bar{\chi}^2$ is ultimately increased, decreased, or eventually becomes negative is determined by the interplay of these factors. The critical limit $\kappa^2-\bar{\chi}^2=0$ is attained if

$$G_{-1}(a[s,b])-(1/a[s,b]) \times\{G_0(a[s,b]-G_{-2}(a[s,b])\}=1/sb. \quad (67)$$

The general behavior of the $b(s)$ curve separating stable and unstable regions (see Fig. 7) determined from (68) is such that it defines a critical temperature (b_c^{-1}) a critical density (s_c) and a maximum density (s_m).

$$b_c^{-1}=0.210,\quad s_c=0.760,\quad s_m=2.254. \quad (68)$$

The interpretation of the occurrence of $\kappa^2-\chi^2<0$ as either an instability or a phase transition follows from rather obvious considerations. In simple physical terms, if a static potential $\Phi(k)$, with real $k\neq 0$ can develop, this should be regarded as the heralding of the spontaneous breakup of the homogeneous system into chunks of finite size which, in turn, can be regarded as a process of phase transition, provided that the new density within the droplets can be such that stability ensues. Mathematically speaking, the existence of a real k for $\omega=0$ as the solution of the dispersion relation is the indication of a more general solution $\omega(k)=i\gamma(k)$, having the property that $\gamma(k)\to 0$ as $k\to 0$; this can be inferred from the analytic behavior[11] of $\epsilon(\mathbf{k}\omega)$ for $\omega\to 0$:

$$\lim_{k\to 0}\epsilon(\mathbf{k},i\gamma(k))=\lim_{k\to 0}\epsilon(\mathbf{k}0), \quad (69)$$

while

$$\lim_{k\to 0}\epsilon(\mathbf{k},\omega(k))\neq\lim_{k\to 0}\epsilon(\mathbf{k}0). \quad (70)$$

The two critical curves Fig. 4 and Fig. 6 are not identical.[13] This should not be surprising: they ought to coincide in an exact theory only. Although both the equilibrium Hartree approximation and the Vlasov approximation for the calculation of the linear-response function correspond to the same order of the perturbation theory (order g^2n), an equation of state, if it were based on the linear-response function, would contain a higher-order contribution also (order g^4n). Thus the equilibrium criteria and those based on the linear-response function pertain to two different levels of approximation. That the two results are not qualitatively different and order-of-magnitude-wise provide the same critical constants is rather comforting and seems to indicate that the refinement of the perturbation calculation would not modify too much the equilibrium results obtained.

V. CONCLUSIONS

How much relevance do the calculations presented in this paper have to the behavior of actual physical systems? Not very much, probably. The most serious question, of course, is whether a system of strongly interacting particles could be described in terms of such a simple dynamical picture as particles interacting through a field.[14] Even admitting that this can be done, the interaction is much more intricate than the simple scalar-field model adopted in this paper. Moreover, the model of the pure neutron gas is not very tenable, since competing creation and annihilation processes play a significant role.[14,15] But putting aside these difficulties, which are common to many calculations pertaining to the behavior of superdense matter,[1,2,3,5] there remains this question: To what extent can the neglect of quantum mechanics and of collisional effects be justified? Quantum mechanics is expected to intervene in two different ways: (a) when degeneracy is relevant, i.e., when $pn^{-1/3}<\hbar$, this leading to the condition $b^{-1}<(s/\gamma\mu^2)^{\lambda/3}$, with $1<\lambda<2$ and γ being the coupling constant ($\gamma=g^2/hc$), and μ the baryon-meson mass ratio ($\mu=mc/\hbar$); and (b) when pair-creation effects that modify the effective interaction become appreciable; this happens whenever $\beta^{-1}>mc^2$ or $\hbar\omega_0>mc^2$: the corresponding conditions are $b^{-1}>1$ and $s>\mu^2$. In order to fix ideas about orders of magnitude, let us adopt $\gamma=15$, $\mu=7$, and $\kappa^{-1}=2\times10^{-13}$ cm; then $s=1$ will correspond to 0, 5×10^{37} cm^{-3} and $s=\mu^2$ to 2, 5×10^{38} cm^{-3}. Degeneracy enters, of course, at much lower densities and it changes the picture significantly. Although the temperature dependence of the present

[13] The agreement, actually, is surprisingly good: for $b^{-1}>0.1$, the relative deviation between the b values calculated through equilibrium and linear-response function considerations is less than 10^{-3}.

[14] J. N. Bahcall and R. A. Wolff, Phys. Rev. Letters **14,** 343 (1965).

[15] B. M. Barker, M. S. Bhatia, G. Szamosi, Phys. Rev. **158,** 1498 (1967).

results should give some indication of the role the Fermi energy plays in that case, classical calculations can not replace a proper quantum treatment. On the other hand, there are collisions to restrict the Hartree approximation through the requirement that either the density should be limited by the $\kappa^{-3}n\leq 1$, or if it exceeds this value, $\chi^{-3}n>1$ should hold, guaranteeing that collisions are rare. This can be formulated as $b^{-3}>\gamma^2\mu^{-2}s$. The domain so determined is well above the degeneracy limit, showing that the justification of the Hartree approximation for a classical system is much easier than for a degenerate one. This is in contrast to the situation prevailing in electron plasmas, which can be regarded as weakly interacting almost always if they are degenerate; the difference stems from the big difference in the coupling constants.

The main interest of this paper is, however, not in the applicability of the results to actual physical systems, but in the elucidation of the consequences of the transformation properties of the field and in the systematic relativistic handling of the Hartree approximation. From this point of view the results for a system characterized by attractive interaction can be summarized as follows:

(1) The system is thermodynamically stable for any value of the density and for any interaction range;

(2) the "relativistic hard core," predictable from a one-particle model, never shows up, and the effective mass is always positive;

(3) for not very high temperatures two phases exist: at a maximum density $s(b)$ the attractive interaction leads to the instability of the "gaseous" phase; for densities higher than s_m (~ 2), however, the system is stable for any density or temperature.

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Langevin Method for Damped Quantum-Mechanical Systems

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It is shown that the Langevin forces needed to convert expectation-value equations into operator equations for a two-level system cannot be Gaussian, and arguments are presented for extending this result to other nonlinear systems. In order to satisfy quantum-mechanical requirements, a generalization to higher-order moments of a procedure used for specifying second-order moments of the Langevin forces is proposed.

THE Langevin procedure of introducing, in a phenomenological manner, random process (Langevin) forces is well known in the classical theory of Brownian motion.[1] The force is described by its statistical properties, and being Gaussian, only its first- and second-order moment functions need be specified. The Langevin equation for a damped quantum-mechanical harmonic oscillator was derived some time ago,[2,3] and it was shown that the expectation-value equation can be converted into an operator equation through the introduction of a (quantum-mechanical) Langevin force. The force was shown to be Gaussian, so that here, too, complete specification of the force is obtained from the first two moment functions. Recently, several authors have applied the Langevin procedure to a general quantum-mechanical system, introducing Langevin forces in order to convert expectation-value equations into operator equations. Haken and Weidlich[4] introduce forces described by a Gaussian Markov process, and specify its properties completely with the first two moment functions. Lax[5] does not require the force to be Gaussian, and, by discussing only the first two moment functions, leaves the description of the Langevin force incomplete. In the present note, a two-level system will be considered, and it will be shown that, firstly, the Langevin force *cannot* be Gaussian, and secondly, a general procedure is available for specifying the higher-moment functions of the Langevin force.

In order to bring out the essential aspects of the present argument, we consider the simplest possible

[1] See, for instance, Ming Chen Wang and G. E. Uhlenbeck, Rev. Mod. Phys. **17**, 393 (1945).

[2] I. R. Senitzky, Phys. Rev. **119**, 670 (1960); **124**, 642 (1961).

[3] J. Schwinger, J. Math. Phys. **2**, 407 (1961).

[4] H. Haken and W. Weidlich, Z. Physik **189**, 1 (1966).

[5] M. Lax, Phys. Rev. **145**, 110 (1966).