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Phenomenological Model of Strong and Weak Interactions in Chiral $U(3)\otimes U(3)$*

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A phenomenological model for the strong and weak interactions of an octet or nonet of pseudoscalar mesons in chiral $U(3)\otimes U(3)$ is constructed and discussed. In this model one can study the effect that the partially conserved axial-vector current hypothesis (PCAC) and the transformation properties of the interaction Lagrangian in chiral $U(3)\otimes U(3)$ have on the transition amplitudes for various meson processes. The processes considered in this paper are the leptonic and nonleptonic decays of K mesons, the strong interaction decay $\eta'(959)\to\eta(549)+2\pi$, the s- and p-wave scattering lengths for pion-nucleon scattering, and meson-meson scattering. Low-energy pion-pion scattering is discussed as an illustration of the fact that for processes involving more than one soft pion, PCAC and the algebra of currents are not sufficient to give unique results in general.

I. INTRODUCTION

THE recent success of the combination of the concept of a partially conserved axial-vector current (PCAC) together with the algebra of vector and axial-vector currents in $SU(3)$ suggests that the chiral group $U(3)_L\otimes U(3)_R$ may be a good symmetry group in which to formulate a dynamics of the strong and weak interactions. This is illustrated in this paper by the construction of a phenomenological model for the strong and weak interactions of a nonet of pseudoscalar mesons based upon this chiral group. The amplitudes for various strong- and weak-interaction processes will be calculated and shown to agree with those obtained with current algebra techniques. These results are also in good agreement with experiment.

In Secs. II–IV, a phenomenological model of strong-interaction pseudoscalar-meson dynamics is constructed which is essentially the extension to $U(3)_L\otimes U(3)_R$ of an approach discussed by Gürsey[1] in the context of $U(2)_L\otimes U(2)_R$. The primary ingredient is the construction of a meson-coupling matrix $M_i{}^j(\Phi)$ as a function of the 3×3 pseudoscalar-meson matrix Φ. This coupling matrix is defined to transform according to the representation $(3_L,3_R{}^*)$ of $U(3)_L\otimes U(3)_R$. This function is not unique, the only constraints being

$$M^\dagger M=I \quad\text{and}\quad M^\dagger(\Phi)=M(-\Phi).$$

Two forms of M which seem to be of special significance are discussed. In particular, it is found that only with M equal to $e^{2if\Phi}$ is it possible to have only eight pseudoscalar mesons in chiral $SU(3)\otimes SU(3)$. Explicit expressions for the meson part of the vector and axial-vector currents which appear in the weak interactions are derived and PCAC is discussed.

In Sec. V this model is applied to meson-meson scattering and found to be identical to results obtained from the current algebra and PCAC. A calculation of the rate for the strong-interaction decay of the η' (also known as the X^0) to $\eta+2\pi$ is given and found to be in approximate agreement with experiment, although slightly large.

In Secs. VI and VII, the vector and axial-vector currents of this model are applied to the leptonic and nonleptonic decays of K mesons. Of the nonleptonic decays, only those with $|\Delta I|=\frac{1}{2}$ and CP conserving are discussed. The interaction is taken to be of the current$\times$current form and transforming like the sixth component of $(8_L,1_R)$ under $U(3)_L\otimes U(3)_R$. The results are in agreement with experiment and those obtained with the algebra-of-currents method.

In Sec. VIII an invariant coupling of the pseudoscalar mesons to a nonet of baryons is constructed. The Goldberger-Treiman relation follows directly from this chiral invariant coupling. Pion-nucleon scattering is calculated in lowest order and shown to give excellent agreement with experiment for the s- and p-wave scattering lengths in all channels except that containing the $N^*(1236)$ resonance. The amplitude calculated from this coupling is also shown to be identical to that obtained from the conventional pseudovector coupling together with ρ exchange at low-momentum transfers, which is the result obtained from PCAC and the algebra of currents.

Finally, various aspects of this investigation are summarized and discussed.

II. CONSTRUCTION OF THE MODEL

In order to construct the transformation properties of an octet (or nonet) of pseudoscalar mesons under chiral $U(3)\otimes U(3)$, we consider a particular model.[1] This model is specified by a coupling of these mesons to a triplet of quarks given by

$$\mathcal{L}_{\text{int}}=-m_0\{\bar{q}_LM(f\Phi)q_R+\bar{q}_RM^\dagger(f\Phi)q_L\}. \tag{1}$$

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[1] K. Nishijima, Nuovo Cimento **11**, 698 (1959); F. Gürsey, *ibid.* **16**, 230 (1960); Ann. Phys. (N. Y.) **12**, 91 (1961).

In this expression, Φ is the 3×3 Hermitian pseudoscalar-meson matrix, while the coupling matrix M is expanded as a power series in Φ;

$$M=\sum a_n(if\Phi)^n. \qquad (2)$$

The expansion coefficients a_n are considered to be independent of Φ, and the parameter f with dimensions of $(\text{mass})^{-1}$ is chosen to be real. This parameter will be determined from the decay $\pi\to\mu+\nu$, where one finds $f\simeq m_\pi^{-1}$. The notation q_L (q_R) denotes a left-(right-) handed quark, i.e., $\gamma_5 q_L=q_L$ and $\gamma_5 q_R=-q_R$. The terms $\bar{q}_R M q_R$ and $\bar{q}_L M q_L$ do not appear in the coupling because $\bar{q}_{iR}q_{jR}\equiv\bar{q}_{iL}q_{jL}\equiv0$. The expansion given by Eq. (2) may represent either a polynomial or an infinite series. For example, one obtains a conventional Yukawa-type coupling by choosing only a_1 to be nonzero. However, as will be shown, this choice is not consistent with the symmetries demanded of the coupling.

In addition to the full Poincaré group (including parity and time-reversal invariances), this coupling is required to be invariant under charge conjugation and the group $U(3)_L\otimes U(3)_R$. This latter group is defined by the transformation properties of the quarks:

$$U(3)_L:\quad q_L\to e^{i\alpha_k\lambda_k/2}q_L \text{ and } q_R\to q_R, \qquad (3a)$$

$$U(3)_R:\quad q_L\to q_L \text{ and } q_R\to e^{i\beta_k\lambda_k/2}q_R. \qquad (3b)$$

The generators of these two groups define the algebra of $U(3)_L\otimes U(3)_R$;

$$[F_i^+,F_j^-]=0\,,$$
$$[F_i^\pm,F_j^\pm]=if_{ijk}F_k^\pm.$$

Under the parity transformation, we define

$$Pq_R(\mathbf{x},t)P^{-1}=\gamma_4 q_L(-\mathbf{x},t) \text{ and } Pq_L(\mathbf{x},t)P^{-1}=\gamma_4 q_R(-\mathbf{x},t)\,;$$

therefore, parity invariance requires that M satisfy the equation

$$PM[f\Phi(\mathbf{x},t)]P^{-1}=M^\dagger[f\Phi(-\mathbf{x},t)]. \qquad (4)$$

For pseudoscalar mesons $P\Phi(\mathbf{x},t)P^{-1}=-\Phi(-\mathbf{x},t)$; therefore, invariance requires the coefficients a_n in Eq. (2) to be real. Likewise, charge-conjugation invariance requires

$$CM_i{}^jC^{-1}=M_j{}^i.$$

For the pseudoscalar mesons we have $C\Phi_i{}^jC^{-1}=\Phi_j{}^i$, thus charge conjugation places no constraint on the coefficients a_n. Similarly, time-reversal invariance requires

$$TM[f\Phi(\mathbf{x},t)]T^{-1}=M[f\Phi(\mathbf{x},-t)].$$

Since T is an antiunitary operator and the a_n are real from parity invariance, time-reversal invariance may be satisfied by taking either

$$T\Phi(\mathbf{x},t)T^{-1}=-\Phi(\mathbf{x},-t) \quad \text{with } a_n \text{ arbitrary}$$

or

$$T\Phi(\mathbf{x},t)T^{-1}=\Phi(\mathbf{x},-t)\,,$$

with all $a_n=0$ for n being an odd integer. However, this latter choice is not compatible with the symmetry $U(3)_L\otimes U(3)_R$ because one cannot set all the a_n equal to zero for n being an odd integer. The reason is that invariance under chiral $U(3)\otimes U(3)$ is quite demanding on the coefficients a_n in so far as it requires the matrix M to be unitary.

In order to ensure invariance under chiral $U(3)\otimes(3)$, the mesons must transform in the following manner:

$$U(3)_L\colon\ \Phi\to\Phi',\quad \text{where}\quad M(f\Phi')=e^{i\alpha_k\lambda_k/2}M(f\Phi)\,; \qquad (5a)$$

$$U(3)_R\colon\ \Phi\to\Phi'',\ \text{where}\ M(f\Phi'')=M(f\Phi)e^{-i\beta_k\lambda_k/2}. \qquad (5b)$$

From Eq. (5) it follows that

$$M^\dagger(f\Phi')M(f\Phi')=M^\dagger(f\Phi)M(f\Phi) \qquad (6a)$$

and

$$M(f\Phi')M^\dagger(f\Phi')=e^{i\alpha_k\lambda_k/2}M(f\Phi)M^\dagger(f\Phi)e^{-i\alpha_k\lambda_k/2}. \qquad (6b)$$

With Φ and Φ' being Hermitian, it is easy to verify that $M^\dagger(f\Phi)$ commutes with $M(f\Phi)$ as does $M^\dagger(f\Phi')$ with $M(f\Phi')$. Thus Eqs. (6a) and (6b) imply that $M^\dagger M$ commutes with all λ_k which is only possible if $M^\dagger M$ is a multiple of the identity matrix.[2] We shall take M to be normalized such that $M^\dagger M=1$.

This unitarity restricts the allowed values of a_n in the expansion of M. Without loss of generality, we shall choose $a_0=1$, while the parameter a_1 may be absorbed into the definition of f. For convenience we shall choose $a_1=2$, then the expansion for M becomes

$$M(f\Phi)=1+2if\Phi+2(if\Phi)^2+a_3(if\Phi)^3 +2(a_3-1)(if\Phi)^4+\cdots. \qquad (7)$$

M is thus determined by two parameters, f and a_3 to fourth order in Φ.

Although Eq. (1) represents a certain model of meson-quark interactions which may or may not have something to do with reality, the important point is that the pseudoscalar mesons are contained in $M_i{}^j$ which transforms like the representation $(3_L,3_R^*)$ of $U(3)_L\otimes U(3)_R$. An effective Lagrangian will be constructed as a function of M which, when expanded in powers of f, will be used to calculate the S matrix for multiple-meson processes. We shall use this model only phenomenologically, calculating the amplitude for various meson processes to lowest order in f.

Although M belongs to the representation $(3_L,3_R^*)$, the pseudoscalar mesons by themselves do not belong

[2] The constraint that $M^\dagger M$ be a multiple of the identity may be used to show that one cannot construct the coupling given by Eq. (1) if scalar rather than pseudoscalar mesons are used. This follows from parity invariance which requires that $M^\dagger(f\Phi)=M(f\Phi)$ if Φ is a scalar field. Thus $M=\sum b_n f^n\Phi^n$ with real coefficients, b_n. The unitarity condition $M^\dagger M=cI$ can be satisfied only by choosing $b_0{}^2=c$ with all other $b_n=0$ for $n\neq0$. This shows that M is a multiple of the identity. Furthermore Eqs. (5a) or (5b) then requires that $M=0$.

to a linear representation of this group. This may be seen by solving Eqs. (5a) and (5b) for the infinitesimal transformations $\delta_L\Phi$ and $\delta_R\Phi$ where one substitutes $\Phi'=\Phi+\delta_L\Phi$ and $\Phi''=\Phi+\delta_R\Phi$ and solves to first order in α_k and β_k. It is more convenient, however, to work with the combinations $\delta_V\Phi\equiv\delta_L\Phi+\delta_R\Phi$ and $\delta_A\Phi\equiv\delta_L\Phi-\delta_R\Phi$ obtained from simultaneous transformation under $U(3)_L$ and $U(3)_R$ with $\alpha_k=\beta_k$ and $\alpha_k=-\beta_k$, respectively. One finds to lowest order in α_k that

$$M(f\Phi+f\delta_V\Phi)=M(f\Phi)+i\alpha_k[\tfrac{1}{2}\lambda_k,M(f\Phi)] \tag{8a}$$

and

$$M(f\Phi+f\delta_A\Phi)=M(f\Phi)+i\alpha_k\{\tfrac{1}{2}\lambda_k,M(f\Phi)\}_+. \tag{8b}$$

The solution to Eq. (8) is easily found to be

$$\delta_V\Phi=i\alpha_k[\tfrac{1}{2}\lambda_k,\Phi], \tag{9}$$

which shows that under ordinary $SU(3)$ [where both left and right quarks undergo the same $SU(3)$ transformation] the mesons transform like an octet and a singlet.

The solution for $\delta_A\Phi$ is more difficult to obtain and is most easily calculated in a basis in which Φ is diagonal (with eigenvalues x_i)

$$\delta_A\Phi_i{}^j=\tfrac{1}{2}i\alpha_k(\lambda_k')_{ij}\left[\frac{M(fx_i)+M(fx_j)}{M(fx_i)-M(fx_j)}\right](x_i-x_j). \tag{10}$$

One may then expand this expression in powers of f and express the result in an arbitrary basis. Thus,

$$\delta_A\Phi=\frac{\alpha_k}{2f}\{\lambda_k-f^2(\Phi^2\lambda_k+\lambda_k\Phi^2)+\tfrac{1}{2}f^2a_3(\lambda_k\Phi^2+\Phi\lambda_k\Phi+\Phi^2\lambda_k)+\text{terms of order }(f\Phi)^4\}. \tag{11}$$

III. PROPERTIES OF VARIOUS FORMS FOR M

Except for the fact that M must be unitary there is a great deal of freedom in the form used for M. We have found the following two forms to be particularly interesting:

$$e^{2ib\Phi} \tag{12a}$$

$$\frac{1+if\Phi}{1-if\Phi}. \tag{12b}$$

From Eq. (11) for $\delta_A\Phi$ it can be seen that it is not consistent in general to use only eight mesons. Even if one restricts the transformations to $SU(3)$ rather than the full $U(3)$ group by setting $\alpha_0=0$, one finds that the trace of $\delta_A\Phi$ is not, in general, zero. This means that one cannot impose the traceless condition on Φ necessary to restrict the model to an octet of mesons. Thus a set of nine pseudoscalars must be used, in general. Actually, the requirement that only eight pseudoscalar mesons be consistent is so strong that it determines the coupling matrix uniquely. From Eq. (10), one obtains the expression

$$\mathrm{Tr}(\delta_A\Phi)=i\sum_{j,k}\alpha_k(\lambda_k')_{jj}\frac{M(fx_j)}{M'(fx_j)}. \tag{13}$$

It is easy to show that the trace of $\delta_A\Phi$ can be zero for all α_k ($k=1, 2, \cdots, 8$) only if

$$\frac{M'(fx_1)}{M(fx_1)}=\frac{M'(fx_2)}{M(fx_2)}=\frac{M'(fx_3)}{M(fx_3)}. \tag{14}$$

Since at least two of the x_i's are linearly independent (if Φ is traceless the third is determined from $x_1+x_2+x_3=0$), it follows that

$$M'/M=C,$$

where C is a constant independent of x. To be compatible with the expansion in Eq. (7), we choose $C=2if$ and hence

$$M=e^{2if\Phi}. \tag{15}$$

It is interesting to note that only at the $SU(3)$ level is Eq. (14) so restrictive. At the $SU(2)$ level there are only two eigenvalues x_1 and x_2 and they satisfy the equation $x_1+x_2=0$ if Φ is traceless. In this case Eq. (14) becomes

$$\frac{M'(fx)}{M(fx)}=\frac{M'(-fx)}{M(-fx)}.$$

However, this is true for any function satisfying the equation

$$M(fx)M(-fx)=1.$$

Hence, in chiral $SU(2)\otimes SU(2)$, $\delta_A\Phi$ will be traceless so long as M is unitary and Φ is traceless. However, in chiral $SU(3)\otimes SU(3)$ this will be so only if M is given by Eq. (15). It should be noted, however, that although M must be given by (15) if only eight pseudoscalars are to be used, there is nothing to prevent one from using this expression for M with nine pseudoscalar mesons in the model, in which case the singlet meson is invariant under chiral $SU(3)\otimes SU(3)$.

If M is given by

$$M=\frac{1+if\Phi}{1-if\Phi}, \tag{16}$$

then the series expansion for $\delta_A\Phi$ given by Eq. (11) ends with the term of order $(f\Phi)^2$. This is easily shown by using Eq. (10) from which one obtains

$$\delta_A\Phi=\frac{\alpha_k}{2f}\{\lambda_k+f^2\Phi\lambda_k\Phi\}. \tag{17}$$

In fact one can prove that *if $\delta_A\Phi$ is to be a polynomial* *in Φ, then Eq. (17) is the only possibility and M must be given by Eq. (16).*[3]

IV. MESON LAGRANGIAN AND THE VECTOR AND AXIAL-VECTOR CURRENTS

The next stage in the construction of our phenomenological Lagrangian density is the addition of the kinetic term. At this level we still require invariance under the chiral group and define

$$-\mathcal{L}_k(\text{mesons})=\frac{1}{8f^2}\operatorname{Tr}(\partial_\mu M^\dagger\partial_\mu M). \quad (18)$$

Invariance of this Lagrangian density is easily verified provided the group parameters α_k and β_k in Eqs. (3a) and (3b) do not depend on space-time coordinates. Possible extensions to transformations where these parameters depend on the space-time coordinates are not considered in this paper.

The coefficient of $\operatorname{Tr}(\partial_\mu M^\dagger\partial_\mu M)$ is chosen so that upon expanding it the leading term is the free Lagrangian for noninteracting mesons, i.e.,

$$\frac{1}{8f^2}\operatorname{Tr}(\partial_\mu M^\dagger\partial_\mu M)=\tfrac{1}{2}\operatorname{Tr}(\partial_\mu\Phi\partial_\mu\Phi) +\tfrac{1}{2}f^2\operatorname{Tr}(\partial_\mu\Phi^2\partial_\mu\Phi^2-a_3\partial_\mu\Phi\partial_\mu\Phi^3)+\cdots. \quad (19)$$

The terms of order Φ^4 and higher are interpreted as meson-meson interactions and contribute to such processes as pion-pion scattering. That this is a valid interpretation will be shown when this model is actually applied to such scatterings.

With this expression for $\mathcal{L}_k$ one may now obtain the currents $j_\mu{}^p(x)$ associated with an arbitrary transformation $\delta\Phi$

$$\alpha_p j_\mu{}^p(x)\equiv-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)}\delta\phi_k =\frac{1}{8f^2}\operatorname{Tr}(\partial_\mu M^\dagger\delta M+\delta M^\dagger\partial_\mu M). \quad (20)$$

To obtain the vector current, one substitutes the expression

$$\delta_V M=i\alpha_p[\tfrac{1}{2}\lambda_p,M]$$

in (20) and obtains

$$V_\mu=\frac{i}{4f^2}[M,\partial_\mu M^\dagger]=i[\Phi,\partial_\mu\Phi]+\cdots. \quad (21)$$

In a similar manner one obtains the axial-vector current

$$A_\mu=\frac{i}{4f^2}\{\partial_\mu M^\dagger,M\}_+ =\frac{\partial_\mu\Phi}{f}+f(2\Phi\partial_\mu\Phi\Phi-\tfrac{1}{2}a_3\partial_\mu\Phi^3)+\cdots. \quad (22)$$

In order to complete the model, we now add a meson "mass" term to the Lagrangian. We treat the general case for a nonet of mesons and require that the combination

$$-\tfrac{1}{2}\operatorname{Tr}\{(a+b\lambda_8)\Phi^2\}-\tfrac{1}{2}c\phi_0{}^2-\tfrac{1}{2}d\phi_0\phi_8-\tfrac{1}{2}e\phi_8{}^2 \quad (23)$$

be contained in $\mathcal{L}_m$ with

$$\begin{aligned}
a&=\tfrac{1}{3}(2m_K{}^2+m_\pi{}^2),\\
b&=(2/\sqrt{3})(m_\pi{}^2-m_K{}^2),\\
c&=m_{\eta'}{}^2\cos^2\lambda+m_\eta{}^2\sin^2\lambda-\tfrac{1}{3}(2m_K{}^2+m_\pi{}^2),\\
d&=-2\sin\lambda\cos\lambda(m_{\eta'}{}^2-m_\eta{}^2)+\tfrac{4}{3}\sqrt{2}(m_K{}^2-m_\pi{}^2),\\
e&=m_{\eta'}{}^2\sin^2\lambda+m_\eta{}^2\cos^2\lambda-\tfrac{1}{3}(4m_K{}^2-m_\pi{}^2).
\end{aligned} \quad (24)$$

In the above formulas, λ is the $\eta'(959)$ and $\eta(548)$ mixing angle defined by

$$\begin{aligned}
\eta'&=\phi_0\cos\lambda-\phi_8\sin\lambda,\\
\eta&=\phi_0\sin\lambda-\phi_8\cos\lambda,
\end{aligned} \quad (25)$$

which expresses the physical particles η' and η (of masses 959 and 548 MeV, respectively) in terms of the $SU(3)$ singlet state and the $I=0$ member of the octet.

It is necessary that the "mass" part of the Lagrangian contain the terms in Eq. (23) so that when it is combined with the free meson part of the kinetic Lagrangian given in Eq. (19), together they represent the total free Lagrangian for a nonet of mesons.

If one requires that the above mass term transforms at most like a singlet and an octet under $SU(3)$, then the coefficient e must be zero. From Eq. (24) one finds in such a case that the mixing angle is given by

$$\sin^2\lambda=\frac{\tfrac{1}{3}(4m_K{}^2-m_\pi{}^2)-m_\eta{}^2}{(m_{\eta'}{}^2-m_\eta{}^2)}=0.034\pm0.004. \quad (26)$$

This fixes the $\eta'-\eta$ mixing angle at $\lambda=\pm11°$.

The above mass term is easily generalized to include meson interactions. We require that these interactions transform simply under chiral $U(3)\otimes U(3)$ and

[3] From the above expressions for $\delta_V\Phi$ and $\delta_A\Phi$, one may construct vector and axial-vector charges which formally satisfy the equal-time commutation relations of chiral $U(3)\otimes U(3)$. By defining variable $\pi_i(x)$ which satisfy cannonical commutation relations with the fields Φ_i, one obtains an expression for the axial-vector charge density by defining

$$-i\alpha_k A_0{}^k\equiv\tfrac{1}{2}i[\pi_j(\delta_A\Phi_j)+(\delta_A\Phi_j)\pi_j].$$

In particular, if $\delta_A\Phi$ is given by Eq. (17), the axial-charge density is

$$A_a{}^b=-\left(\frac{\pi}{f}+f\Phi\pi\Phi\right)_a^b.$$

This is precisely the form constructed by T. K. Kuo and M. Sugawara, Phys. Rev. **151**, 1181 (1966).

choose[4]

$$\mathcal{L}_m=\frac{1}{8f^2}\operatorname{Tr}\{(a+b\lambda_8)(M+M^\dagger)\}+\frac{c}{64f^2}\operatorname{Tr}\{\lambda_0(M-M^\dagger)\}$$
$$\times\operatorname{Tr}\{\lambda_0(M-M^\dagger)\}$$
$$+\frac{d}{64f^2}\operatorname{Tr}\{\lambda_0(M-M^\dagger)\}\operatorname{Tr}\{\lambda_8(M-M^\dagger\}$$
$$+\frac{e}{64f^2}\operatorname{Tr}\{\lambda_8(M-M)^\dagger\}\operatorname{Tr}\{\lambda_8(M-M^\dagger)\}. \quad (27)$$

This expression may be expanded in powers of Φ, the terms quadratic in Φ being given by Eq. (23). The terms of order Φ^4 and higher represent meson-meson interactions and contribute to such processes as π-π and π-K scattering and the decay $\eta'\to\eta+2\pi$.

PCAC

The addition of a term such as $\mathcal{L}_m$ to the Lagrange density destroys its invariance under $U(3)_L\otimes U(3)_R$, and the vector and axial-vector currents in (21) and (22) are no longer conserved. A particularly successful hypothesis when used in conjunction with the algebra of currents has been that the symmetry is broken in a manner such that one obtains the PCAC equations

$$\partial_\mu A_\mu{}^i=\frac{m_\pi{}^2}{f\sqrt{2}}\phi_i,\quad (i=1,2,3),$$
$$\partial_\mu A_\mu{}^j=\frac{m_K{}^2}{f\sqrt{2}}\phi_j\quad (j=4,5,6,7). \quad (28)$$

For simplicity we first consider PCAC when the mesons are degenerate. If the "mass" Lagrangian density is not required to transform in some definite manner under chiral $U(3)\otimes U(3)$, then one may choose

$$\mathcal{L}_m=-\tfrac{1}{2}m^2\operatorname{Tr}[\mu(\Phi)], \quad (29)$$

where the only constraint on $\mu(\Phi)$ is that it be an even function of Φ (parity invariance) and that the leading term in an expansion of μ be Φ^2 which represents the free meson mass Lagrangian. From the definition of the axial-vector current in Eq. (20), it follows that the divergence of the axial-vector current is given by

$$-\alpha_p\partial_\mu A_\mu{}^p=\delta_A\mathcal{L}=\mathcal{L}(\Phi+\delta_A\Phi)-\mathcal{L}(\Phi)$$

with $\delta_A\Phi$ given by Eq. (11). With the help of Eq. (10) it is not difficult to prove that if μ satisfies the differential equation

$$\mu'(x)=-ixM'(x)/fM(x), \quad (30)$$

one obtains the PCAC equation

$$\partial_\mu A_\mu{}^i=\frac{m^2}{f\sqrt{2}}\phi_i.$$

Thus for each form used for M, one can construct a model with PCAC. In particular, if one uses the exponential form for M, then $\mu=\Phi^2$, *in which case* $\mathcal{L}_m$ *does not contain any meson interactions.*

For nondegenerate mesons the situation is not nearly so simple. With the mass Lagrangian in (27), we find that

$$-\alpha_p\partial_\mu A_\mu{}^p=\frac{i\alpha_p}{16f^2}\operatorname{Tr}[(a+b\lambda_8)\{(M-M^\dagger),\lambda_p\}_+]$$
$$+\text{similar terms in } c, d, \text{ and } e. \quad (31)$$

The terms in c, d, and e are proportional to the ϕ_0 and ϕ_8 fields. With regard to PCAC we wish only to point out that with the parameter a_3 in the expansion of M taken to be zero, Eq. (31) yields the PCAC equations in (28) when contributions of order f^3 and from the ϕ_0 and ϕ_8 states are neglected. The amplitudes for π-π and π-K scattering, and the weak interactions of K mesons will be calculated only to lowest order in f (which at most is f^2). Therefore, we shall be able to show that with a_3 equal to zero these amplitudes satisfy all the limits required by PCAC as various four-momenta go to zero.

V. MESON-MESON INTERACTIONS

In applying the above model to various meson processes, we take the S matrix to be given by

$$S=T\exp\left[i\int d^4x\,(\mathcal{L}_S+\mathcal{L}_W)\right]$$

and *calculate only to the lowest contributing order in f.* If one attempts to calculate these amplitudes to all orders in f, one is confronted with difficulties associated with the nonlinear character of the model. Without a solution to such difficulties we cannot regard the above model as an entirely satisfactory theory of chiral dynamics. Nevertheless, the above model may be used to calculate the amplitude for an arbitrary meson process to the lowest contributing order in f, the difficulties being encountered only if one attempts to go beyond the lowest-order contribution. However, even in lowest order. these amplitudes reflect the chiral symmetry of the Lagrangian and will be shown to be identical to those that have been obtained with the algebra-of-currents method.

[4] The addition of terms such as $[\operatorname{Tr}(M+M^\dagger)]^2$ and $\operatorname{Tr}(M+M^\dagger)\times\operatorname{Tr}\{\lambda_8(M+M^\dagger)\}$ together with more complicated terms is also possible. In the absence of any strong motivation for their inclusion we prefer to delete them. With the choice made in Eq. (27) the terms proportional to c, d, and e do not contribute to π-π and π-K scattering in lowest order. The simplest choice, however, is to keep only the terms in a and b or Eq. (27) in which case L_m transforms like the zero and eight components of $(3,3^*)+(3^*,3)$. It is the presence of the ninth meson which forces one to use the more complicated terms proportional to c, d, and e.

From the previous discussion we take

$$\mathcal{L}_S=-\frac{1}{8f^2}\mathrm{Tr}(\partial_\mu M^\dagger\partial_\mu M)+\frac{1}{8f^2}\mathrm{Tr}\{(a+b\lambda_8)(M+M^\dagger)\}$$
$$+\frac{c}{64f^2}\mathrm{Tr}\{\lambda_0(M-M^\dagger)\}\,\mathrm{Tr}\{\lambda_0(M-M^\dagger)\}$$
$$+\frac{d}{64f^2}\mathrm{Tr}\{\lambda_0(M-M^\dagger)\}\,\mathrm{Tr}\{\lambda_8(M-M^\dagger)\}$$
$$+\frac{e}{64f^2}\mathrm{Tr}\{\lambda_8(M-M^\dagger)\}\,\mathrm{Tr}\{\lambda_8(M-M^\dagger)\}\,. \quad (32)$$

(The weak-interaction Lagrangian density L_W will be constructed later.)

For meson-meson scattering and the decay $\eta'\to\eta+2\pi$, the lowest-order terms (of order f^2) arise from the expansion of L to fourth order in Φ

$$\begin{aligned}-\mathcal{L}_4=&\tfrac{1}{2}f^2\,\mathrm{Tr}(\partial_\mu\Phi^2\partial_\mu\Phi^2-a_3\partial_\mu\Phi\partial_\mu\Phi^3)\\&+\tfrac{1}{2}f^2(1-a_3)\,\mathrm{Tr}\{(a+b\lambda_8)\Phi^4\}\\&-\tfrac{1}{4}cf^2a_3\,\mathrm{Tr}(\lambda_0\Phi)\,\mathrm{Tr}(\lambda_0\Phi^3)\\&-\tfrac{1}{8}df^2a_3\,\mathrm{Tr}(\lambda_0\Phi)\,\mathrm{Tr}(\lambda_8\Phi^3)\\&-\tfrac{1}{8}df^2a_3\,\mathrm{Tr}(\lambda_0\Phi^3)\,\mathrm{Tr}(\lambda_8\Phi)\\&\qquad-\tfrac{1}{4}ef^2a_3\,\mathrm{Te}(\lambda_8\Phi)\,\mathrm{Tr}(\lambda_8\Phi^3)\,. \quad (33)\end{aligned}$$

This Lagrangian yields the following amplitudes[5] to lowest order in perturbation theory.

π-π scattering: $\pi_a(q_1)+\pi_c(q_2)\to\pi_b(q_3)+\pi_d(q_4)$

$$\begin{aligned}A_{\pi\pi}=&2f^2\delta_{ab}\delta_{cd}(m_\pi^2-t)+2f^2\delta_{ad}\delta_{cb}(m_\pi^2-\mu)\\&+2f^2\delta_{ac}\delta_{bd}(m_\pi^2-s)-\tfrac{1}{2}f^2a_3\{\delta_{ab}\delta_{cd}+\delta_{ad}\delta_{cb}+\delta_{ac}\delta_{bd}\}\\&\qquad\times(4m_\pi^2+q_1^2+q_2^2+q_3^2+q_4^2)\,; \quad (34)\end{aligned}$$

π-K scattering: $\pi_a(q_1)+K_\alpha(q_2)\to\pi_b(q_3)+K_\beta(q_4)$

$$\begin{aligned}A_{\pi K}=&\tfrac{1}{2}f^2(2m_K^2+2m_\pi^2-2t-s-u)\delta_{ab}\delta_{\alpha\beta}\\&\qquad+\tfrac{1}{2}f^2(u-s)i\epsilon_{bal}\sigma_{\beta\alpha}{}^l\\&-\tfrac{1}{2}f^2a_3(2m_K^2+2m_\pi^2+q_1^2+q_2^2+q_3^2+q_4^2)\delta_{ab}\delta_{\alpha\beta}\,; \quad (35)\end{aligned}$$

K-K scattering: $K_a(q_1)+K_c(q_2)\to K_b(q_3)+K_d(q_4)$

$$\begin{aligned}A_{KK}=&f^2(2m_K^2-t-u)(\delta_{bc}\delta_{ad}+\delta_{cd}\delta_{ab})\\&-\tfrac{1}{2}f^2a_3(4m_K^2+q_1^2+q_2^2+q_3^2+q_4^2)(\delta_{bc}\delta_{ad}+\delta_{cd}\delta_{ab})\,, \quad (36)\end{aligned}$$

where $s=-(q_1+q_2)^2$, $t=-(q_1-q_3)^2$, and $u=-(q_1-q_4)^2$.

The Adler "self-consistency" condition[6] which (follows from PCAC) requires that the above amplitudes vanish when any one of the four meson momenta goes to zero and the other three remain on the mass shell. Thus the above amplitudes should satisfy the conditions

$$\begin{aligned}&A_{\pi\pi}=0 \text{ at } s=t=u=m_\pi^2\,,\\&A_{\pi K}=0 \text{ at } s=u=m_\pi^2,\ t=m_K^2;\text{ and at}\\&\qquad\qquad s=u=m_K^2,\ t=m_\pi^2\,,\\&A_{KK}=0 \text{ at } s=t=u=m_K^2\,.\end{aligned} \quad (37)$$

It can be seen that the above amplitudes satisfy these conditions if a_3 is zero, in agreement with our previous discussion of PCAC. It should also be noted that when all particles are on the mass shell these amplitudes are independent of a_3. In addition, the above amplitudes are in agreement with calculations carried out for $A_{\pi\pi}$ and $A_{\pi K}$ using the algebra of currents and PCAC.[7,8]

From the above amplitudes one obtains the following scattering lengths[9] in the various isospins channels (at threshold):

$$a_0(\pi\pi)=\frac{7}{16\pi}(fm_\pi)^2m_\pi^{-1}=(0.15\pm0.02)m_\pi^{-1}\,,$$
$$a_1(\pi\pi)=0\,,$$
$$a_2(\pi\pi)=-\frac{1}{8\pi}(fm_\pi)^2m_\pi^{-1}=-(0.04\pm0.004)m_\pi^{-1}\,,$$
$$a_{1/2}(\pi K)=\frac{1}{2\pi}(fm_\pi)^2\left(1+\frac{m_\pi}{m_K}\right)^{-1}m_\pi^{-1}=(0.13\pm0.02)m_\pi^{-1}\,,$$
$$a_{3/2}(\pi K)=-\frac{1}{4\pi}(fm_\pi)^2\left(1+\frac{m_\pi}{m_K}\right)^{-1}m_\pi^{-1}=-(0.07\pm0.01)m_\pi^{-1}\,,$$
$$a_0(KK)=0\,,$$
$$a_1(KK)=-\frac{1}{8\pi}\left(\frac{m_K}{m_\pi}\right)(fm_\pi)^2m_\pi^{-1}=-(0.15\pm0.02)m_\pi^{-1}\,.$$

The above value of $a_0(\pi\pi)$ is to be compared with the experimental value obtained from K_{e4} decays

$$a_0(\pi\pi)=(0.6_{-0.5}^{+0.6})m_\pi^{-1}\,.$$

Besides giving the scattering amplitudes as discussed above, the Lagrangian density in (33) also gives the amplitude for $\eta'\to\eta+2\pi$. In terms of the mixing angle defined in Eq. (25) one finds

$$A(\eta'\to\eta+2\pi)=-\tfrac{2}{3}\sqrt{2}f^2(m_{\eta'}^2+m_\eta^2-m_\pi^2)\times\left(\cos2\lambda+\frac{\sin2\lambda}{2\sqrt{2}}\right)\delta_{ij}\,.$$

[5] Our amplitudes are defined by $S_{fi}=\delta_{fi}-i(2\pi)^4\delta^4(P_f-P_i)\Pi_j\times(2E_j)^{-1/2}A_{fi}$ which in lowest-order perturbation theory reduces to $A_{fi}=\Pi_j(2E_j)^{1/2}\langle f|-\mathcal{L}(0)|i\rangle$. Where applicable the K_1 and K_2 states are defined as $\sqrt{2}K_1=K^0-\bar{K}^0$; $\sqrt{2}K_2=K^0+\bar{K}^0$.

[6] S. L. Adler, Phys. Rev. **137**, B1022 (1965); **139**, B1638 (1965).

[7] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966); N. N. Khuri, Phys. Rev. **153**, 1477 (1967).

[8] Y. Tomozawa, Princeton Report, 1966 (unpublished).

[9] When leptonic decays of K mesons are discussed it will be found that $f=(1.03\pm0.05)m_{\pi^+}^{-1}$.

This amplitude yields the following rate for $\eta' \to \eta+2\pi$.

$$\Gamma(\eta' \to \eta+2\pi)=\left[\cos 2\lambda+\frac{\sin 2\lambda}{2\sqrt{2}}\right]^2 (10.8\pm 2)\ \mathrm{MeV}.$$

With $\lambda=-11°$, the predicted rate is (6.8 ± 1.5) MeV. Although this value is slightly large, it is encouraging that we obtain the right order of magnitude (1 MeV.).

VI. LEPTONIC MESON DECAYS

For the leptonic decays of the π and K we shall assume that the weak-interaction Lagrangian density is given by

$$\mathcal{L}_W(\text{leptonic})=\frac{G}{\sqrt{2}}J_\alpha(x)l_\alpha(x)+\text{h.c.}, \tag{38}$$

where $J_\alpha(x)$ and $l_\alpha(x)$ are the hadronic and leptonic currents. In particular[10]

$$J_\alpha=\cos\theta(V_1{}^2+A_1{}^2)_\alpha+\sin\theta(V_1{}^3+A_1{}^3)_\alpha,$$

with $V_\alpha(x)$ and $A_\alpha(x)$ given by the expansions in Eqs. (21) and (22). As was done for meson-meson interactions, these decays are calculated only to the lowest order in f contributing to a given process.

To lowest order the matrix elements $\langle 0|J_\alpha(0)|K^+\rangle$ and $\langle 0|J_\alpha(0)|\pi^+\rangle$ are given by

$$\begin{aligned}(2q_0)^{1/2}\langle 0|J_\alpha(0)|\pi^+\rangle &=\frac{\cos\theta}{f}(2q_0)^{1/2}\langle 0|\partial_\alpha\pi^+|\pi^+\rangle \\ &=\frac{i\cos\theta}{f}q_\alpha, \\ (2k_0)^{1/2}\langle 0|J_\alpha(0)|K^+\rangle &=\frac{\sin\theta}{f}(2k_0)^{1/2}\langle 0|\partial_\alpha K^+|K^+\rangle \\ &=\frac{i\sin\theta}{f}k_\alpha.\end{aligned} \tag{39}$$

From Eqs. (38) and (39) one obtains the rates for $\pi^+\to\mu^++\nu_\mu$ and $K^+\to\mu^++\nu_\mu$

$$\Gamma(\pi\to\mu+\nu)=\frac{G^2}{8\pi f^2}m_\pi m_\mu{}^2\cos^2\theta(1-m_\mu{}^2/m_\pi{}^2)^2,$$

$$\Gamma(K\to\mu+\nu)/\Gamma(\pi\to\mu+\nu)=\tan^2\theta\left(\frac{m_K}{m_\pi}\right)\frac{(1-m_\mu{}^2/m_K{}^2)^2}{(1-m_\mu{}^2/m_\pi{}^2)^2}.$$

From the experimental rates one obtains a value of θ and f appropriate for the axial-vector current[11]

$$\sin\theta_A=0.263\pm 0.002,\quad |f|=(1.03\pm 0.05)m_\pi{}^{-1}.$$

[10] R. P. Feynman and M. Gell Mann, Phys. Rev. **109**, 193 (1958); E. C. G. Sudarshan and R. E. Marshak, Phys. Rev. **109**, 1860 (1958); J. J. Sakurai, Nuovo Cimento **7**, 649 (1958); M. Gell Mann, Phys. Rev. **125**, 1067 (1962); N. Cabibbo, Phys. Rev. Letters, **10**, 531 (1963).

[11] W. J. Willis, in *Proceedings of the Argonne International Conference on Weak Interactions, 1965* (Argonne National Laboratories, Argonne, Illinois, 1966), Report No. 7130.

In the leptonic decays of K mesons for which there is one pion in the final state, e.g., $K^+\to\pi^0+\bar{l}+\nu_l$, the matrix elements of $J_\alpha(x)$ may be parametrized by the following form:

$$\begin{aligned}(4q_0k_0)^{1/2}&\langle\pi(q)|J_\alpha(0)|K(k)\rangle \\ &=\sin\theta\,\{f_+(k+q)_\alpha+f_-(k-q)_\alpha\},\end{aligned} \tag{40}$$

where $f_\pm$ in general are functions of q^2, k^2, and $k\cdot q$. Since the total number of mesons in the initial and final states is even, only the vector part of $J_\alpha(x)$ contributes and we find

$$f_+=-\frac{1}{\sqrt{2}}\quad\text{and}\quad f_-=0 \tag{41}$$

for both $K^+\to\pi^0+e^++\nu_e$ and $K_2{}^0\to\pi^-+e^++\nu_e$.

With these values for $f_\pm$ and the measured rate for $K^+\to\pi^0+e^++\nu_e$, one obtains a value of θ appropriate for the vector current

$$\sin\theta_V=0.222\pm 0.006\quad(\text{from } K_{e3}).$$

For numerical purposes, we shall simply take $\theta=\theta_V$ (or θ_A) when referring to the vector current (or axial-vector current).

It is interesting to compare this and later results with that expected from PCAC. If $B(0)$ is any operator which is invariant under right-handed isotopic spin transformations, it follows from PCAC that the matrix elements $\langle\alpha,\pi_i(q)|B(0)|K\rangle$ and $\langle\alpha|B(0)|K\rangle$ are related by[12]

$$\begin{aligned}\lim_{q\to 0}\ &(2q_0)^{1/2}\langle\alpha,\pi_i(q)|B(0)|K\rangle \\ &=-if\sqrt{2}\langle\alpha|[I_i,B(0)]|K\rangle,\end{aligned} \tag{42}$$

where I_i ($i=1, 2, 3$) is the isotopic spin operator. For leptonic K decays $B(0)$ is taken to be $J_\alpha(0)$, while for nonleptonic decays $B(0)$ is taken to be the nonleptonic weak Lagrangian density, both of which are assumed invariant under right-handed transformations.

With Eq. (42) one obtains the relation

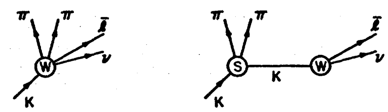
$$\sin\theta_V\ (f_++f_-)_{q_\pi=0}=-\frac{\sin\theta_A}{\sqrt{2}}.$$

Neglecting renormalization effects $\theta_A=\theta_V$, and it is seen that the values of $f_\pm$ in (41) satisfy the above equation.

For K_{l4} the matrix elements of $J_\alpha(0)$ may be parametrized by the following form:

$$\begin{aligned}(8q_0p_0k_0)^{1/2}&\langle\pi_a(q)\pi_b(p)|J_\alpha(0)|K_c(k)\rangle \\ &=\frac{i}{m_K}\{(q+p)_\alpha F_1+(q-p)_\alpha F_2 \\ &\quad+(k-p-q)_\alpha F_3+\epsilon_{\alpha\beta\gamma\delta}k_\beta p_\gamma q_\delta F_4\},\end{aligned}$$

[12] For a derivation of this formula, see, for example, C. Callan and S. B. Treimann, Phys. Rev. Letters **16**, 153 (1966). Also N. Cabibbo, Rapporteur's Talk at the 13th International Conference on High Energy Physics at Berkeley, 1966 (unpublished).



(a) (b)

FIG. 1(a). Direct contribution to $J_\alpha(0)$ from the term $(2f\Phi\partial_\mu\Phi\Phi - \frac{1}{2}a_3\partial_\mu\Phi^3)$ in A_μ; (b) combination of a strong-interaction π-K vertex of order f^2 and the weak current $\langle 0|A_\mu|K\rangle$ of order $1/f$.

where in general the F's are functions of the kinematical variables q^2, p^2, k^2, $q\cdot k$, $p\cdot k$, $q\cdot p$ and the isospin indices a, b, c. Since the total number of mesons in the initial and final states is odd, only the axial-vector part of $J_\alpha(0)$ contributes in the model we are using. Therefore, F_4 will be taken to be zero.

In lowest order, contributions to the F's arise from two sources. First, there is the direct three-meson term from the expansion of the axial-vector current in powers of f given in Eq. (22)

$$A_\mu = \frac{\partial_\mu\Phi}{f} + f(2\Phi\partial_\mu\Phi\Phi - \tfrac{1}{2}a_3\partial_\mu\Phi^3) + \cdots.$$

This contribution is of order f and is illustrated by the Feynman diagram in Fig. 1(a). In addition to this direct contribution to $J_\alpha(0)$, there is also the contribution from a strong π-K vertex (of order f^2, was calculated in Sec. V) together with the vertex $\langle 0|J_\alpha(0)|K\rangle$ (of order $1/f$). These are illustrated by the Feynman diagram in Fig. 1(b).

Together these two diagrams yield the following values for the form factors:

$K^+(k) \to \pi^+(q) + \pi^-(p) + l^+ + \nu_l$:

$$F_1 = A;\quad F_2 = A;\quad F_3 = A\frac{[(p+q)^2 + (k-q)^2 + m_K^2 + m_\pi^2]}{(k-p-q)^2 + m_K^2} - \tfrac{1}{2}Aa_3\frac{(m_K^2 + 2m_\pi^2 + k^2 + p^2 + q^2)}{m_K^2 + (k-p-q)^2};\qquad(43)$$

$K^+(k) \to \pi^0(q) + \pi^0(p) + l^+ + \nu_l$:

$$F_1 = A;\quad F_2 = 0;\quad F_3 = \tfrac{1}{2}A\frac{[(k-p)^2 + (k-q)^2 + 2(p+q)^2 + 2m_K^2 + 2m_\pi^2]}{(k-p-q)^2 + m_K^2} - \tfrac{1}{2}Aa_3\frac{(k^2 + p^2 + q^2 + m_K^2 + 2m_\pi^2)}{(k-p-q)^2 + m_K^2};\qquad(44)$$

$K_2 \to \pi^0(q) + \pi^-(p) + l^+ + \nu_l$:

$$F_1 = 0;\quad F_2 = -A;\quad F_3 = \tfrac{1}{2}A\frac{[(k-p)^2 - (k-q)^2]}{(k-p-q)^2 + m_K^2},\qquad(45)$$

where $A = -fm_K\sin\theta$.

It is interesting to compare these results with the requirements of PCAC given in Eq. (42). For $K^+ \to \pi^+(q) + \pi^-(p) + l^+\nu_l$, Eq. (42) requires that

at $q_\mu = 0$,

$$F_1 = F_2,$$
$$F_3 = 0;$$

at $p_\mu = 0$,

$$F_1 + F_2 = 2fm_K\sin\theta(\sqrt{2}f_+),$$
$$F_3 = \sqrt{2}fm_K\sin\theta(f_+ + f_-),$$

where f_+ and f_- are defined in Eq. (40). It is easily seen that the form factors in Eq. (43) satisfy the above limits when $a_3 = 0$. On the mass shell where $k^2 = -m_K^2$ and $p^2 = q^2 = -m_\pi^2$, it is seen that the F's are independent of a_3. When terms of order m_π^2, p^2, q^2, and $p\cdot q$ are neglected in Eqs. (43)–(45), these results reduce to those obtained by Weinberg[13] using current algebra techniques and PCAC.

In comparing with the data on K_{e4}, we neglect F_3 compared to F_1 and F_2 because in the limit as $m_e \to 0$ it does not contribute. For $K^+ \to \pi^+ + \pi^- + e^+ + \nu_e$, we calculate

$$F_1 = F_2 = 0.96 \pm 0.05 \quad (\text{theory with } \theta = \theta_A).$$

From Table I of Cabibbo and Maksymowicz,[14] one finds that with $F_1 = F_2$, one needs a value

$$F_1 = F_2 = 1.2 \pm 0.2 \quad (\text{experiment})$$

to fit the experimental rate[15] with the π-π, $I=0$, $J=0$ scattering length taken to be zero. In addition, the phase-space average of F_1 and F_2 has been measured[15] and found to be given by $\langle F_1\rangle/\langle F_2\rangle = 0.8 \pm 0.3$ in good agreement with (43).

The rates for the other K_{e4} decays have not been measured yet and cannot be compared with the predictions of (44) and (45) at present.

VII. NONLEPTONIC K DECAYS

For the nonleptonic K decays we take the weak-interaction Lagrangian density to be

$$\mathcal{L}_W(\text{nonleptonic}) = \frac{cG}{4\sqrt{2}f^4}\,\text{Tr}\{\lambda_6\partial_\mu M\partial_\mu M^\dagger\}.\qquad(46)$$

Thus $\mathcal{L}_W$(n.l.) is taken to be of the current×current

[13] S. Weinberg, Phys. Rev. Letters **17**, 336 (1966); C. Callan and S. B. Treimann, *ibid.* **16**, 153 (1966).

[14] N. Cabibbo and A. Maksymowicz, Phys. Rev. **137**, B438 (1965). As pointed out by Weinberg (Ref. 13) there are numerical errors in Eqs. (12) and (A2) of this paper. These equations for the rate should be multiplied by a factor of 4. F_1 and F_2 are denoted by f and g in this paper.

[15] R. Birge *et al.*, Phys. Rev. **139**, B1600 (1965).

form with the property that it transforms like the sixth component of $(8_L,1_R)$ under $U(3)_L\otimes U(3)_R$.

We consider first the $K_1\to\pi^++\pi^-$ and $K_1\to\pi^0+\pi^0$ decays in order to determine the parameter c. From the third-order terms in Φ of an expansion of $\mathcal{L}_W$ in powers of Φ, one obtains the following amplitudes:

$K_1(k)\to\pi^+(q)+\pi^-(p)$:

$$A(+-)=-\frac{icG}{2f}(2k^2-q^2-p^2).\tag{47}$$

$K_1(k)\to\pi^0(q)+\pi^0(p)$:

$$A(00)=-\frac{icG}{2f}(2k^2-q^2-p^2).$$

The experimental value of $A(+\ -)$ obtained from the K_1 rate is

$$|A(+\ -)|=(2.81\pm0.04)\times10^{-6}\,m_\pi.$$

This value requires

$$c=1.1\pm0.1.$$

From the Cabibbo form for the charged currents, one might expect that $c=\cos\theta\sin\theta$. Thus in sharp contrast to semileptonic processes, the Cabibbo angle may not be needed in nonleptonic decays.[16]

With the parameter c now determined, one may calculate the amplitudes for $K\to3\pi$ and the $K\to\pi$ spurion.[16,17] From the terms of second order in Φ contained in $\mathcal{L}_W$, one obtains the following amplitudes:

$$A(K_2\to\pi^0)=-A(K^+\to\pi^+)=\frac{cG}{f^2\sqrt{2}}q(\pi)\cdot q(K).\tag{48}$$

The above $K\to\pi$ amplitudes will be needed to calculate the $K\to3\pi$ amplitudes.

In lowest order the amplitude for $K\to3\pi$ consists of two parts. First there is the direct four-meson weak interaction obtained from the terms of order Φ^4 in the expansion of Eq. (46)

$$\mathcal{L}_W^{(4)}=\frac{cG}{\sqrt{2}}\mathrm{Tr}\{\lambda_6(\partial_\mu\Phi^2\partial_\mu\Phi^2-\tfrac{1}{2}a_3\partial_\mu\Phi\partial_\mu\Phi^3-\tfrac{1}{2}a_3\partial_\mu\Phi^3\partial_\mu\Phi)\}.\tag{49}$$

The amplitude calculated from this Lagrangian density

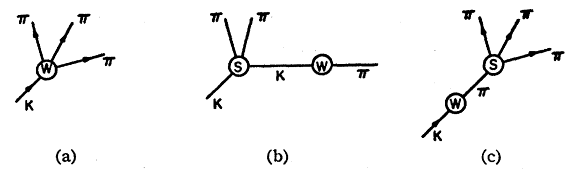


FIG. 2(a). Direct contribution from L_W given in Eq. (49); (b) and (c) combination of a strong meson-meson vertex of order f^2 and a weak K-π vertex of order $1/f^2$.

is zeroth order in f and is illustrated by the Feynman diagram in Fig. 2(a). In addition, the amplitudes obtained by the combination of a strong K-π vertex of order f^2 and a weak K-π spurion of order $1/f^2$, as in Fig. 2(b); or a weak K-π spurion of order $1/f^2$ and a strong π-π vertex of order f^2 as in Fig. 2(c), are also zeroth order in f. These are the only contributions to zeroth order in f. The amplitudes corresponding to the respective diagrams in Fig. 2 will be denoted by $A(a)$, $A(b)$, and $A(c)$, with the total amplitude being given by

$$A_t=A(a)+A(b)+A(c).$$

We calculate only the amplitude for $K^+(k)\to\pi^+(q_1)+\pi^+(q_2)+\pi^-(q_3)$. The other $K\to3\pi$ amplitudes may be obtained from this one by using the $|\Delta I|=\frac{1}{2}$ rule, when electromagnetic mass differences are neglected. From Eqs. (34), (35), (48), and (49) we find

$$\begin{aligned}A(a)&=-\frac{cG}{\sqrt{2}}\{q_1^2+q_2^2+2q_3\cdot(q_1+q_2+q_3)\\&\qquad-\tfrac{1}{2}a_3[2k^2+q_1^2+q_2^2]\},\\A(b)&=\frac{cG}{\sqrt{2}}\left\{\frac{q_2^2}{q_2^2+m_K^2}+\frac{q_1^2}{q_1^2+m_K^2}\right\}\{m_K^2+m_\pi^2+(q_1+q_3)^2\\&\qquad+(q_2+q_3)^2-\tfrac{1}{2}a_3[2m_K^2+2m_\pi^2+k^2+q_1^2+q_2^2+q_3^2]\},\\A(c)&=\frac{cG}{\sqrt{2}}\frac{k^2}{k^2+m_\pi^2}\{4m_\pi^2+2(q_1+q_3)^2+2(q_2+q_3)^2\\&\qquad-a_3[4m_\pi^2+q_1^2+q_2^2+q_3^2+k^2]\}.\end{aligned}\tag{50}$$

PCAC [Eq. (42)] requires that the total amplitude $A(++-)=A(a)+A(b)+A(c)$ satisfy the equations

$$\lim_{q_3\to0}A(++-)=0,\tag{51}$$

$$\lim_{q_1\to0}A(++-)=\frac{if}{\sqrt{2}}A(K_1\to\pi^++\pi^-),\tag{52}$$

where the remaining particles are kept on the mass shell. It is easily verified that these relations are obeyed, thus demonstrating consistency of our results with PCAC.

[16] J. J. Sakurai, Phys. Rev. 156, 1508 (1967).

[17] Y. Hara and Y. Nambu, Phys. Rev. Letters 16, 875 (1966). The constant c defined in their Eq. (6) is related to our f by $c=\sqrt{2}m_\pi^2/f$ except that they use the Goldberger-Treiman relation to evaluate c numerically whereas we use π decay. See also D. K. Elias and J. C. Taylor, Nuovo Cimento 44, 518 (1966); S. K. Bose and S. N. Biswas, Phys. Rev. Letters 16, 330 (1966); B. M. K. Nefkens, Phys. Letters 22, 94 (1966); H. D. I. Abarbanel, Phys. Rev. 153, 1547 (1967).

On the mass shell we shall write the amplitudes for $K \to 3\pi$ in the following form[18]:

$$A = A_{\text{av}}\{1 + a/m_\pi^2(S_3 - S_0)\},$$

where

$$S_i = [q(K) - q(\pi_i)]^2 = -m_K^2 - m_{\pi_i}^2 + 2m_K E(\pi_i),$$
$$3S_0 = S_1 + S_2 + S_3 = -m_K^2 - m_1^2 - m_2^2 - m_3^2.$$

The third pion π_3 is defined as the odd pion in K^+ decays and the π^0 in $K_2 \to \pi^0 + \pi^+ + \pi^-$. From Eq. (50) one obtains

$$A_{\text{av}}(++-) = -\tfrac{1}{3}\sqrt{2} c m_K^2 G = -(1.43 \pm 0.1) \times 10^{-6},$$
$$a(++-) = 3m_\pi^2/2m_K^2 = 0.12.$$

Neglecting the mass differences $m_{K^+} - m_K$ and $m_{\pi^+} - m_{\pi^0}$, the other $K \to 3\pi$ amplitudes are obtained from the $|\Delta I| = \frac{1}{2}$ rule.[18] Thus the predictions of this model are

$$\begin{aligned} A_{\text{av}}(++-) &= -(1.43 \pm 0.1) \times 10^{-6}, & a(++-) &= 0.12; \\ A_{\text{av}}(+00) &= -(0.77 \pm 0.05) \times 10^{-6}, & a(+00) &= -0.24; \\ A_{\text{av}}(+-0) &= (0.77 \pm 0.05) \times 10^{-6}, & a(+-0) &= -0.24; \\ A_{\text{av}}(000) &= (2.15 \pm 0.15) \times 10^{-6}, & a(000) &= 0; \end{aligned}$$

while the experimental values are[19]

$$\begin{aligned} |A_{\text{av}}(+--)|_{\text{expt}} &= (1.93 \pm 0.04) \times 10^{-6}, & a(++-)_{\text{expt}} &= 0.093 \pm 0.01; \\ |A_{\text{av}}(+00)| &= (0.96 \pm 0.03) \times 10^{-6}, & a(+00) &= -0.25 \pm 0.2; \\ |A(+=0)| &= 0.89 \pm 0.03) \times 10^{-6}, & a(+=0) &= -0.24 \pm 0.2; \\ |A(000)| &= (2.8 \pm 0.2) \times 10^{-6}, & a(000) &= 0. \end{aligned}$$

The agreement with experiment is in general good, although the theoretical amplitudes are about 20% too low.

VIII. CHIRAL INVARIANT MESON-BARYON COUPLING

We now construct an invariant coupling of an octet (or nonet) of low-lying baryons (spin $\frac{1}{2}$) to the pseudoscalar mesons in analogy with Eq. (1). The advantage of this coupling is that the baryon mass does not break the chiral symmetry and the Goldberger-Treiman relation is given directly.

If such a coupling is to be possible then these baryons must belong to the (3,3*) and (3*,3) representations of $U(3)_L \otimes U(3)_R$ as opposed to the (8,1) and (1,8). This is because the product $\bar{B} \otimes B$, where $B_i{}^j$ is the baryon field, must contain the (3,3*) and (3*,3) representations in order to form an invariant coupling with M.

Assuming that the low-lying $J = \frac{1}{2}$ baryons belong to the (3,3*) and (3*,3) representations, then the states

$$B_i{}^{(+)j} \equiv \tfrac{1}{2}(1+\gamma_5) B_i{}^j \text{ and } B_i{}^{(-)j} \equiv \tfrac{1}{2}(1-\gamma_5) B_i{}^j$$

may be taken to transform under $U(3)_L \otimes U(3)_R$ like $(3_L, 3_R^*)$ and $(3_R, 3_L^*)$, respectively. With the baryons belonging to this representation, the trace of B is not invariant under $F_5{}^i$ and thus a set of nine baryons is required.

Because of the above transformation properties of $B^{(+)}$ and $B^{(-)}$ the bilinear combination

$$U_i{}^j \equiv \epsilon_{ikl} \epsilon^{j\beta\gamma} \bar{B}_\beta{}^{(-)k} B_\gamma{}^{(+)l}$$

transforms like $(3_R, 3_L^*)$. The following coupling is then invariant under $U(3)_L \otimes U(3)_R$:

$$L_{\text{int}} = m \,\mathrm{Tr}(MU) + \text{h.c.} \tag{53}$$

To zeroth order in f

$$\mathfrak{L}_{\text{int}}{}^{(0)} = -m\{\sum_{i=1}^{6} \bar{B}_i B_i - 2\bar{B}_0 B_0\}. \tag{54}$$

If one, therefore identifies the ninth baryon field B_9 with $\gamma_5 B_0$, then $L_{\text{int}}{}^{(0)}$ represents the mass term for an octet of baryons ($J^P = \frac{1}{2}^+$) with mass m and an $SU(3)$ singlet baryon ($J^P = \frac{1}{2}^-$) with mass $2m$. Perhaps this ninth baryon is the $\Lambda(1405)$. The spin and parity is correct; however, the mass is quite a bit lower than twice the average mass of the octet. Ignoring difficulties associated with mass splittings, the Lagrangian in (53) could be used to calculate meson-baryon scattering to lowest order in f. However, only the simplest case of pion-nucleon elastic scattering will be discussed here.

Neglecting all other particles except the pion and nucleon, Eq. (53) reduces to

$$\mathfrak{L}_{\text{int}}(\pi N) = -m\bar{N}M(-f\gamma_5 \boldsymbol{\pi} \cdot \boldsymbol{\tau}/2)N, \tag{55}$$

where

$$N = \begin{pmatrix} p \\ n \end{pmatrix}$$

is the nucleon field and π the pion field. From the ex-

[18] S. Weinberg, Phys. Rev. Letters **4**, 87, 585 (1960); G. Barton, C. Kacser, and S. P. Rosen, Phys. Rev. **130**, 738 (1963).

[19] G. H. Trilling, in *Proceedings of the Argonne International Conference on Weak Interactions, 1965* (Argonne National Laboratory, Argonne, Illinois, 1966), Report No. 7130.

pansion of M in Eq. (7) one obtains as the first few terms

$$\mathcal{L}_{\mathrm{int}}(\pi N)=-m\bar{N}N+i(mf\sqrt{2})\bar{N}\boldsymbol{\pi}\cdot\boldsymbol{\tau}\gamma_5 N+mf^2(\boldsymbol{\pi}^2)\bar{N}N+\cdots. \tag{56}$$

The Goldberger-Treiman relation is contained in term linear in f which relates the strong-interaction πNN coupling to the pion decay constant f

$$g_{\pi NN}=mf\sqrt{2}.$$

To order f^2 the elastic pion-nucleon scattering amplitude is calculated from the Feyman diagrams in Fig. 3. From these, one obtains the following amplitudes in the various isospin channels for $\pi(q_1)+N(p_1)\to\pi(q_2)+N(p_2)$:

$$T=\tfrac{1}{2}:\quad A(\tfrac{1}{2})=i\bar{N}\left\{m^2f^2\left(\frac{3}{m^2-s}+\frac{1}{m^2-u}\right)\times(\boldsymbol{q}_1+\boldsymbol{q}_2)+2imf^2\right\}N,$$

$$T=\tfrac{3}{2}:\quad A(\tfrac{3}{2})=\bar{N}\left\{\frac{-2im^2f^2}{m^2-u}(\boldsymbol{q}_1+\boldsymbol{q}_2)-2mf^2\right\}\bar{N}, \tag{57}$$

where $s=-(p_1+q_1)^2$; $u=-(p_1-q_2)^2$; $t=-(p_1-p_2)^2$.

These amplitudes yield the following s- and p-wave scattering lengths:

S wave:

$$a_3=-\frac{m^2\mu f^2(2m+\mu)}{2\pi(m+\mu)(4m^2-\mu^2)}=(-0.08\pm0.01)\mu^{-1},$$

$$a_1=\frac{m^2\mu f^2(4m-\mu)}{\pi 2(m+\mu)(4m^2-\mu^2)}=(0.144\pm0.01)\mu^{-1}. \tag{58}$$

P wave: This scattering length is denoted by $a_{2T,2J}$ where T is the total isotopic spin and J the total angular momentum.

$$a_{11}=-\frac{m^3f^2}{3\pi\mu(m+\mu)(2m-\mu)^2}-\frac{f^2(8m^3-4m\mu^2-\mu^3)}{8\pi\mu(m+\mu)(4m^2-\mu^2)}=(-0.10\pm0.01)\mu^{-3},$$

$$a_{13}=-\frac{m^3f^2}{3\pi\mu(m+\mu)(2m-\mu)^2}=(-0.029\pm0.003)\mu^{-3},$$

$$a_{31}=\frac{2m^2f^3}{3\pi\mu(m+\mu)(2m-\mu)^2}-\frac{mf^2(4m^2+4m\mu-\mu^2)}{8\pi\mu m(m+\mu)(2-m\mu)}=(-0.034\pm0.003)\mu^{-3}, \tag{59}$$

$$a_{33}=\frac{2m^3f^2}{3\pi\mu(m+\mu)(2m-\mu)^2}=(0.057\pm0.006)\mu^{-3}.$$

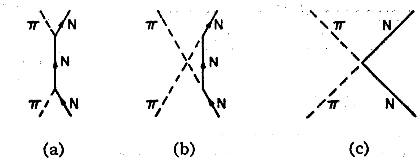


FIG. 3(a) and (b). Diagrams arising from the term linear in π of $L(\pi N)$ giving a contribution of order f^2 to the amplitude; (c) the diagram corresponding to the term quadratic in π of $L(\pi N)$ also of order f^2.

The experimental values as obtained from dispersion relations[20] are

$$a_1=0.171\pm0.005,\qquad a_3=-0.088\pm0.004,$$
$$a_{11}=-0.101\pm0.007,\quad a_{13}=-0.029\pm0.005,$$
$$a_{31}=-0.380\pm0.005,\quad a_{33}=0.215\pm0.005.$$

The agreement between experiment and theory is good except in the case of a_{33}.

It is interesting to note that the amplitude calculated from (56) to *identical*[21] to that obtained from

$$\mathcal{L}'=if^2\bar{N}\gamma_\mu\left(-\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}+\gamma_5\frac{\sqrt{2}}{f}\partial_\mu\boldsymbol{\pi}\right)\cdot\frac{\boldsymbol{\tau}}{2}N \tag{60}$$

to order f^2.

The vector part of this effective Langrangian is identical to ρ exchange at low-momentum transfer if one makes the identification[22]

$$f_\rho^2/m_\rho^2=f^2\simeq m_\pi^{-2}.$$

Numerically, this equation for f_ρ^2 is very good.

IX. DISCUSSION

In conclusion we would like to add the following remarks:

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[20] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).

[21] As first pointed out by S. Weinberg [Phys. Rev. Letters **18**, 188 (1967)] nonderivative-type couplings as in (55) are easily transformed into derivative-type couplings. If one defines a new nucleon field by the transformation

$$N\equiv U(f\gamma_5\boldsymbol{\pi}\cdot\boldsymbol{\tau}/\sqrt{2})N',\quad\text{with } U^\dagger U=I$$

and if U is defined by the equation

$$U^\dagger(-f\gamma_5\boldsymbol{\pi}\cdot\boldsymbol{\tau}/\sqrt{2})M(-f\gamma_5\boldsymbol{\pi}\cdot\boldsymbol{\tau}/\sqrt{2})U(f\gamma_5\boldsymbol{\pi}\cdot\boldsymbol{\tau}/\sqrt{2})=I,$$

then the nucleon part of the Lagrangian

$$\mathcal{L}_N=-\bar{N}\gamma_\mu\partial_\mu N-m\bar{N}M(-f\gamma_5\boldsymbol{\pi}\cdot\boldsymbol{\tau}/\sqrt{2})N$$

becomes

$$\mathcal{L}_{N'}=-\bar{N}'\gamma_\mu\partial_\mu N'-m\bar{N}'N'-\bar{N}'\gamma_\mu U^\dagger\partial_\mu UN'.$$

and the interaction has been transformed into a derivative-type coupling. With $U=M^{1/2}$ and expanded to second order in f, one obtains the Lagrangian (60). The effective Lagrangian constructed by Weinberg in this reference corresponds to the model presented here in chiral $SU(2)\otimes SU(2)$ with M given by (12b). A similar model has also been discussed by J. Schwinger [Phys. Letters **24B**, 473 (1967)] with the meson mass Lagrangian determined by PCAC [Eq. (30)].

[22] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966). Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966); J. J. Sakurai, Phys. Rev. Letters **17**, 552 (1966).

(1) The success of the above model in describing low-energy pseudoscalar physics would seem to indicate that some sort of chiral dynamics may well be appropriate at least at low energies. In particular, it is striking that the chiral-invariant coupling in (56) reproduces low-energy pion-nucleon scattering so well. In this regard, it is interesting to point out that the coupling in (56) is the minimum needed so that the nucleon mass does not violate the chiral symmetry. An additional chiral-invariant term is possible by coupling the vector and axial-vector current of the nucleon to the corresponding currents of the pseudoscalars. The fact that very little of this additional coupling is needed suggests that there is "just enough" low-energy pion-nucleon dynamics to make the nucleon mass compatible with a chiral symmetry.

(2) In this model the breaking of chiral $U(3)\otimes U(3)$ was assumed to be due solely to the meson mass Lagrangian L_m in (27). This was written for the general case in which Φ contained a nonet of particles and a rate for $\eta'\rightarrow\eta+2\pi$ could be calculated. A simpler possibility for the transformation properties of L_m is to restrict Φ to an octet of mesons and neglect the η'. In this case one may choose

$$\mathcal{L}_m=\frac{1}{8f^2}\operatorname{Tr}\{(a+b\lambda_8)(M+M^\dagger)\}.$$

M must now be of the exponential type in (12a) and the chiral group restricted to $SU(3)\otimes SU(3)$. Of course now nothing can be said about the rate for $\eta'\rightarrow\eta+2\pi$; however, on the mass shell none of the other amplitudes calculated above are changed.

(3) For all amplitudes calculated it was found that they were independent of the form used for M on the mass shell. It may be that in general the amplitudes calculated from an effective Lagrangian which is a function of M alone (i.e., not an explicit function of Φ) are independent of the form used for M. If this is true in general, the amplitudes for other meson processes such as low-energy multiple pion production could be calculated without the introduction of new parameters, namely the expansion coefficients of M.

(4) In chiral $SU(2)\otimes SU(2)$ where the pions may be considered degenerate there are many ways of obtaining a model with PCAC. As discussed in Sec. IV, one obtains PCAC equations for any M as long as the mass term satisfies Eq. (30). For example, if one chooses $M=e^{2if\Phi}$ then $\mu=\Phi^2$ and L_m *does not contain any meson interactions*. In this case the kinetic part of the Lagrangian gives in lowest order

$$\mathcal{L}^2=\tfrac{1}{3}f^2(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})^2, \tag{61}$$

which at low energies gives the same amplitude[23] as ρ exchange with

$$f_\rho{}^2=\tfrac{2}{3}(fm_\rho)^2=1.7\pm0.1.$$

In general, different choices for M give different π-π scattering amplitudes. Since M determines how Φ transforms under the axial-vector charge [Eq. (10)], we see that the ambiguity associated with different choices for M is the analog of what one assumes about the equal-time commutator $[A_0{}^i,\partial_\mu A_\mu{}^j]$ in current algebra calculations.

Note added in proof. If higher powers of M are used in constructing a chiral invariant baryon mass then in addition to the coupling in (53),

$$m'\operatorname{Tr}[\bar{B}^{(+)}MB^{(-)}M]+\text{H.c.}$$

is also an invariant for a set of baryons belonging to the (3,3*) and (3*,3) representations, while for baryons belonging to the (8,1) and (1,8) representations

$$m\operatorname{Tr}[\bar{B}^{(+)}MB^{(-)}M^\dagger]+\text{H.c.}$$

is invariant. All of these couplings reduce to the pion-nucleon coupling in (55) which follows from $SU(2)\otimes SU(2)$ invariance alone, the nucleon belonging to the (2,1) and (1,2) representations.

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[23] To obtain this result from PCAC and the current algebra, one need only make the ansatz that the π-π scattering amplitude vanish at $s=t=u=0$ in the calculations of Weinberg (Ref. 7). In his notation, this means $A=0$, while the Adler self-consistency requires $C=-2B$. Using his Eq. (17) for $B-C$, one then obtains the amplitude calculable from (61). This amplitude gives $a_0=(0.11\pm0.01)m_\pi{}^{-1}$ and $a_2=-(0.06\pm0.005)m_\pi{}^{-1}$ for the $I=0$, 2 scattering lengths at threshold.