Bootstrap Model with Hadrons of Both Parities*

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A bootstrap condition, used commonly to study internal symmetries, is generalized so that it may be applied to states of different spins and parities. A bootstrap model of vector mesons and spin-$\frac{1}{2}$ baryons, developed by Cutkosky and Polkinghorne, is generalized to $SU(6)_W$ symmetry and to include mesons and baryons of both parities. A solution is found involving odd- and even-parity mesons and even- and odd-parity baryons that correspond to the $SU(6)_W$ representations **35**, **35**$\oplus$**1**, **56**, and **70**, respectively. The existence of the solution depends on certain exact properties of the $35\otimes56$ and $35\otimes35$ crossing matrices of $SU(6)$. The physical interpretation of the solution is discussed; the odd-parity baryon multiplet corresponds physically to the $SU(6)\otimes O(3)$ representation (70, 3).

I. INTRODUCTION

IN 1963, Cutkosky considered a bootstrap model involving the trilinear interactions of a set of degenerate V (vector) mesons, in which the V are VV composites produced by V exchange forces.[1] The self-consistency conditions in this model can be satisfied only if the V correspond to the regular representation of a compact, semisimple Lie group. Later, Polkinghorne extended this model by introducing a set of degenerate spin-$\frac{1}{2}$ baryons.[2] The baryons B are assumed to be VB composites produced by B and V exchange forces. Polkinghorne showed that if the VVV interactions satisfy the requirements of Ref. 1, the baryons must correspond to a representation of the group, and the VBB interactions must be proportional to matrix elements of the group generators.

The Cutkosky-Polkinghorne (CP) model suggests an elegant way in which the requirement of self-consistency may induce an interaction symmetry. However, the model is very incomplete dynamically, since attention is limited to VV and VB states transforming under space rotations as vectors and spin-$\frac{1}{2}$ spinors. The exchange mechanisms considered induce strong potentials in other partial waves as well. It is reasonable to neglect states of large angular momenta, because the centrifugal barrier in these states is expected to inhibit the potential in the low-energy region. However, in the CP model, states are neglected that correspond to orbital angular momenta no larger than that of the included states. The model is inconsistent if these states are considered. For example, it has been shown that if the group is $SU(n)$, the force in the P-wave, pseudoscalar VV state corresponding to an $SU(n)$ singlet is attractive and stronger than the force in the P-wave states in which the V are bootstrapped.[3] Thus, the assumption of no pseudoscalar mesons is inconsistent.

If one can find a modification of the CP model, in which all partial waves of orbital angular momenta smaller than some fixed value are considered, this type of inconsistency would be removed. The main purpose of this paper is to point out that a symmetry of the $SU(6)_W$ type leads to such a modification, provided that meson and baryon multiplets of both intrinsic parities are present.

The self-consistency conditions of the CP model are expressed in convenient form in Sec. II of the paper. In Sec. III, these conditions are generalized to $SU(6)_W$ symmetry. Exact solutions are found that involve mesons and baryons of both parities. The nature of the potentials and some effects of symmetry breaking are discussed in Sec. IV.

II. THE STRONG BOOTSTRAP CONDITION

We consider a set of degenerate mesons M and a set of degenerate baryons B. In this section the M and B are vector mesons and spin-$\frac{1}{2}$ baryons; the derivation of the self-consistency condition is a modification of that of Ref. 2. We concentrate on the baryon bootstrap condition, since it is more complicated than the corresponding meson condition.

The baryon and meson exchange potentials that act on the amplitude $M_i+B_\alpha \rightarrow M_j+B_\beta$ are represented by Figs. 1(a) and 1(b). [Figure 1(c) is not relevant for this section.] For convenience, we choose the mesons to correspond to Hermitian fields; the coupling constant $G_{\beta\alpha}{}^i$ corresponds to either of the vertices, $B_\alpha \rightarrow B_\beta+M_i$ or $B_\alpha+M_i \rightarrow B_\beta$. In general, $G_{\alpha\beta}{}^i=(G_{\beta\alpha}{}^i)^*$ (a representation in which the G are not real may be used). The baryon exchange potential V^B is the sum of the contributions of all possible intermediate baryons, and is given by

$$V^B(\beta j,\alpha i)=v^B \sum_\gamma G_{\beta\gamma}{}^i G_{\gamma\alpha}{}^j, \tag{1}$$

where v^B is a function of energy determined by making the appropriate partial-wave projection of the full potential (left-hand cut in a dispersion formalism). If the MMM interaction constants are denoted by g, the meson exchange potential may be written similarly,

$$V^M(\beta j,\alpha i)=v^M \sum_k G_{\beta\alpha}{}^k g_{ji}{}^k. \tag{2}$$

A key assumption of the model is that the B and M

* Supported in part by the National Science Foundation.
[1] R. E. Cutkosky, Phys. Rev. 131, 1888 (1963).
[2] J. C. Polkinghorne, Ann. Phys. (N.Y.) 34, 153 (1965).
[3] R. H. Capps, in Proceedings of the Summer Institute for Theoretical Physics, University of Colorado, Boulder, Colorado, 1966 (to be published). See Ref. 11.

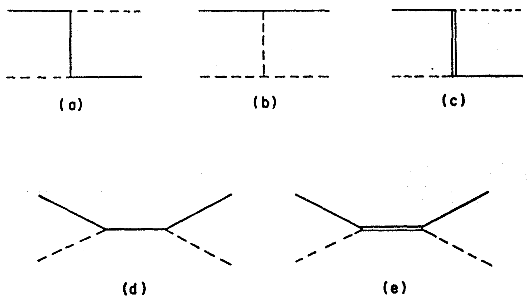

FIG. 1. Exchange poles and direct poles in MB states. Dashed lines, single solid lines, and double solid lines denote mesons (M), even-parity baryons (B), and odd-parity baryons (R), respectively.

exchange potentials may be considered proportional, i.e.,

$$v^M = \kappa v^B, \tag{3}$$

where κ is a constant. Since we are concerned with the low-energy region, in which the shape of the potential is not expected to be important, this assumption is justifiable, even though the actual shapes of v^B and v^M may be quite different.

The entire potential ($V^B + V^M$) may be expanded in terms of its eigenvectors, i.e.,

$$V(\beta j,\alpha i) = v^B[\sum_\gamma \lambda_\gamma u_\gamma(\beta j)^* u_\gamma(\alpha i)], \tag{4}$$

where the λ_γ are the eigenvalues, and the eigenvectors u_γ are normalized by the condition $\sum_{\beta j} u_\gamma(\beta j)^* u_{\gamma'}(\beta j) = \delta_{\gamma\gamma'}$. It is well known that in the static limit, v^B is an attractive potential, so that a positive eigenvalue λ_γ corresponds to an attractive potential in the eigenstate γ.

We define the "weak bootstrap condition" by the two following requirements:

(i) If composites are assumed formed in the eigenstates with the largest positive eigenvalues λ_γ, these composites are of the proper quantum numbers so that they may be identified with the B.

(ii) If this identification is made, the coefficients $u_\gamma(\alpha i)$ are proportional to the appropriate MBB coupling constants, i.e.,

$$G_{\gamma\alpha}{}^i = u_\gamma(\alpha i) A_\gamma, \tag{5}$$

where the A_γ are constants.

We define the "strong bootstrap condition" to include the weak condition and the additional requirement that all nonzero eigenvalues λ are positive and correspond to composite baryons. The strong condition also includes the assumption that the ratio $A_\gamma^2/\lambda_\gamma$ is independent of the composite γ, i.e.,

$$A_\gamma^2/\lambda_\gamma = \Lambda^2. \tag{6}$$

If the B are degenerate, one might require λ_γ to be independent of γ and assume that A_γ^2 is independent of γ. However, the less strict condition of Eq. (6) is sufficient for the derivation of the bootstrap condition.

If Eqs. (3) through (6) are substituted into the sum of Eqs. (1) and (2), the result may be written

$$\sum_\gamma G_{\beta\gamma}{}^j G_{\gamma\alpha}{}^i = \Lambda^2 (\sum_\gamma G_{\beta\gamma}{}^i G_{\gamma\alpha}{}^j + \kappa \sum_k G_{\beta\alpha}{}^k g_{ji}{}^k). \tag{7}$$

An important advantage of this strong bootstrap equation is that one may consider a fixed pair of external states at a time, and need not compute the eigenvectors of the full potential.

Both the strong and weak conditions are used frequently in bootstrap models, although the distinction is sometimes not drawn clearly. The weak condition permits potentials that are repulsive or attractive and weaker than the strongest to exist and not be associated with composites. The strong condition does not permit this, and so is more difficult to satisfy. The strong condition is more consistent with the basic bootstrap philosophy, since the distinction between resonances and nonresonating states with attractive interactions is not a sharp one. The validity of the strong condition does not require that all attractive potentials produce observable resonances, but rather that the effect of exchanging states with attractive interactions may be approximated by single-particle exchange. The contribution to the left-hand cut of the interaction in nonresonating states is neglected if one uses the weak condition. The strong condition will be used in the rest of the paper.[4]

If the baryon exchange forces are treated in the static approximation, the residue of the pseudopole resulting from the u-channel pole is a convenient measure of the strength of V^B. It is well known that if one makes the effective-range approximation as well as the static approximation, the magnitudes of the residues of the pseudopole and composite pole in the s channel must be the same.[5] This implies the condition

$$\Lambda^2 = 1. \tag{8}$$

If this relation is substituted into Eq. (7), the result is equivalent to Eq. (12) of Ref. 2. It is pointed out by Polkinghorne that if the $g_{ij}{}^k$ are proportional to the structure constants of the group, as is required in the Cutkosky model, this equation implies that the baryons correspond to a representation of the group, and that the MBB interaction constants are proportional to matrix elements of the group generators.

If the mesons are all pseudoscalar mesons, so that no MMM interaction exists, and attention is limited to states of baryon number one, the strong bootstrap condition requires an infinite number of baryon states. This is the strong-coupling model of Goebel and collaborators.[6] The strong-coupling assumption of this model requires the strong-bootstrap condition, and may

[4] In Ref. 1, Cutkosky shows that in the VVV model, the weak condition implies the strong condition.

[5] See, for example, G. F. Chew, Phys. Rev. Letters **9**, 233 (1962); R. H. Capps, Nuovo Cimento **34**, 932 (1964).

[6] C. J. Goebel, Phys. Rev. Letters **16**, 1130 (1966). This paper contains references to previous works on the subject.

be regarded simply as one way of imposing this condition.

We wish to make one more point concerning strong-bootstrap conditions in general, in order to clear up a common misconception about the bootstrap hypothesis. Frequently, it is pointed out that some coupling-constant relations that result from bootstrap models also may be derived from an alternate set of assumptions that do not include a "compositeness" assumption.[7] It is clear from these nonbootstrap arguments that a dynamical condition on the residue of an s-channel pole associated with a particle A in the (BC) scattering amplitude does not imply that A is a composite of B and C. It is commonly thought that this result is foreign to bootstrap models. This is not so. In order to illustrate this point, let us consider a channel BC coupled to only one direct pole (A) in a strong bootstrap model, and assume that A is coupled to other channels much more strongly than to the BC channel. Clearly, A is not simply a BC composite. This does not alter the validity or usefulness of the strong bootstrap condition in the BC state. By extending this argument, we see that in a strong bootstrap model, it is not necessary for the entire set of two-particle states considered to dominate the composite wave functions. For example, if the strong bootstrap hypothesis for all states is valid approximately, Eq. (7) remains valid even if the most important states in the composite wave function of the B are three-quark states, or something else not considered here. Of course, if one uses the bootstrap arguments to predict which particles exist, and the mass differences among the composites, this does involve the assumption that the mass relations depend more on the states considered than on the states omitted.

An important result of this section is that one can generalize the strong bootstrap condition to many systems by applying the proportionality approximation of Eq. (3) to many potentials. Such a generalized strong bootstrap condition offers a possible way out of one of the dilemmas of bootstrap theory, that of finding a set of particles that provides a reasonable first approximation to a complete theory of hadrons. It is clear that one must start with only a subset of the existing particles. If the model is to describe reality, additional particles must be generated by the forces. The dilemma results from the requirement that all particles be treated equivalently, so that the generation of new particles requires that one redo the first approximation, including the new particles.

In order to see how the strong bootstrap formulation may help with this problem, let us denote by A all the particles to be included in the first approximation. The A are AA composites produced by A exchange forces. In general, one must postulate an additional set of particles B, with interactions of the type BAA, in order that the consistency conditions of the type of Eq. (7) are satisfied. (Finding such a B set is the program of Sec. III.) The resulting model is not complete, since BB and BA states must also be considered. Consideration of these states, and further iterations of this general procedure may lead to other particles, denoted by C. It may turn out that no CAA interactions are necessary, in which case the AA states need not be re-examined. The AA equations are correct, and the A set is suitable first approximation to a complete bootstrap theory.

III. $SU(6)_W$ SOLUTIONS FOR BARYONS AND MESONS

A. Application of the Bootstrap Conditions to $SU(6)_W$

It is pointed out in Sec. I that in the CP model, one neglects partial waves with orbital angular momenta as small or smaller than the l considered. This flaw can be eliminated if the spin and internal symmetry correspond to the same Lie group. The semisimplicity requirement on the group in the CP model rules out a group of the type $SU(3)\otimes SU(2)$. The $SU(6)$ group is suggested, although the bootstrap conditions do not favor this over similar groups that correspond to internal symmetries other than $SU(3)$ symmetry. We will consider $SU(6)_W$ symmetry, since this is the only form of $SU(6)$ that describes nonzero MMM interactions simply.[8] The odd-parity mesons (P and V nonets) are assumed to correspond to the direct sum of the regular and identity representations. The singlet state (V singlet with spin component zero in the direction of the interaction vertex) does not participate in the MMM interaction, and so may be neglected here, if the only external mesons considered correspond to the $SU(6)_W$ representation **35**. It has been shown previously that in a meson bootstrap model involving only the $SU(6)_W$ representation **35**, all 36 physical meson states are bootstrapped.[9]

The assumption of $SU(6)_W$-symmetric MMM and MBB vertices implies symmetric forward and backward one-particle-exchange amplitudes T_f and T_b. The potentials corresponding to even and odd orbital angular momenta will be taken proportional to the linear combinations, $\frac{1}{2}(T_f \pm T_b)$. (The terms potential, one-particle-exchange amplitude, and Born-approximation amplitude will be used interchangeably.) We will make the assumptions that S waves and S-D transitions dominate the even-l amplitudes, and that P waves dominate the odd-l amplitudes. With these assumptions, the sum T_f+T_b determines the S-S and S-D potentials completely, while T_f-T_b determines

[7] See, for example, F. E. Low, Phys. Rev. Letters **9**, 277 (1962); in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, California, 1967), p. 241.

[8] H. J. Lipkin and S. Meshkov, Phys. Rev. **143**, 1269 (1966).

[9] R. H. Capps, Phys. Rev. **148**, 1332 (1966).

the P-wave potentials nearly completely.[10] That term of the P-wave amplitude that is proportional to $\mathbf{L}\times\mathbf{L}'$ (where $\mathbf{L}$ and $\mathbf{L}'$ are the initial and final orbital angular momenta) does not contribute in the collinear directions and so is not included in the model.[9] However, this only means that the model is incomplete, and does not invalidate the consistency conditions used.

In the generalization of the CP model to $SU(6)_W$, the assumption of Eq. (3) is extended, i.e., it is assumed that all contributions to the MB potential (or to the MM potential) may be taken proportional to each other. If attention were restricted to the P-wave MM and MB states, identification of the M and B with the representations **35** and **56** of $SU(6)_W$ would lead to a solution of the bootstrap condition of Eq. (7), and the corresponding meson condition. Furthermore, all P-wave amplitudes (except the $\mathbf{L}\times\mathbf{L}'$ amplitudes mentioned above) are considered in this model. However, the M exchange potentials are strong in states of both parities. Clearly, the bootstrap condition for states of even l cannot be satisfied unless additional particles are present. The baryon and meson bootstrap conditions for states of both parities are discussed in Secs. III B and III C.

B. The Baryon Bootstrap

We attempt to satisfy the bootstrap condition in the odd-parity (even-l) MB states by introducing an odd-parity baryon multiplet corresponding to the irreducible representation R, in addition to M and B. Since the strong bootstrap condition is used, we need not treat MB and MR states at the same time. Scattering in the MR states is complicated, and is not treated in this paper. The potentials in the MB states result from B, M, and R exchange, as shown in Figs. 1(a), 1(b), and 1(c). The B and R composites correspond to Figs. 1(d) and 1(e).

We treat the u-channel (B and R exchange) potentials in the static approximation. In this approximation, $k_u^2=k_s^2$ and $\cos\theta_u=\cos\theta_s$, where k is the momentum in the center-of-mass system. This implies that B exchange affects only the even-parity states, while R exchange affects only the odd-parity states. Furthermore, if the effective range approximation is made, Eq. (8) may be used for the u-channel contributions in the states of both parities. The even-parity bootstrap condition is then the straightforward generalization of the Polkinghorne condition to $SU(6)_W$, and may be satisfied if B is identified with any irreducible representation other than the identity, provided that the MBB interaction constants are of F type, i.e., are proportional to the matrix elements of the generators. For such an interaction, Eq. (7) (with $\Lambda^2=1$) may be written in terms of the irreducible representations i in the direct product $M\otimes B$, i.e.,

$$\delta_{iB}G_B{}^2=C_{iB}G_B{}^2+\tfrac{1}{2}[(X_M+X_B-X_i)/X_B]\kappa_{\rm even}G_Bg\,, \quad (9)$$

where X is the eigenvalue of the quadratic Casimir operator, C is the u-channel crossing matrix, and $\kappa_{\rm even}$ is the proportionality constant analogous to the κ of Eq. (7). The symbols G_B and g denote total MBB and MMM interaction constants, defined by the relations $G_B{}^2=\sum_{\beta j}(G_{\alpha\beta}{}^i)^2$ and $g^2=\sum_{jk}(g_{ij}{}^k)^2$, where α and i denote any states in the B and M multiplets. The coefficient $\frac{1}{2}(X_M+X_B-X_i)/X_B$ is the t-channel crossing matrix element. The form of the bootstrap condition for odd-parity states is similar, i.e.,

$$\delta_{iR}=C_{iR}G_R{}^2+\tfrac{1}{2}[(X_M+X_B-X_i)/X_B]\kappa_{\rm odd}G_Bg\,. \quad (10)$$

We use the convention that a universal M interaction (one in which the M interactions with both mesons and baryons are proportional to matrix elements of the generators, with the same proportionality constant) corresponds to a positive G_Bg. With this convention, both $\kappa_{\rm even}$ and $\kappa_{\rm odd}$ are positive, as is shown in the literature,[11] and discussed briefly in Sec. IV. The even-parity equation, Eq. (9), is satisfied if $\kappa_{\rm even}g=G_B$; in this case the equation is equivalent to a well-known expression for the crossing matrix elements C_{iB}.[12] Consistency of both conditions may be obtained if the B and R columns of the matrix $(1-C)$ are proportional to each other, with a positive proportionality constant. Recently, it has been proved that if B is any representation such that the MBB interaction is unique, all columns of $1-C$ are proportional; thus, the number of solutions is very large.[13] We write below the crossing matrix for the physically relevant $\mathbf{56}\otimes\mathbf{35}$ case.[14]

$$C=\begin{array}{c} \\ 56 \\ 70 \\ 1134 \\ 700 \end{array}\!\!\begin{array}{c}\begin{array}{cccc} 56 & 70 & 1134 & 700\end{array}\\ \left[\begin{array}{cccc} 132 & -90 & -162 & 300 \\ -72 & 45 & -243 & 450 \\ -8 & -15 & 153 & 50 \\ 24 & 45 & 81 & 30 \end{array}\right]\end{array}\times\frac{1}{180}\,. \quad (11)$$

In this case, either of the assignments **56**, **70**, or **1134** for R leads to a solution.

Physically, the resonances R correspond to the representation $(R,3)$ of $SU(6)\otimes O(3)$ [rather than to the representation $(R,1)$].[10] Thus, the assignment of R to the **70** of $SU(6)_W$ leads to 210 predicted spin states in the R multiplet. This phenomenon is discussed in detail in Ref. 10. It results from the fact that an MB state of W spin w is a superposition of states of spins $w-1$, w, and $w+1$, in general. Since all odd-parity MB

[10] R. H. Capps, Phys. Rev. **158**, 1433 (1967).

[11] R. H. Capps, Phys. Rev. Letters **14**, 842 (1965).

[12] J. G. Belinfante and R. E. Cutkosky, Phys. Rev. Letters **14**, 33 (1965).

[13] R. H. Capps, Ann. Phys. (N. Y.) **43**, 428 (1967).

[14] V. Singh and B. M. Udgaonkar, Phys. Rev. **139**, B1585 (1965).

amplitudes in this model involve at least one S state, the intrinsic spin spectrum of the MB states corresponds to the predicted spin spectrum of R. On the other hand, this argument does not apply to the P-wave MB states. The B multiplet does correspond simply to the representation B of $SU(6)$.[15]

Because of this difference in the interpretation of B and R, there is no reason to expect nature to choose the "apparently symmetrical" solution of Eqs. (9) and (10), in which B and R are the same representation. The experimental evidence seems to indicate that nature has chosen the **70**.[10] This brings up the question of whether or not some simple dynamical principle is behind this choice. In previous $SU(6)$ or $SU(6)_W$ symmetric models of baryon resonances, where only the meson exchange force was considered in odd-parity states, the **70** seemed the most likely resonance candidate because the meson exchange force is most attractive in this state.[10,11] A related argument exists in the present model, if one assumes that the average R mass is greater than the average B mass. (This assumption agrees with experiment.) The mass difference increases the energy denominator corresponding to R exchange [Fig. 1(c)], and is expected to decrease the importance of this process. Hence, a particular assignment of a representation to R is especially stable to the R-B mass difference if the R exchange potentials are small compared to the largest M exchange potentials. This is equivalent to the condition that the elements of the R column of Eq. (11) are small compared to $1-C_{RR}$. The assignment of R to the **70** is the most stable in this sense.

C. The Meson Bootstrap

We now turn to the MM states, where M corresponds to the regular representation of the group $SU(n)_W$, and n is even. In these states, also, the M exchange potential vanishes in the forward direction, indicating that potentials in states of different parities must be comparable. We assume the existence of even-parity mesons, denoted by N. There are two types of potential diagrams in the MM states, corresponding to M and N exchange, and also there are M and N direct-pole diagrams. The dynamics are quite different from those of the baryon case. No static approximation is applicable, and statistics limits the MM states to those of the proper combined orbital and $SU(n)_W$ symmetry.

For each amplitude, both M and N exchange may contribute in both the t and u channels. However, if symmetrized MM states are used, it is sufficient to consider only the t channel, since inclusion of the u channel would simply multiply the potential by two. The t-channel MM crossing matrix has been computed by Neville.[16] The columns corresponding to the exchange of the singlet (I), the symmetric regular representation (D), and the antisymmetric regular representation (M) are listed in Table I. The notation for the representations other than I, D, and M is that of Ref. 16. If $n=6$, $P_s{}^A$, $P_A{}^s$, $P_A{}^A$, and $P_s{}^s$ are the representations **280**, **280***, **189**, and **405**, respectively. The t-channel crossing matrix differs from the corresponding u-channel matrix only in the signs of the elements connecting the symmetric and antisymmetric states. One could derive the bootstrap conditions equally well by using the u-channel matrix, provided the signs of the potentials were chosen correctly.

TABLE I. The $MM\to MM$ crossing matrix elements corresponding to the exchange of the states I, D, and M.

s-channel state	Exchanged (t-channel) state I	D	M
I	$1/(n^2-1)$	1	1
D	$1/(n^2-1)$	$(n^2-12)/2(n^2-4)$	$\frac{1}{2}$
M	$1/(n^2-1)$	$\frac{1}{2}$	$\frac{1}{2}$
$P_s{}^A$	$1/(n^2-1)$	$-2/(n^2-4)$	0
$P_A{}^s$	$1/(n^2-1)$	$-2/(n^2-4)$	0
$P_A{}^A$	$1/(n^2-1)$	$-1/(n-2)$	$1/n$
$P_s{}^s$	$1/(n^2-1)$	$1/(n+2)$	$-1/n$

We write the consistency equations in terms of the states of Table I. The indices r and s will be used for the antisymmetric and symmetric states, respectively. (The antisymmetric states are M, $P_s{}^A$, and $P_A{}^s$.) It is assumed that M is the only antisymmetric state in which composites occur, while the even-parity mesons may correspond to more than one irreducible representation. The potentials V in the various states are

$$\tfrac{1}{2}V_r=v_{oo}C_{rM}F_M{}^2+\textstyle\sum_{s'} v_{oe}C_{rs'}F_{s'}{}^2\,,$$
$$\tfrac{1}{2}V_s=v_{eo}C_{sM}F_M{}^2+\textstyle\sum_{s'} v_{ee}C_{ss'}F_{s'}{}^2\,, \qquad (12)$$

where the C are the crossing matrix coefficients, $F_M{}^2$ is the coupling constant associated with the MMM interaction, $F_{s'}{}^2$ are the coupling constants associated with the MMN interactions, and v_{oe} is the form of the potential in the odd-parity state resulting from even-parity meson exchange, etc. The factors of $\frac{1}{2}$ are included because the effect of including the u-channel exchange contribution would be to double the contribution of the t-channel exchange. The generalization of the proportionality assumption of Eq. (3) is

$$v_{oe}=\kappa_o v_{oo}\,,$$
$$v_{ee}=\kappa_e v_{eo}\,. \qquad (13)$$

An advantage of using the t-channel crossing matrix (rather than the u-channel matrix) is that the four potentials v_{oo}, v_{oe}, v_{eo}, and v_{ee} are attractive. Thus, κ_o and κ_e are positive. The attractive nature of the M

[15] This follows from the extension to $SU(6)_W$ of the $SU(6)$-symmetric bootstrap model treated by R. H. Capps [Phys. Rev. Letters 14, 31 (1965)] and by Belinfante and Cutkosky (Ref. 12). The extension to $SU(6)_W$ is simple and is discussed in Ref. 3.

[16] Donald E. Neville, Phys. Rev. **132**, 844 (1963), Table III. The elements referring to the $P_s{}^A$ and $P_A{}^s$ exchange potentials in the representation $P_A{}^A$ are in error in this table. They should be $-(2+n)/(4n)$ rather than $-(2-n)/(4n)$.

exchange potentials (v_{oo} and v_{eo}) has already been used in Sec. III B. (The M-exchange potentials in MM and MB states are similar.) The signs of the N-exchange potentials are demonstrated by the following simple argument. An S-wave direct pole in the t channel corresponds to an amplitude of the form $F^2\mu^2/(\mu^2-t)$, where μ is the mass of an N meson. The corresponding s-channel amplitude is

$$\frac{CF^2\mu^2}{\mu^2+2k_s^2(1-\cos\theta_s)},$$

where C is the appropriate crossing matrix element. If $C>0$, the integral over $\cos\theta_s$ of this amplitude multiplied by either 1 or $\cos\theta_s$ is positive in the physical s-channel region. Thus, v_{oe} and v_{ee} are attractive. These potentials are discussed further in Sec. IV.

Since the potentials are already diagonalized, the analogs of Eqs. (4), (5), and (6) are simply

$$V_r=v_{oo}F_r^2/\Lambda_o^2\,,\quad V_s=v_{eo}F_s^2/\Lambda_e^2\,, \tag{14}$$

where

$$\Lambda_o^2=A_M^2/\lambda_M\,,\quad \Lambda_e^2=A_s^2/\lambda_s\,. \tag{15}$$

The constant F_i^2 is to be taken as zero if no composite corresponding to the state i exists.

We look for a solution involving composites only in the states M, I, and D. Thus, $F_r^2=\delta_{rM}F_M^2$, and $F_s^2=\delta_{sI}F_I^2+\delta_{sD}F_D^2$. Combination of these relations with Eqs. (12), (13), and (14) yields the bootstrap equations

$$\tfrac{1}{2}\delta_{rM}F_M^2/\Lambda_o^2=C_{rM}F_M^2+\kappa_o(C_{rI}F_I^2+C_{rD}F_D^2)\,, \tag{16}$$

$$\tfrac{1}{2}(\delta_{sI}F_I^2+\delta_{sD}F_D^2)/\Lambda_e^2=C_{sM}F_M^2+\kappa_e(C_{sI}F_I^2+C_{sD}F_D^2)\,. \tag{17}$$

It is seen from the crossing matrix elements of Table I that the potentials in the antisymmetric representations $P_s{}^A$ and $P_A{}^s$ and in the symmetric representations $P_s{}^s$ and $P_A{}^A$ will vanish if the following conditions hold:

$$F_I^2=[2(n^2-1)/(n^2-4)]F_D^2\,, \tag{18}$$

$$\kappa_eF_D^2=[(n^2-4)/n^2]F_M^2\,. \tag{19}$$

Thus, the total potential is nonzero in the states M, I, and D. If the F_i^2 satisfy Eqs. (18) and (19), the bootstrap conditions of Eqs. (16) and (17) will be consistent for the M, I, and D states also if

$$\Lambda_e^{-2}=2\kappa_e\,,\quad \Lambda_o^{-2}=1+(\kappa_o/\kappa_e)\,. \tag{20}$$

These equations are equivalent to the usual bootstrap condition that the over-all output coupling constant is equal to the input coupling constant. Since we have no accurate dispersion theory, we do not attempt to use these conditions to calculate the over-all magnitudes of the MMM and MMN interaction constants. Instead, we simply assume that there would be sufficient parameters in a complete theory so that over-all constants for each type of interaction could be chosen that would lead to consistency in all the bootstrap relations. This assumption implies that the Λ^2 may be chosen in accordance with Eq. (20).

The consistency condition may be stated in a more familiar way. The eigenvalues of C are ±1. The bootstrap equations are consistent because an eigenvector corresponding to the eigenvalue 1 exists completely within the M-I-D subspace, with relative components along the three axes that are positive. This form of the bootstrap condition is used frequently in models of baryons.[17]

We now specialize to $SU(6)_W$. Our solution implies that even-parity meson multiplets corresponding to the representations **1** and **35** exist. When the amplitudes are analyzed in terms of spin, rather than W spin, these multiplets must also be classifiable according to the group $SU(6)\otimes O(3)$, as is the baryon multiplet R. On the other hand, the $SU(6)_W$ representation **35** for the odd-parity states corresponds to the 36 physical meson states (P and V nonets), as is explained in Ref. 9. A thorough study of the properties of the predicted meson resonances will be published shortly.

The M-I-D solution to the bootstrap equations is not the only solution, but it is the simplest one. If $n>2$, one may use the crossing matrix of Ref. 16 to show that there are no solutions involving M and only one other multiplet. There are two solutions involving M and two other multiplets, the M-I-D solution and an M-I-$P_A{}^A$ solution. The second involves many more particles than the first, e.g., $P_A{}^A$ contains 189 states if $n=6$. Furthermore, the potential is comparatively weak in the $P_A{}^A$ state in this solution; if $n=6$, $F_{189}{}^2/F_I^2=8/63$.

We assume that the N mesons do not interact with the B baryons and so do not influence the baryon bootstrap.

IV. DISCUSSION OF THE POTENTIALS

Our $SU(6)_W$-symmetric model is more complicated than one involving internal symmetry only, because the symmetry cannot be exact. The collinear amplitudes in Born approximation may satisfy the symmetry. However, the virtual momenta of the two particles in the intermediate states associated with the right-hand (unitarity) cut need not be parallel to the external momenta. Therefore, this cut destroys the exact $SU(6)_W$ symmetry. Although we do not refer to the unitarity cut explicitly in formulating the bootstrap conditions, the presence of the cut is implied by the assumption that bound states or resonances are formed. This cut violates the symmetry of the model in still another way; since the unitarity condition is different in states of even and odd orbital angular momenta, the model is

[17] This condition was first used by Chew in Ref. 5. See also I. S. Gerstein and K. T. Mahanthappa, Nuovo Cimento 32, 239 (1964).

not truly symmetric with respect to the two parities. Because of these effects, we study the potentials more carefully. We first discuss the signs and relative magnitudes of the potentials in MM states, assuming $SU(6)_W$ symmetry. Effects that break the symmetry are then discussed. The fact that the N and R multiplets correspond physically to representations of $SU(6)\otimes O(3)$ does not lead to any difficulty in the treatment of MM and MB states, since the bootstrap equations and crossing relations are formulated in terms of $SU(6)_W$ states.

The relation between the channel variables in MM states is $s+t+u=4m^2$. The simplest way of obtaining reasonable M-exchange potentials is to use Feynmann rules, with conventional vertex functions. It has been shown that for a t-channel amplitude corresponding to a direct P-meson pole (or V pole in one of the spin states $S_z=\pm 1$), this procedure leads to collinear amplitudes of the form

$$T=G_iG_j(\tfrac{1}{4}t/m^2)(\pm k_t^2)/(m^2-t)\,, \qquad (21)$$

where the $\pm$ signs refer to the forward and backward directions ($s=0$ and $u=0$), k_t is the magnitude of the particle momenta in the center-of-mass system, and the G are coupling constants.[9] This amplitude vanishes in the forward direction in the s channel, because of the $(\frac{1}{4}t/m^2)$ factor. The s-channel backward amplitude T_b is given by

$$T_b=-CG_iG_j\frac{s}{4m^2}\frac{k_s^2}{m^2+4k_s^2}\,, \qquad (22)$$

where C is the appropriate t-channel crossing matrix element.

If C is positive and $i=j$, the P-wave potential is attractive, since the collinear P-wave amplitude is equal to $\frac{1}{2}(T_f-T_b)/k^2\cos\theta$, and a positive Born-approximation amplitude corresponds to attraction. It is not immediately obvious whether or not a positive C corresponds to an S-wave attraction, because the k^2 factor in Eq. (22) implies that the S-wave amplitude changes sign at threshold. A simple way to resolve this ambiguity is to use the technique of Ref. 11, i.e., to treat the spin wave functions of the real particles nonrelativistically and write the potential in configuration space. Tensor and central potentials result.[11,18] The central potential corresponding to the amplitude T is

$$\mathcal{V}=-\frac{CG_iG_j}{12}\left[\frac{e^{-mr}}{r}-\frac{4\pi\delta(\mathbf{r})}{m^2}\right], \qquad (23)$$

where a negative potential corresponds to attraction. The volume integral of the potential vanishes. Since the model is not accurate at extremely small distances, we take for the sign of $\mathcal{V}$ the sign of the Yukawa term. Hence, a positive C corresponds to attraction; the potentials in corresponding S and P states are of the same sign.

The situation is somewhat different if the virtual meson is a V meson with $S_z=0$. The t-channel amplitude corresponding to such a V pole does not contain the $\frac{1}{4}t/m^2$ factor of Eq. (21).[9] The static potential corresponding to the exchange of a zero-helicity V is

$$\tfrac{1}{4}CG_iG_jm^2/(m^2-t)\,, \qquad (24)$$

where $t=-2k_s^2(1-\cos\theta_s)$. In configuration space, this potential contains a Yukawa term similar to that of Eq. (23), but no δ-function term. Thus, the M-exchange potential computed this way is not $SU(6)_W$ symmetric. However the deviation results from the nonsymmetric manner of analytic continuation from the mass shell that is implied by the conventional vertex functions. It may be remedied by inserting a delta function into the zero-helicity V-exchange potential. If $\frac{1}{4}s/m^2\approx 1$, this is equivalent to making the following threshold subtraction in Eq. (24):

$$(m^2-t)^{-1}\rightarrow (m^2-t)^{-1}-(m^2)^{-1}. \qquad (25)$$

The M-exchange potential in MB states is similar to the potential in MM states. It vanishes in the forward direction, and in the backward direction (near threshold) is proportional to the factor $k_s^2/(m^2+4k_s^2)$ of Eq. (22). It follows that the constants κ_{even} and κ_{odd} of Eq. (9) and Eq. (10) are positive.

The separation of the potential into two types (called here central and tensor) is discussed extensively in Ref. 10. This separation does not depend on the form of the vertex functions: it may be made from a spin analysis of the collinear amplitude $A+B\rightarrow C+D$, where the symbols denote either mesons or baryons. One writes for the amplitude

$$(AB)_s{}^m\longrightarrow (CD)_{s'}{}^m L_\Delta{}^0\,, \qquad (26)$$

where the subscripts and superscripts are total spins and z components, and L is a "spin-spurion," defined to conserve total spin. For $MB\rightarrow MB$ amplitudes, the spin-exchange Δ must be 0 or 2; these are the central and tensor-type potentials.

We now consider the $SU(6)_W$-symmetric, N-exchange potentials in the MM states. These also contain t factors, and so vanish in the forward direction in the s channel. This may be seen from the following argument. The potentials vanish at the s-channel threshold energy, because the threshold potential cannot contain tensor ($\Delta=2$) terms, yet $SU(6)_W$ symmetry generally requires these terms. In order to vanish at threshold, the potential must contain either the factor t, or $s-4m^2$, or a linear combination of the two. However, the factor $s-4m^2$ corresponds to an energy-dependent mixture of partial waves in the t channel, and thus cannot result from simple meson exchange. Therefore, the factor t is present.

A t-channel S-D transition amplitude containing a

[18] R. H. Capps, Phys. Rev. 150, 1263 (1966).

direct pole may be written in a form similar to that of Eq. (21), i.e., in the forward and backward directions:

$$T=-F_iF_j(\tfrac{1}{4}t/\mu^2)k_t^2/(\mu^2-t)\,, \tag{27}$$

where μ is the mass of the even-parity meson. In the s channel, the forward amplitude vanishes, and the backward amplitude is

$$T_b=-CF_iF_j\frac{s}{4\mu^2}\frac{k_s^2}{\mu^2+4k_s^2}\,. \tag{28}$$

If the t-channel S-S amplitudes with direct poles are written in the form of Eq. (27), these contributions to the s-channel potential are also of the form of Eq. (28). The sign has been chosen correctly, because if $i=j$, the S-wave coupling constant squared (the negative of the residue of the t-channel pole) is $-F_i^2(k_t^2)_{t=\mu^2}=F_i^2\times(m^2-\frac{1}{4}\mu^2)>0$. (It is assumed that the N are bound, i.e., $\mu<2m$. Otherwise, our simple pole treatment would be inadequate.) Comparison of Eqs. (22) and (28) indicates that the signs of the corresponding M and N exchange potentials v_{io} and v_{ie} are the same. In fact, if $\mu=m$, the potentials are of the same form in this approximation.

It is more conventional to assume that a t-channel S-S amplitude with a direct pole is of the form $\frac{1}{4}F_iF_j\mu^2/(\mu^2-t)$. With such an amplitude, one needs only to make a subtraction similar to that of Eq. (25) in order to obtain a potential similar in form to Eq. (28) when $s\approx 4m^2$.

We have shown that with suitable MMM and MMN vertex functions $SU(6)_W$ symmetry of the coupling constants will lead to symmetry of the potentials, and that the four potentials v_{ij} of Eq. (12) are attractive. We now turn to the question of the deviations from the symmetry that are caused by the unitarity cut. It is reasonable to assume that these deviations do not alter the results concerning the multiplets of composites that are favored. However, the deviations may affect the relative strengths of different potentials, and the relative coupling constants. We omit the B exchange force from the discussion, since this force is treated very simply in the usual static approximation.

The relative size of the $\Delta=0$ and $\Delta=2$ (central and tensor-type) potentials may be quite different in the M, N, and R exchange mechanisms, and may not be predictable from $SU(6)_W$ symmetry. The presence of this ambiguity in the M-exchange forces is seen from the fact that the radial dependences of the central and tensor potentials are different.[18] In the momentum representation, the equivalent statement is that the $\Delta=0/\Delta=2$ ratio (called here $\mathcal{R}$) in the Born-approximation amplitudes is energy-dependent. Fortunately, very few of the predictions of the model depend on $\mathcal{R}$. It has been shown in Ref. 18 that the ratios of output MMM coupling constants do not depend on $\mathcal{R}$. Furthermore, the relative predicted masses and branching ratios (except for the over-all D-wave/S-wave ratio) of the odd-parity baryon resonances do not depend on $\mathcal{R}$.[10]

V. CONCLUDING REMARKS

We have pointed out that in any bootstrap model involving meson exchange forces, either in MM or MB states, composite particles of both parities should be expected. Generalization of the proportionality approximation of Eq. (3) allows one to apply the strong bootstrap condition to states of different parities and different spins.

If one assumes vertices invariant to $SU(6)_W$ symmetry, it is straightforward to separate the one-particle-exchange potentials into even- and odd-parity parts. The bootstrap model involving the 36-fold odd-parity meson multiplet and the 56-fold baryon multiplet has been extended to include even orbital angular momenta, and is consistent if even-parity meson resonances corresponding to the $SU(6)_W$ representations **35** and **1**, and odd-parity baryon resonances corresponding to the $SU(6)_W$ representation **70**, are added.

In a previous paper it was shown that the quantum numbers of such an odd-parity baryon multiplet must correspond to the (70,3) representation of $SU(6)\otimes O(3)$.[10] This assignment fits the experimental data very well, much better than does the assumption that these resonances correspond to a superposition of representations.[19] One of the principle motivations of the present paper is to provide a justification for the assumption that only the (70,3) is involved.

A detailed study of the implications of the model with respect to the meson resonances will be published shortly.

An important question for the future of this type of model is that of whether or not scattering states involving the meson and baryon resonances may be included in a manner that is consistent theoretically, and leads to predicted composites that may be identified with the observed Regge recurrences of the lighter meson and baryon states.

[19] R. H. Capps, Phys. Rev. **153**, 1503 (1967).