decomposed in a form such as $\sum_\nu |\nu\rangle(W-E_\nu)^{-1}\langle\nu|$; then (26) or (27) would imply that

$$\sum_\nu |\nu\rangle(W-E_\nu)^{-1}\langle\nu|\Phi\rangle=0\,, \tag{29}$$

or, using the orthogonality of the $|\nu\rangle$, that $\langle\nu|\Phi\rangle=0$ for all $|\nu\rangle$, and so, by completeness, $|\Phi\rangle\equiv0$. That is, the solutions of (26) or (27) are identically zero. Therefore, we have proved that the only discrete solutions of (25), or equivalently, of (22), are bound states of the Schrödinger equation.

We may conjecture that there are infinitely many acceptable formulations of the three-body problem, not equivalent to the Faddeev equations, and without spurious homogeneous solutions.[15] One counter-example is, however, enough to settle the question of the Faddeev equations's uniqueness.

I would like to acknowledge several useful conversations with Professor R. D. Amado, and helpful correspondence with Professor R. G. Newton.

[15] The Faddeev kernel and that given in Eq. (22) differ both in their spectra and in the fact that the kernel of (22) may have simple poles in W corresponding to two noninteracting particles bound to a third. In the theory outlined here (Refs. 8–11), the bound-state problem thus resembles the usual perturbation-theoretic formalism.

PHYSICAL REVIEW VOLUME 161, NUMBER 5 25 SEPTEMBER 1967

Four-Body Leptonic Decays of Hyperons

Y. Singh
Tata Institute of Fundamental Research, Bombay, India
(Received 21 April 1967)

Using the techniques of current algebra, we have investigated the four-body leptonic decays of hyperons of the type $B\rightarrow B'+\pi+e^-+\bar\nu_e$. The predicted branching ratios for the favorable cases $\Sigma^+\rightarrow p+\pi^++e^-+\bar\nu_e$ and $\Sigma^-\rightarrow N+\pi+e^-+\bar\nu_e$ are 0.35×10^{-7} and 0.85×10^{-7}, respectively. The covariant phase-space calculation for the four-body leptonic decay rate is given in an Appendix.

I. INTRODUCTION

The recently developed methods[1] based on the algebra of currents and the hypothesis of partially conserved axial-vector current (PCAC) have been successfully applied to a wide variety of phenomena involving strong, weak, and electromagnetic interactions. Using these techniques, we have analyzed the yet unobserved four-body leptonic decays of hyperons of the type

$$B\rightarrow B'+\pi+e+\bar\nu_e\,.$$

The calculated branching ratios for the favorable cases $\Sigma^-\rightarrow N+\pi+e^-+\bar\nu_e$ and $\Sigma^+\rightarrow p+\pi^++e^-+\bar\nu_e$ are 0.85×10^{-7} and 0.35×10^{-7}, respectively. For Λ and Ξ decay the branching ratios for such modes are much smaller. In the next two sections we give the details of our calculations. In the last section we present the numerical results for the seven energetically possible four-body leptonic decay modes of hyperons:

$$\Lambda\rightarrow p+\pi^0+e^-+\bar\nu_e\,,\quad \Sigma^+\rightarrow p+\pi^++e^-+\bar\nu_e\,,$$
$$\Lambda\rightarrow n+\pi^++e^-+\bar\nu_e\,,\quad \Sigma^-\rightarrow p+\pi^-+e^-+\bar\nu_e\,,$$
$$\Xi^0\rightarrow \Lambda+\pi^++e^-+\bar\nu_e\,,\quad \Sigma^-\rightarrow n+\pi^0+e^-+\bar\nu_e\,,$$
$$\Xi^-\rightarrow \Lambda+\pi^0+e^-+\bar\nu_e\,.$$

The possible counterparts of these modes where positrons instead of electrons could be emitted are forbidden since they need $\Delta S/\Delta Q=-1$ current. The covariant phase-space calculation for the four-body leptonic decay rate is given in detail in an Appendix.

[1] For an extensive list of references on current algebra methods, see B. Renner, Rutherford Laboratory Report No. RHEL/R-126, 1966 (unpublished); J. S. Bell, CERN Report No. 66-29 (unpublished); N. Cabibbo, in *Proceedings of the 12th International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, California, 1967).

II. THE MATRIX ELEMENTS

We are interested in the weak decays of the type

$$B(p)\rightarrow B'(p_1)+\pi(p_2)+e^-(p_3)+\bar\nu_e(p_4)\,. \tag{1}$$

B and B' are the initial and final baryons, and the p's are the four-momenta of the corresponding particles. The matrix elements for such a process is

$$\mathfrak{M}=\langle B'(p_1)\pi(p_2)e^-(p_3)\bar\nu_e(p_4)|H_w(0)|B(p)\rangle\,. \tag{2}$$

Using the usual current x current form for the weak Hamiltonian H_w and the Cabibbo form for the hadronic

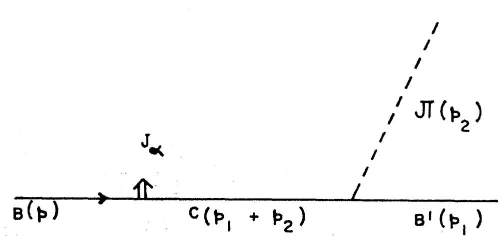

FIG. 1. Born diagram for $B_\alpha^{(1)}(p_2)$.

current, we have

$$\mathfrak{M}=\frac{G\sin\theta}{\sqrt{2}}\langle B'(p_1)\pi(p_2)|J_\alpha{}^{|\Delta S|=1}(0)|B(p)\rangle \times\bar{u}_e(p_3)\gamma_\alpha(1+\gamma_5)u_\nu(p_4)\,, \quad (3)$$

where θ is the Cabibbo angle and G is the universal Fermi coupling constant. For the hadronic part of the matrix element, we choose to write[2]

$$\langle B'(p_1)\pi(p_2)|J_\alpha{}^A(0)|B(p)\rangle=\bar{u}_{B'}(p_1)\times[a\gamma_\alpha+c(\gamma\cdot p_2)\gamma_\alpha+e\gamma_\alpha(\gamma\cdot p_2)]u_B(p)\,,$$
$$\langle B'(p_1)\pi(p_2)|J_\alpha{}^V(0)|B(p)\rangle=\bar{u}_{B'}(p_1)\times[b\gamma_\alpha+d\gamma_\alpha(\gamma\cdot p_2)+f(\gamma\cdot p_2)\gamma_\alpha]\gamma_5u_B(p)\,, \quad (4)$$

where J^V and J^A are the vector and axial vector currents, respectively. The coefficients $a,b,\cdots$ are functions of the invariants that can be constructed from the momenta p, p_1, and p_2. The explicit expressions for these coefficients are obtained in the next section. Combining Eqs. (3) and (4) and performing the sum over spins, we obtain

$$\sum_{\text{spins}}|\mathfrak{M}|^2\equiv|\bar{\mathfrak{M}}|^2=(32\beta^2/\eta\eta_1\eta_2\eta_3\eta_4)\times(A+B+C+D+E+F)\,, \quad (5)$$

where

$$A=(a^2+b^2)[(p\cdot p_3)(p_1\cdot p_4)+(p\cdot p_4)(p_1\cdot p_3)]-(a^2-b^2)mm_1(p_3\cdot p_4)\,,$$
$$B=(c^2+d^2)\{2(p_1\cdot p_2)[(p\cdot p_3)(p_2\cdot p_4)+(p\cdot p_4)(p_2\cdot p_3)]-m_2{}^2[(p\cdot p_3)(p_1\cdot p_4)+(p\cdot p_4)(p_1\cdot p_3)]\}-(c^2-d^2)mm_1m_2{}^2(p_3\cdot p_4)\,,$$
$$C=(e^2+f^2)\{2(p\cdot p_2)[(p_1\cdot p_3)(p_2\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)]-m_2{}^2[(p\cdot p_3)(p_1\cdot p_4)+(p\cdot p_4)(p_1\cdot p_3)]\}-(e^2-f^2)mm_1m_2{}^2(p_3\cdot p_4)\,,$$
$$D=2(ac+bd)m_1[(p\cdot p_3)(p_2\cdot p_4)+(p\cdot p_4)(p_2\cdot p_3)]-2(ac-bd)m[(p_1\cdot p_2)(p_3\cdot p_4)]\,,$$
$$E=-2(ae+bf)m_1[(p\cdot p_2)(p_3\cdot p_4)]+2(ae-bf)m[(p_1\cdot p_3)(p_2\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)]\,,$$
$$F=2(ce+df)\{(p_2\cdot p_3)[2(p\cdot p_1)(p_2\cdot p_4)-(p\cdot p_2)(p_1\cdot p_4)-(p\cdot p_4)(p_1\cdot p_2)]-(p_2\cdot p_4)[(p\cdot p_3)(p_1\cdot p_2)+(p\cdot p_2)(p_1\cdot p_3)]+m_2{}^2[(p\cdot p_3)(p_1\cdot p_4)+(p\cdot p_4)(p_1\cdot p_3)-(p\cdot p_1)(p_3\cdot p_4)]\}+4(ce-df)mm_1(p_2\cdot p_3)(p_2\cdot p_4)\,.$$

The η's are the usual normalization factors[3] and $\beta=(G\sin\theta)/\sqrt{2}$. In expression (5) we have neglected interference terms like ab, cd, $\cdots$, since, being odd under parity, these do not contribute to the total decay rate.

III. DETERMINATION OF THE COEFFICIENTS a, b, $\cdots$

We define the invariant amplitude for the semileptonic matrix element ($B\to B'+\pi$) by

$$M_\alpha(B\to B'+\pi)=(-i)\langle B'(p_1)\pi(p_2)|J_\alpha(0)|B(p)\rangle\,. \quad (6)$$

Using the standard reduction technique, PCAC, and performing a partial integration, one obtains the familiar relation[4]

$$\lim_{p_2\to 0}[p_{2\mu}T_{\mu,\alpha}+f_\pi M_\alpha]=i\langle B(p_1)|[F_5(0),J_\alpha(0)]|B(p)\rangle\,, \quad (7)$$

where f_π is the well-known PCAC coefficient, defined by

$$i\partial_\mu A_\mu{}^i(x)=m_\pi{}^2f_\pi\phi^i(x)\,,$$
$$f_\pi=-\frac{(m_B+m_{B'})g_A{}^{B\to B'}}{G(\bar{B}'B\pi)}\,, \quad (8)$$

and

$$F_5{}^i(0)=\int d^3x\,A_0{}^i(x,0)\,,$$
$$T_{\mu,\alpha}\equiv T_{\mu,\alpha}{}^{(1)}+T_{\mu,\alpha}{}^{(2)}=\int d^4x\,e^{ip_2\cdot x}\theta(x_0)\times\langle B'(p_1)|[A_\mu(x),J_\alpha(0)]|B(p)\rangle\,. \quad (9)$$

$T^{(1)}$ and $T^{(2)}$ correspond to the two terms of the commutator in Eq. (9). As is well known, if there does not exist any pole term in $T_{\mu,\alpha}$ as $p_2\to 0$, the left-hand side of Eq. (7) reduces to $f_\pi M_\alpha(p_2\to 0)$. To judge the presence or absence of pole terms in $T_{\mu,\alpha}$ we insert a complete set of states between the commutator in Eq. (9). If the intermediate state C or D (C is the intermediate state in $T^{(1)}$ and D in $T^{(2)}$) is degenerate in mass with the external baryons (B or B'), then there will be a pole in $T_{\mu,\alpha}$ as $p_2\to 0$. In this case the ambiguous term of $p_{2\mu}T_{\mu,\alpha}$ is exactly cancelled by that of

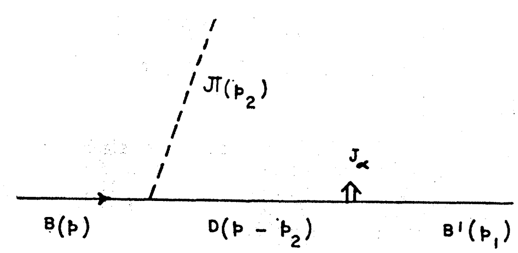

FIG. 2. Born diagram for $B_\alpha{}^{(2)}(p_2)$.

[2] Out of the many possible terms which could be written down on general arguments we have kept only those which are nonvanishing in the soft-pion limit. Normalization factors are suppressed. Notations: γ_0 and γ_5 are Hermitian; γ_1, γ_2, γ_3, are anti-Hermitian; the scalar product $a\cdot b=a_0b_0-\mathbf{a}\cdot\mathbf{b}$.

[3] The normalization factor is $[\eta/(2\pi)^3 2E]^{1/2}$, where $\eta=2m$ for a fermion and $\eta=1$ for a boson (see Appendix).

[4] V. A. Alessandrini, M. A. B. Bég, and L. S. Brown, Phys. Rev. **144**, 1137 (1967).

the Born term $f_\pi B_\alpha(p_2\to 0)$ [the Born terms are defined by Figs. 1 and 2], so that the sum of the two is well defined. On the other hand, if the intermediate state C or D is not degenerate in mass with the external baryons, then there exists no pole term in $T_{\mu,\alpha}$ in the limit $p_2\to 0$, so that $p_{2\mu}T_{\mu,\alpha}\to 0$, but, in this case the Born term itself is well defined. Following Brown and Sommerfield,[5] we split M_α into a sum of Born term B_α and a remainder term R_α defined by

$$M_\alpha(p_2)=B_\alpha(p_2)+R_\alpha(p_2). \tag{10}$$

We shall assume that the remainder term $R_\alpha(p_2)$ varies slowly with p_2 so that its physical value is well approximated by $R_\alpha(p_2=0)$. Combining Eqs. (6), (7), and (10) we can write

$$\langle B'(p_1)\pi(p_2)|J_\alpha|B(p)\rangle=\frac{-1}{f_\pi}\{\langle B'(p_1)|[F_5,J_\alpha(0)]|B(p)\rangle+(i)K_\alpha(p_2\to 0)+(-i)f_\pi B_\alpha(p_2)\}, \tag{11}$$

where

$$K_\alpha(p_2\to 0)=\lim_{p_2\to 0}[p_{2\mu}T_{\mu,\alpha}+f_\pi B_\alpha(p_2)]. \tag{12}$$

We emphasize that the result for $K_\alpha(p_2\to 0)$ is independent of whether or not the intermediate states in $T_{\mu,\alpha}$ and B_α are degenerate in mass with the external baryons. Evaluating the various terms of Eq. (11) we have the following results:

$$\langle B'(p_1)|[F_5,J_\alpha{}^{V,A}]|B(p)\rangle=xg_{A,V}{}^{B\to B'}\bar{u}_{B'}(p_1)\gamma_\alpha\begin{pmatrix}\gamma_5\\1\end{pmatrix}u_B(p), \tag{13}$$

$$\begin{aligned}
&K_\alpha=K_\alpha{}^{(1)}+K_\alpha{}^{(2)},\\
&K_\alpha{}^{(1)V,A}(p_2\to 0)=+ig_{V,A}{}^{B\to C}g_A{}^{C\to B'}\bar{u}_{B'}(p_1)\gamma_\alpha\begin{pmatrix}\gamma_5\\1\end{pmatrix}u_B(p),\\
&K_\alpha{}^{(2)V,A}(p_2\to 0)=-ig_A{}^{B\to D}g_{V,A}{}^{D\to B'}\bar{u}_{B'}(p_1)\gamma_\alpha\begin{pmatrix}\gamma_5\\1\end{pmatrix}u_B(p),
\end{aligned} \tag{14}$$

$$\begin{aligned}
&B_\alpha=B_\alpha{}^{(1)}+B_\alpha{}^{(2)},\\
&f_\pi B_\alpha{}^{(1)V,A}(p_2)=+ig_{V,A}{}^{B\to C}g_A{}^{C\to B'}\bar{u}_{B'}(p_1)\left[\frac{(m_1{}^2-m_c{}^2)\gamma_\alpha+(m_1+m_c)(\gamma\cdot p_2)\gamma_\alpha}{(m_1{}^2-m_c{}^2)+2p_1\cdot p_2+m_2{}^2}\right]\begin{pmatrix}\gamma_5\\1\end{pmatrix}u_B(p),\\
&f_\pi B_\alpha{}^{(2)V,A}(p_2)=-ig_A{}^{B\to D}g_{V,A}{}^{D\to B'}\bar{u}_{B'}(p_1)\left[\frac{(m^2-m_d{}^2)\gamma_\alpha\pm(m+m_d)\gamma_\alpha(\gamma\cdot p_2)}{(m^2-m_d{}^2)-2p\cdot p_2+m_2{}^2}\right]\begin{pmatrix}\gamma_5\\1\end{pmatrix}u_B(p),
\end{aligned} \tag{15}$$

where

$$\begin{aligned}x&=1\text{ for }\pi^+\\&=0\text{ for }\pi^-\\&=1/\sqrt{2}\text{ for }\pi^0.\end{aligned} \tag{16}$$

m_c and m_d are the masses of the intermediate single-particle states in the two Born diagrams, Figs. 1 and 2. We have made use of the Goldberger-Treiman relation [Eq. (8)] to write the strong-coupling constants $G(\bar{\alpha}\beta\pi)$ in terms of the corresponding axial-vector coupling constants $g_A{}^{\alpha\to\beta}$. The g_A's and g_V's have the standard definitions

$$\lim_{(p-p')\to 0}\langle\beta(p')|J_\alpha{}^{V,A}(0)|\alpha(p)\rangle=\left(\frac{m_\alpha m_\beta}{E_\alpha E_\beta}\right)^{1/2}g_{V,A}{}^{\alpha\to\beta}\bar{u}_\beta(p')\gamma_\alpha\begin{pmatrix}1\\\gamma_5\end{pmatrix}u_\alpha(p). \tag{17}$$

$g_V{}^{n\to p}\simeq 1.0$ and $g_A{}^{n\to p}\simeq 1.18$. All the other g's are related to these two by $SU(3)$ invariance. From Eqs. (13), (14), and (15) it is easy to derive the coefficients a, b, c, d, e, and f. They are tabulated in Table I. The coefficients $a,b,\cdots$ have an over-all common factor $(-1/f_\pi)$ [see Eq. (11)] which we include in the definition of β in Eq. (5), so that

$$\beta=-\frac{G\sin\theta}{\sqrt{2}f_\pi}\quad\text{and}\quad f_\pi=-\frac{\sqrt{2}m_N g_A{}^{n\to p}}{g_{NN\pi}}. \tag{18}$$

IV. RESULTS

As mentioned earlier, all the g_V's and g_A's are related by $SU(3)$ symmetry. The g_V's have only the f-type coupling, while the g_A's involve both f- and d-type couplings. From the leptonic decays of hyperons it seems that the ratio $d/f\simeq 1.95$ gives a reasonable fit to the experimental data.[6] Using $a,b,\cdots$ from Table I we have calculated the decay rates for the seven energetically possible four-body leptonic decays. Values of the various parameters used are the following: (a) Cabibbo angle $\theta=0.26$; (b) $g_V{}^{n\to p}=1.0$ and $g_A{}^{n\to p}=1.18$; (c) three different values for the ratio d/f, namely, 1.68 1.95, and 2.3 (the results are insensitive to the variations

[5] L. S. Brown and C. Sommerfield, Phys. Rev. Letters **16**, 751 (1966).

[6] N. Brene *et al.*, Phys. Rev. **149**, 1288 (1966).

TABLE I. The contributions to the various terms of the semileptonic hadronic matrix element, Eq. (4), obtained from the current algebra relation, Eq. (11). x is defined in Eq. (16).

Contribution from	Coeff. of γ_α	Coeff. of $\gamma_\alpha\gamma_5$
Equal-time commutator $(i)K_\alpha(p_2\to 0)$	$x g_A^{B\to B'} - g_A^{B\to C} g_A^{C\to B'} + g_A^{B\to D} g_A^{D\to B'}$	$x g_V^{B\to B'} - g_V^{B\to C} g_A^{C\to B'} + g_A^{B\to D} g_V^{D\to B'}$
$(-i)f_\pi B_\alpha^{(1)}(p_2)$	$g_A^{B\to C} g_A^{C\to B'}\left[\dfrac{m_1^2-m_c^2}{m_1^2-m_c^2+2p_1\cdot p_2+m_2^2}\right]$	$g_V^{B\to C} g_A^{C\to B'}\left[\dfrac{m_1^2-m_c^2}{m_1^2-m_c^2+2p_1\cdot p_2+m_2^2}\right]$
$(-i)f_\pi B_\alpha^{(2)}(p_2)$	$-g_A^{B\to D} g_A^{D\to B'}\left[\dfrac{m^2-m_d^2}{m^2-m_d^2-2p\cdot p_2+m_2^2}\right]$	$-g_A^{B\to D} g_V^{D\to B'}\left[\dfrac{m^2-m_d^2}{m^2-m_d^2-2p\cdot p_2+m_2^2}\right]$
	Coeff. of $(\gamma\cdot p_2)\gamma_\alpha$	Coeff. of $(\gamma\cdot p_2)\gamma_\alpha\gamma_5$
$(-i)f_\pi B_\alpha^{(1)}(p_2)$	$g_A^{B\to C} g_A^{C\to B'}\left[\dfrac{m_1+m_c}{m_1^2-m_c^2+2p_1\cdot p_2+m_2^2}\right]$	$g_V^{B\to C} g_A^{C\to B'}\left[\dfrac{m_1+m_c}{m_1^2-m_c^2+2p_1\cdot p_2+m_2^2}\right]$
	Coeff. of $\gamma_\alpha(\gamma\cdot p_2)$	Coeff. of $\gamma_\alpha(\gamma\cdot p_2)\gamma_5$
$(-i)f_\pi B_\alpha^{(2)}(p_2)$	$g_A^{B\to C} g_A^{C\to B'}\left[\dfrac{m+m_d}{m^2-m_d^2-2p\cdot p_2+m_2^2}\right]$	$g_A^{B\to D} g_V^{D\to B'}\left[\dfrac{m+m_d}{m^2-m_d^2-2p\cdot p_2+m_2^2}\right]$

of this ratio; (d) physical masses for all the external and intermediate hadrons; and (e) other constants: $G=1.0\times10^{-5}/m_p^2$, $g_{NN\pi}^2/4\pi=14.6$. The calculated decay rates are given in Table II.

The small decay rates for such modes can be understood in terms of the small phase space available and cancellations[7] between the various terms in the matrix elements. For instance, in $\Sigma^-\to p+\pi^-+e^-+\bar{\nu}_e$ decay, each one of the two Born terms will give a decay rate about 25–30 times larger than the one obtained here. But the two together almost cancel each other, leaving a small net contribution. In conclusion, the predicted rates appear to be rather small, so that the observation of four-body leptonic decays of hyperons (even for the most favorable case) will not be feasible unless the number of hyperon decay events is increased by at least two orders of magnitude.

TABLE II. The calculated four-body leptonic decay rates of hyperons.

Decay mode	Decay rate $\Gamma_{B\to B'+\pi+e^-+\bar{\nu}_e}$ (sec^{-1}) $d/f=1.68$	$d/f=1.95$	$d/f=2.28$	Experimental total decay rate (sec^{-1})
$\Lambda\to p+\pi^0+e^-+\bar{\nu}_e$	0.47	0.46	0.45	4.0×10^9
$\Lambda\to n+\pi^++e^-+\bar{\nu}_e$	0.35	0.35	0.34	
$\Sigma^+\to p+\pi^++e^-+\bar{\nu}_e$	4.52×10^2	4.68×10^2	4.83×10^2	1.3×10^{10}
$\Sigma^-\to n+\pi^0+e^-+\bar{\nu}_e$	4.72×10^2	4.87×10^2	5.05×10^2	6.2×10^9
$\Sigma^-\to p+\pi^-+e^-+\bar{\nu}_e$	1.35×10^2	1.40×10^2	1.47×10^2	
$\Xi^0\to\Lambda+\pi^++e^-+\bar{\nu}_e$	1.01×10^1	1.01×10^1	$1.0\ \times10^1$	3.4×10^9
$\Xi^-\to\Lambda+\pi^0+e^-+\bar{\nu}_e$	1.61×10^1	1.62×10^1	1.63×10^1	5.7×10^9

[7] We note that our expression, Eq. (11), for the amplitude in terms of the Born terms is a consequence of soft-pion extrapolation together with current algebra. As it is not clear how well the amplitude is approximated by the soft-pion limit, it is not certain that the smallness of the amplitude is a feature independent of extrapolation. However, we feel that the cancellations may not be related to the errors involved in the soft-pion extrapolation since, in a similar analysis of nonleptonic decays of hyperons, [A. Kumar and J. C. Pati, Phys. Rev. Letters **18**, 1230 (1967)] large cancellations between the Born terms produce results which are in very good agreement with the experiment.

ACKNOWLEDGMENT

The author is very much indebted to Professor J. C. Pati for suggesting this investigation and for his guidance during the course of this work.

APPENDIX

Decay Rate and Phase-Space Integrations

The decay rate for the process (1), in the rest frame of the decaying particle, is given by

$$\Gamma=(2\pi)^4\int\frac{\eta_1 d^3P_1}{(2\pi)^3 2E_1}\frac{\eta_2 d^3P_2}{(2\pi)^3 2E_2}\frac{\eta_3 d^3P_3}{(2\pi)^3 2E_3}\frac{\eta_4 d^3P_4}{(2\pi)^3 2E_4}\times\delta^4(p-p_1-p_2-p_3-p_4)|\mathfrak{M}|^2.\quad\text{(A1)}$$

Using Eq. (5) for $|\mathfrak{M}|^2$, we have

$$\Gamma=\frac{1}{(2\pi)^8}\left(\frac{\beta^2}{m}\right)\int\frac{d^3P_1}{E_1}\frac{d^3P_2}{E_2}\frac{d^3P_3}{E_3}\frac{d^3P_4}{E_4}\times\delta^4(p-p_1-p_2-p_3-p_4)\times(A+B+C+D+E+F).\quad\text{(A2)}$$

$A,B,\cdots$ are given in Eq. (5) and β is given in Eq. (18). Integrations over d^3P_3 and d^3P_4 can be easily performed by using the standard covariant techniques.[8] The end result of these two integrations for the matrix element

[8] See, for example, J. D. Jackson, Brandeis Lectures Vol. 1, (1962); D. Loebbaka, Ph.D. thesis, University of Maryland, 1966 (unpublished).

given by Eq. (5) can be stated as follows:

$$\int \frac{d^3P_3}{E_3}\frac{d^3P_4}{E_4}\delta^4(N-p_3-p_4)[(q\cdot p_3)(s\cdot p_4)] = \tfrac{1}{6}\pi[2(q\cdot N)(s\cdot N)+(q\cdot s)N^2],$$

$$\int \frac{d^3P_3}{E_3}\frac{d^3P_4}{E_4}\delta^4(N-p_3-p_4)[(p_3\cdot p_4)]=\pi N^2. \quad \text{(A3)}$$

q and s are four-vectors independent of p_3 and p_4 and $N=p-p_1-p_2$. Here we have neglected the mass of the electron and have put $m_\nu=0$. Making use of Eq. (A3) in Eq. (A2) to take account of the d^3P_3 and d^3P_4 integrations, we obtain

$$\Gamma=\frac{1}{2(2\pi)^7}\left(\frac{\beta^2}{m}\right)\int \frac{d^3P_1}{E_1}\frac{d^3P_2}{E_2} \times(\bar{A}+\bar{B}+\bar{C}+\bar{D}+\bar{E}+\bar{F}), \quad \text{(A4)}$$

where

$$\bar{A}=(a^2+b^2)(\tfrac{1}{3})[2(p\cdot N)(p_1\cdot N)+(p\cdot p_1)N^2]-(a^2-b^2)mm_1N^2,$$

$$\bar{B}=(c^2+d^2)(\tfrac{1}{3})\{2(p_1\cdot p_2)[2(p\cdot N)(p_2\cdot N)+(p\cdot p_2)N^2]-m_2^2[2(p\cdot N)(p_1\cdot N)+(p\cdot p_1)N^2]\}-(c^2-d^2)mm_1m_2^2N^2,$$

$$\bar{C}=(e^2+f^2)(\tfrac{1}{3})\{2(p\cdot p_2)[2(p_1\cdot N)(p_2\cdot N)+(p_1\cdot p_2)N^2]-m_2^2[2(p\cdot N)(p_1\cdot N)+(p\cdot p_1)N^2]\}-(e^2-f^2)mm_1m_2^2N^2,$$

$$\bar{D}=2(ac+bd)(\tfrac{1}{3})m_1[2(p\cdot N)(p_2\cdot N)+(p\cdot p_2)N^2]-2(ac-bd)m(p_1\cdot p_2)N^2,$$

$$\bar{E}=-2(ae+bf)m_1(p\cdot p_2)N^2+2(ae-bf)(\tfrac{1}{3})m[2(p_1\cdot N)(p_2\cdot N)+(p_1\cdot p_2)N^2],$$

$$\bar{F}=2(ce+df)(\tfrac{2}{3})\{(p\cdot p_1)[(p_2\cdot N)^2-\tfrac{1}{2}m_2^2N^2]-(p\cdot p_2)[(p_1\cdot N)(p_2\cdot N)+(p_1\cdot p_2)N^2] -(p\cdot N)[(p_1\cdot p_2)(p_2\cdot N)-m_2^2(p_1\cdot N)]\}+2(ce-df)(\tfrac{1}{3})mm_1[2(p_2\cdot N)^2-m_2^2N^2].$$

To perform the d^3P_2 and d^3P_1 integrations, we choose the rest frame of the decaying particle and fix P_2 along the z axis and P_1 along any (θ,ϕ) direction. Since our integrand has no ϕ dependence, integration over $d\phi$ gives a factor of (2π). Now we can choose P_1 in the x-z plane, so that the momentum vectors are

$$p=(m,0);$$
$$p_2=(E_2,P_2);\quad P_2 \text{ is along the } z \text{ axis};$$
$$p_1=(E_1,P_1);\quad P_1 \text{ is in the } x\text{-}z \text{ plane}, \quad \text{(A5)}$$

and

$$\frac{P_1\cdot P_2}{|P_1||P_2|}=\cos\theta\equiv x.$$

Using the relation

$$d^3x=|x|^2dxd\Omega_x, \quad \text{(A6)}$$

we can write

$$I\equiv\int\frac{d^3P_2}{E_2}\frac{d^3P_1}{E_1}=(2\pi)\int d\Omega_2\frac{|P_2|^2dP_2}{E_2}\frac{|P_1|^2dP_1}{E_1}dx. \quad \text{(A7)}$$

Since there is no preferred direction for P_2, integrating over $d\Omega_2$ gives another factor of 4π.

$$I=(2\pi)(4\pi)\int\frac{|P_2|^2dP_2}{E_2}\frac{|P_1|^2dP_1}{E_1}dx. \quad \text{(A8)}$$

Choosing dE_2 and dE_1 as our new integration variables, we have

$$I=(2\pi)(4\pi)\int|P_2|dE_2|P_1|dE_1dx. \quad \text{(A9)}$$

Now, using Eq. (A9) in Eq. (A4), we obtain

$$\Gamma=\frac{1}{(2\pi)^5}\left(\frac{\beta^2}{m}\right)\int_{m_2}^{E_2^{\max}}|P_2|dE_2\int_{E_1^{\min}}^{E_2^{\max}}|P_1|dE_1\int_{-1}^{x^{\max}}dx \times(\bar{A}+\bar{B}+\bar{C}+\bar{D}+\bar{E}+\bar{F}). \quad \text{(A10)}$$

Limits of Integrations

The minimum value that E_2 can have is obviously m_2. The maximum value of E_2 will occur when all the other particles have their momenta directed against P_2 and are moving with the same velocity. Energy-momentum conservation then gives

$$(P_2^2+m_2^2)^{1/2}+[P_2^2+(m_1+m_3+m_4)^2]^{1/2}=m. \quad \text{(A11)}$$

Solving for E_2 we get

$$E_2=(P_2^2+m_2^2)^{1/2}=[m^2-(m_1+m_3+m_4)^2+m_2^2]/2m.$$

Neglecting the electron mass, the upper limit for dE integration is therefore

$$E_2^{\max}=(m^2-m_1^2+m_2^2)/2m. \quad \text{(A12)}$$

For a fixed value of momentum P_2 the energy E_1 will have its maximum value when all the other momenta are directed opposite to P_1. In such a situation

$$P_1+P_2=-Q,\quad \cos\theta=-1;\quad m=E_1+E_2+|Q|. \quad \text{(A13)}$$

Q is the momentum of the electron and neutrino, $Q=P_3+P_4$. Solving Eq. (A13) for P_1, we get

$$|P_1^{\max}|=\frac{\alpha|P_2|+\gamma[\alpha^2-4m_1^2(\gamma^2-P_2^2)]^{1/2}}{2(\gamma^2-P_2^2)}, \quad \text{(A14)}$$

where $\gamma=m-E_2$ and $\alpha=\gamma^2+m_1^2-P_2^2$. In determining $E_2^{\max}$ and $P_1^{\max}$ we have used $x=-1$; this, being the minimum value of x, is the lower limit of the dx integration. The upper limit for x can be obtained from

$$P_2+P_1\cos\theta+Q\cos\phi=0,$$
$$P_1\sin\theta+Q\sin\phi=0,$$
$$E_1+E_2+E_3+E_4=m,$$
$$E_3+E_4\geqslant|Q|, \quad \text{(A15)}$$

where ϕ is the angle between Q and P_2. Eliminating ϕ from the first two equations, we get

$$x \equiv \cos\theta = (Q^2 - P_1^2 - P_2^2)/2|P_1||P_2|. \quad \text{(A16)}$$

For fixed magnitude of P_1 and P_2, x will be maximum when $|Q|$ is maximum. $|Q|$ will be maximum when P_3 and P_4 are parallel, so that

$$|Q|^{\max} = E_3 + E_4 = m - E_1 - E_2. \quad \text{(A17)}$$

This gives

$$x^{\max} = [(m - E_1 - E_2)^2 - P_1^2 - P_2^2]/2|P_1||P_2|. \quad \text{(A18)}$$

For some values of P_1 and P_2, Eq. (A17) may give an overestimate for $|Q|$ in which case we shall limit $x^{\max}$ to $+1$. From Eq. (A18) we can see that for fixed value of P_2 the momentum P_1 cannot be zero as this will imply unphysical value of $x^{\max}$. Thus the minimum value for P_1 will be decided by the condition that $x^{\max}$ be not less than -1.

With the limits so determined, the three integrations in Eq. (A10) were performed numerically on the computer to obtain the decay rates given in Table II.

PHYSICAL REVIEW VOLUME 161, NUMBER 5 25 SEPTEMBER 1967

Strong and Weak Decays with Meson Emission in the Quark Model

HARRY J. LIPKIN, H. R. RUBINSTEIN, AND H. STERN*
Weizmann Institute of Science, Rehovoth, Israel
(Received 3 April 1967)

Strong, electromagnetic, and weak decays are described in the independent-quark model as single quark transitions with no symmetry higher than isospin assumed for the interaction. Relations between quark model and symmetry predictions, e.g., $SU(6)_W$, and the possibility of experimental tests are discussed. Strong decays of a three-quark baryon supermultiplet with $L=2$ are treated in detail. Many predictions are given which can provide tests of symmetries and of the model when more data are available. Present data indicate rough agreement, but are inconclusive.

IN the same manner that electromagnetic decays are described in the "independent-quark model"[1,2] by considering that each quark in the initial state can radiate a photon, one can also consider a model in which each quark individually can "radiate" a meson to describe certain decays. There is no inconsistency in treating a meson as a "radiated quantum" rather than as a quark-antiquark bound state. Although a justification of this model from first principles is out of the question at present, the general approach is similar to that used in conventional treatment of "soft pion emission." This treatment is based on very general considerations which are independent of the structure of the pion and are consistent with a composite structure.[3]

The transition amplitude is expressed in terms of fundamental quark-quark-meson vertex functions. These can either be left as completely free parameters, or related to one another by symmetries such as isospin and $SU(3)$. In the present treatment we assume isospin symmetry but do not assume $SU(3)$ symmetry for the quark-quark-meson vertex.[4] The conventional $SU(6)$ wave functions, including orbital excitation, are assumed for the mesons and baryons.[1] We consider only decays involving the emission of π and K mesons. For this case the quark-model predictions can be shown to be given as follows[5]:

All decays involving the emission of a pion are related to one another in exactly the same way as required by collinear $SU(6)_W$ symmetry without the quark model. The same is true for all decays involving the emission

* Present address: Centre d'Etudes Nucléaires de Saclay, Gif-sur-Yvette, France.

[1] Y. Nambu, in *Proceedings of the Second Coral Gables Conference on Symmetry Principles at High Energy*, edited by B. Kursunoglu, A. Perlmutter, and I. Sakmar (W. H. Freeman and Company, San Francisco, 1965), p. 274; G. Morpurgo, Physics **2**, 95 (1965); C. Becchi and G. Morpurgo, Phys. Rev. **140**, B687 (1965). Y. Anisovich, A. Anselm, Y. Azimov, G. Danilov, and I. Dyatlov, Phys. Letters **16**, 194 (1965); R. H. Dalitz, in *Proceedings of the XIII International Conference on High-Energy Physics, Berkeley, 1967* (University of California Press, Berkeley, 1967); W. Thirring, Acta Phys. Austriaca Suppl. III, 294 (1966).

[2] H. J. Lipkin, in *Proceedings of the Yalta International School on High-Energy Physics and Elementary Particles, 1966*, edited by V. P. Shelest (Naukova Dumka, Kiev, 1967), p. 639; H. J. Lipkin, F. Scheck, and H. Stern, Phys. Rev. **152**, 1375 (1966).

[3] This point is discussed in detail by R. Van Royen and V. F. Weisskopf, Nuovo Cimento **50A**, 617 (1967).

[4] R. J. Rivers, [Phys. Letters **22**, 514 (1966)]; A. N. Mitra and M. Ross [Phys. Rev. (to be published)] have calculated meson emission for certain baryon states using a detailed model. J. Uretsky [to be published in High Energy Theoretical Physics, edited by Hadi Aly, Beirut] has calculated meson decays with a similar method. Our approach differs from theirs in using only symmetry properties of the quark-model wave function and not assuming $SU(3)$ symmetry for the quark meson vertex. We therefore obtain general results which are independent of the particular details of the model, and which must also be obtained in any specific model such as those above which satisfies our general symmetry assumptions.

[5] H. J. Lipkin, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy*, edited by B. Kurşunoglu, A. Perlmutter, and I. Sakmar (W. H. Freeman and Company, San Francisco, California, 1966), p. 96; Nucl. Phys. **B1**, 597 (1967).