Semirelativistic Quark Model for Mesons*

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We present a quark model of mesons which treats the kinematics, including spin, exactly, but neglects the effects of retardation in the interaction. Starting with the two-particle Dirac equation with a static square-well potential, we find that tightly bound physical solutions exist only for certain special cases. In addition, there are unphysical solutions which correspond to the binding of a positive-energy spinor to a negative-energy spinor. The solution used in the Fermi-Yang model of the pion is unphysical in this sense. For continuously varying potentials, the equations contain singularities which arise from the presence of unfilled negative-energy states. These make the two-particle Dirac equation unreliable for interaction energies which are comparable to the masses involved. Using a reduced form of the Bethe-Salpeter equation in the ladder approximation for an instantaneous interaction, which we represent by an S-wave separable potential, we compute the wave functions of the π and ρ mesons. We calculate the π lifetime, which comes out in good agreement with the observed value for a quark mass of 3.1 BeV. The π-ρ mass splitting has the right sign, but is too large, for all values of the quark mass.

I. INTRODUCTION

THE idea that the hadrons may be bound states of quarks[1] is an esthetically pleasing and natural way to account for the observed symmetries.[2] Nearly all the recent applications of this idea have assumed that the quarks move nonrelativistically in the bound state, although if their mass is very large, as appears to be indicated experimentally,[3] this would seem to be a bad approximation. Nevertheless, the "naive quark model" has had some remarkable successes.[4] As a justification for the assumption, Morpurgo[5] has suggested a specific model in which the quarks are bound in a very deep, but very wide, square-well potential. On the basis of the Schrödinger equation it is reasoned that in such a system the quarks may be very tightly bound and yet possess arbitrarily small kinetic energy. Greenberg[6] has shown that this is not the case for Coulomb, Yukawa, or exponential potentials. Several consequences of the model have been worked out by Becchi and Morpurgo.[7]

Notwithstanding the success of the nonrelativistic models, several attempts have been made to treat the relativistic aspects of the problem. None of these has attempted to deal with the difficult problem of the many-body states which couple to the "bare" $q\bar{q}$ state. Bogolyubov *et al.*[8] have used the Bethe-Salpeter equation, keeping only the largest terms and using a separable potential, to compute the electromagnetic properties of mesons. Lichtenberg and Tassie[9] have computed meson mass formulas using the Klein-Gordon equation with a local potential, and handling the kinematics (neglecting the quark spins) exactly.

In this paper, we shall investigate quark models of mesons, using the two-particle Dirac equation, and a reduced, noncovariant form of the Bethe-Salpeter equation. For the two-particle Dirac equation, with a square-well potential, we show that arbitrarily tightly bound physical solutions do not exist. By "physical" we mean solutions which go over smoothly to a state of two positive-energy spinors as the interaction is turned off. In particular, for the state studied in the Fermi-Yang treatment of the pion[10] (the $^1S_0$ state, bound by a vector-coupled square-well potential), we show that the minimum energy of the bound state is $\frac{2}{3}\sqrt{2}M$, where M is the sum of the masses of the constituent quarks. The solution found by Fermi and Yang is unphysical, in the sense defined above. This fact has been pointed out by Moseley and Rosen.[11]

We extend the analysis to other LS states, and to other couplings. For all states other than $^1S_0$, with vector coupling, tightly bound solutions exist only in the case of equal masses of the constituent particles. The same qualitative features hold with scalar and axial-vector coupling. For tensor coupling, all solutions are well behaved (in the sense that their energy goes to zero as the depth of the potential increases). For pseudoscalar coupling, no bound state exists at all, for any combination of potential depth and range. Explicit calculations with more general potentials than square wells are made difficult by the presence in the differential equation of an unexpected singularity at a particular finite value of r. We are able, through a qualitative analysis of a general nonsingular potential, to show that

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tightly bound solutions can exist. However, their existence requires the presence of the extra singularity. This singularity results from an improper treatment of the negative-energy states in the two-particle Dirac equation, which is therefore of doubtful validity, even as a first approximation, in the relativistic regime.

If one neglects retardation effects in the interaction, the Bethe-Salpeter equation, in the ladder approximation, can be reduced to a simpler, noncovariant one, by a method first used by Salpeter.[12] This equation differs from the Dirac equation by the presence in it of certain energy projection operators, which arise from the imposition of Feynman boundary conditions on the propagators. These projection operators perform the valuable services of eliminating the unphysical solutions, and of allowing the existence of tightly bound states. The reduced Bethe-Salpeter equation is therefore a more promising candidate than the Dirac equation for calculating the properties of tightly bound states of spinors.

In Secs. II and III we treat the $J=0$ and $J=1$ states of the Dirac equation with a vector-coupled square-well potential. In Sec. IV we generalize this to include other couplings, and more general potentials. In Secs. V and VI we consider the model based on the reduced Bethe-Salpeter equation, and apply it to the calculation of the pion lifetime, and the ρ-γ vertex.

II. THE TWO-BODY DIRAC EQUATION: $J=0$ STATES

We consider a system of two spin-$\frac{1}{2}$ particles, of masses m_1 and m_2, interacting through a local potential. The Dirac equation of motion, in the center-of-mass system of the particles, is

$$[H_1(\mathbf{r})+H_2(\mathbf{r})-E]\psi(\mathbf{r})=V(\mathbf{r})\Gamma\psi(\mathbf{r}). \quad (2.1)$$

Here $\mathbf{r}$ is the relative coordinate, and

$$H_1(\mathbf{r})=-i\boldsymbol{\alpha}^{(1)}\cdot\boldsymbol{\nabla}+\beta^{(1)}m_1,$$
$$H_2(\mathbf{r})=i\boldsymbol{\alpha}^{(2)}\cdot\boldsymbol{\nabla}+\beta_2 m_2.$$

The superscripts denote the spin subspaces in which the α's and the β act. Γ is an operator which depends on the coupling. For vector coupling, in our notation,

$$\Gamma=[I^{(1)}I^{(2)}-\boldsymbol{\alpha}^{(1)}\cdot\boldsymbol{\alpha}^{(2)}].$$

Finally, we take $V(\mathbf{r})$ to be a square well:

$$V(r)=V, \quad r<r_0$$
$$=0, \quad r>r_0.$$

With this convention, V is positive (negative) for attractive (repulsive) potentials.

Actually, (2.1) applies to a system of two particles. For a particle-antiparticle system, the correct equation would be (2.1) multiplied through by the operator $C^{(i)}$ which charge conjugates the ith particle. The correct wave function, therefore, differs from the one which appears in (2.1) by just this operator $C^{(i)}$. It is convenient for our purposes to deal with a particle-particle system, remembering to make the required change whenever necessary. We remark that of the five possible couplings (scalar, vector, tensor, axial vector, and pseudoscalar), the vector and tensor have the property that they change sign under charge conjugation of one of the two particles. Thus in these cases the sign of the interaction determines whether $\bar{q}q$ or qq actually bind.

Throughout this paper we shall use the representation

$$\alpha_r=\begin{pmatrix}0 & \sigma_r\\ \sigma_r & 0\end{pmatrix}, \quad \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix},$$

where the σ_r are the usual Pauli matrices. Our notation for the γ matrices will be

$$\gamma_0=\beta, \quad \gamma_r=\beta\alpha_r, \quad \gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3,$$

and they satisfy

$$\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu},$$

with $-g_{00}=g_{11}=g_{22}=g_{33}=-1$.

ψ carries two spinor indices, each of which ranges from 1 to 4. It is convenient to write it as a 4×4 matrix. Furthermore, Eq. (2.1) is invariant under rotations and reflections so ψ is an eigenstate of total angular momentum J, and parity P. With our representation for the α's and β, ψ breaks up into four parts:

$$\psi_P{}^{Jm}(\mathbf{r})=\begin{pmatrix}A_{L_1S_1}{}^{Jm}(\mathbf{r}) & B_{L_2S_2}{}^{Jm}(\mathbf{r})\\ C_{L_3S_3}{}^{Jm}(\mathbf{r}) & D_{L_4S_4}{}^{Jm}(\mathbf{r})\end{pmatrix},$$

in which A, B, C, D are 2×2 matrices, eigenstates of J, J_z, L, and S. Under parity,

$$\psi_P{}^{Jm}(\mathbf{r})\underset{P}{\longrightarrow}\beta^{(1)}\beta^{(2)}\psi_P{}^{Jm}(-\mathbf{r})$$
$$=\begin{pmatrix}(-1)^{L_1}A_{L_1S_1}{}^{Jm}(-\mathbf{r}) & (-1)^{L_2}B_{L_2S_2}{}^{Jm}(-\mathbf{r})\\ (-1)^{L_3}C_{L_3S_3}{}^{Jm}(-\mathbf{r}) & (-1)^{L_4}D_{L_4S_4}{}^{Jm}(-\mathbf{r})\end{pmatrix}$$
$$=(-1)^P\psi_P{}^{Jm}(-\mathbf{r}).$$

This requires

$$L_1=L_4, \quad L_2=L_3=L_1\pm1.$$

In other words the diagonal and off-diagonal quadrants have opposite parity.

For $J=0$ there are two possible states, which we label in the usual spectroscopic notation, according to the LS values of their "large components," i.e., the quadrant we have labeled A above. For states which are formed of two positive-energy spinors (those which we shall call "physical" states below) the quadrants B and C will be of order v/c, and D will be of order $(v/c)^2$, compared to A. For the qq ($q\bar{q}$) system, the $^1S_0$ state has even (odd) parity, and the $^3P_0$ state has odd (even)

[12] E. E. Salpeter, Phys. Rev. 87, 328 (1952).

parity. We shall label our wave functions according to the parity they would have if they represented $q\bar{q}$ states.

A. The $^1S_0$ State

The $^1S_0$ state, which we identify with the pseudoscalar octet, has the wave function

$$\psi_-{}^0(\mathbf{r})=\begin{pmatrix} f_1(r)\chi_{00}{}^0 & ig_1(r)\chi_{11}{}^0 \\ ig_2(r)\chi_{11}{}^0 & f_2(r)\chi_{00}{}^0 \end{pmatrix}, \tag{2.2}$$

where the χ's are 2×2 bispinor matrices, independent of $|\mathbf{r}|$, and we have suppressed the index m for $J=0$. A particular realization of the bispinors is displayed in the Appendix. $f_i(r)$ and $g_i(r)$ are real, continuous functions. Equation (2.1) now breaks up into 4 equations in the f's and g's. It is convenient to express these in terms of the linear combinations $f_\pm=f_1\pm f_2$, $g_\pm=g_1\pm g_2$. Then, defining $M=m_1+m_2$, $m=m_1-m_2$, we find, for $r<r_0$,

$$Mf_+-Ef_-=-2Vf_-, \tag{2.3}$$

$$Mf_--Ef_+-2\left(\frac{d}{dr}+\frac{2}{r}\right)g_+=4Vf_+, \tag{2.4}$$

$$mg_+-Eg_-=2Vg_-, \tag{2.5}$$

$$mg_--Eg_++2\frac{d}{dr}f_+=0. \tag{2.6}$$

Because the potential is a simple square well, the solution to these equations is simply expressible in terms of spherical Bessel functions. The choice which satisfies the boundary conditions at $r=\infty$ is

$$\begin{aligned} f_\pm&=A_\pm j_0(\alpha r), \quad g_\pm=B_\pm j_1(\alpha r), \quad r<r_0 \\ f_\pm&=a_\pm h_0{}^{(1)}(\beta r), \quad g_\pm=b_\pm h_1{}^{(1)}(\beta r), \quad r>r_0. \end{aligned} \tag{2.7}$$

Solving for α and β, we find

$$\alpha^2=-\frac{1}{4}\frac{[(E+4V)(E-2V)-M^2][E(E+2V)-m^2]}{E^2-4V^2}, \tag{2.8}$$

$$\beta^2=-\frac{1}{4}\frac{[E^2-M^2][E^2-m^2]}{E^2}. \tag{2.9}$$

The various coefficients are related by

$$A_-=\frac{M}{E-2V}A_+, \qquad a_-=\frac{M}{E}a_+, \tag{2.10}$$

$$B_-=\frac{M}{E+2V}B_+, \qquad b_-=\frac{M}{E}b_+, \tag{2.11}$$

$$B_+=2\frac{\alpha}{E-m^2/(E+2V)}A_+, \quad b_+=2\frac{\beta}{E-m^2/E}. \tag{2.12}$$

Equations (2.4) and (2.6) imply that f_+ and g_+ are continuous across the boundary, though their first derivatives are not. This determines E by

$$\left[E-\frac{m^2}{E+2V}\right]\left[\frac{1}{1-\alpha r_0\cot\alpha r_0}\right]=\left[E-\frac{m^2}{E}\right]\left[\frac{1}{1+i\beta r_0}\right]. \tag{2.13}$$

The finiteness of the wave function at $r=0$ and $r=\infty$ requires $\alpha^2>0$, $\beta^2<0$. The latter condition places the limits on E:

$$m^2<E^2<M^2. \tag{2.14}$$

α^2 has singularities at $E=\pm2V$, but in this instance these will not have any effect on the solutions of interest. To see this, we set $\alpha^2=0$ and solve for E. (Since α^2 is the eigenvalue of p^2, this corresponds to neglecting the kinetic-energy terms in the Hamiltonian. This is a good approximation in the limit $r_0\gg M^{-1}$, which is the situation envisaged by Morpurgo.[5])

$$\begin{aligned} E&=-V\pm(M^2+9V^2)^{1/2} \\ &=-V\pm(m^2+V^2)^{1/2} \quad (\alpha^2=0). \end{aligned} \tag{2.15}$$

In Fig. 1 we have graphed these curves, together with the cuts in α^2 at $E=\pm2V$.

The only solution which is physically attainable is that for which $E=M$ when $V=0$. The other three solutions contain admixtures of negative-energy states. We note that our condition on α^2 requires that the physical solutions lie *above* the curve $E=-V+(M^2+3V^2)^{1/2}$, which they approach as $r_0\to\infty$. This curve does not cross the cut at $E=2V$, which may therefore be neglected. The minimum in the curve constitutes an *absolute minimum* for the energy of a physical solution. It is easily seen to be $\frac{2}{3}\sqrt{2}M$, and to occur at $V=(1/12)\sqrt{2}M$.

Solutions exist with values of E lower than this minimum, but they are bounded above by $E=2V$. They correspond to products of positive- with negative-energy solutions to the (single-particle) Dirac equation, and are unphysical. Their existence is an unavoidable consequence of the fact that the single-particle Dirac Hamiltonian is not positive-definite. We shall not consider them further, except to note that we must be on guard against the possibility that the two families of solutions may have overlapping energy spectra, in which case the two kinds of eigenstates may mix. This does not happen for a square-well potential, however.

The fact that there exists a definite value of V for which the binding energy is maximized implies that for sufficiently short-range potentials no physical solution exists at all, in contrast to the situation, e.g., with the Schrödinger equation, in which, by increasing the depth of the potential, it is always possible to create a bound state, no matter what the range is. A rough estimate

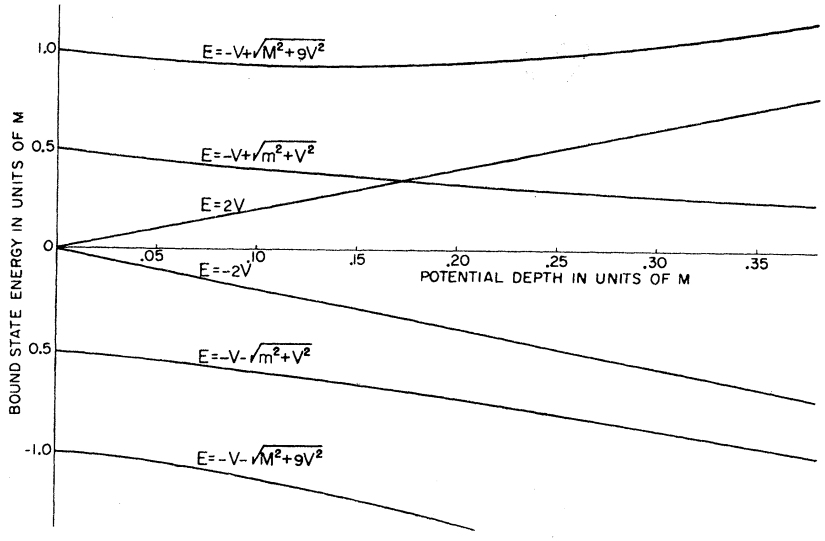

FIG. 1. Locus of poles and zeros of $\alpha^2(E,V)$ for the $^1S_0$ state.

yields the inequality

$$r_0 > 4.4M^{-1}$$

as a condition for the existence of a physical bound state. In Ref. 10, Fermi and Yang took $r_0=2M^{-1}$, so that in fact no physical solution existed, and the one they write down is unphysical. In Fig. 2 we have graphed E against V for two different solutions, one physical and one unphysical, taking $r_0=40M^{-1}$.

B. The $^3P_0$ State

The $^3P_0$ wave function has the general form

$$\psi_+^0(\mathbf{r})=\begin{pmatrix} g_1(r)\chi_{11}^0 & if_1(r)\chi_{00}^0 \\ if_2(r)\chi_{00}^0 & g_2(r)\chi_{11}^0 \end{pmatrix}, \tag{2.16}$$

and the equations which are obtained from (2.1) for this wave function differ from (2.3)–(2.6) only by the transformations $\mathbf{r}\rightarrow -\mathbf{r}$ and $M\leftrightarrow m$. Therefore, for this state we have

$$\alpha^2=-\frac{1}{4}\frac{[E(E+2V)-M^2][(E+4V)(E-2V)-m^2]}{E^2-4V^2}, \tag{2.17}$$

with β^2 still given by Eq. (2.9). Setting $\alpha^2=0$, we obtain

$$\begin{aligned} E&=-V\pm(M^2+V^2)^{1/2} \\ &=-V\pm(m^2+3V^2)^{1/2} \quad (\alpha^2=0). \end{aligned} \tag{2.18}$$

These curves, and the cuts, are graphed in Fig. 3. The physical solutions are constrained to lie above the

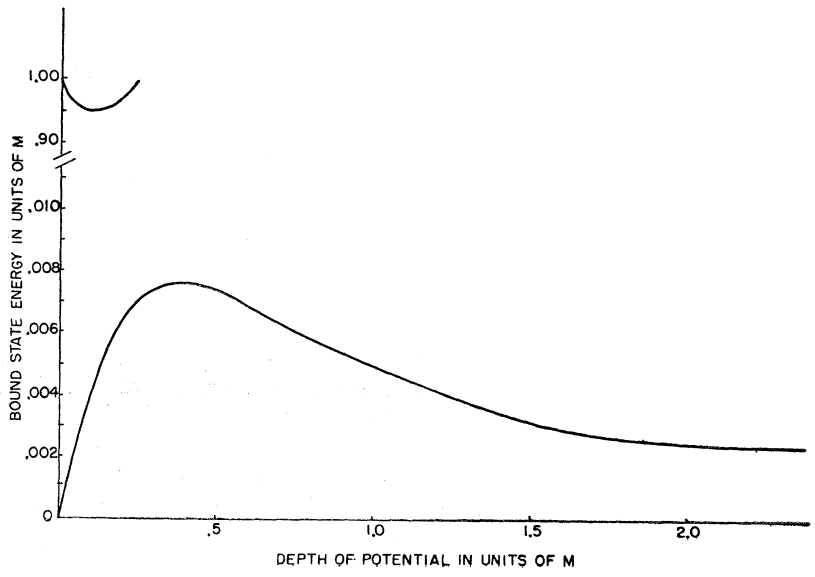

FIG. 2. Physical and unphysical solutions for the $^1S_0$ state, with vector coupling. $r_0=40M^{-1}$. (Note the change of scale for E.)

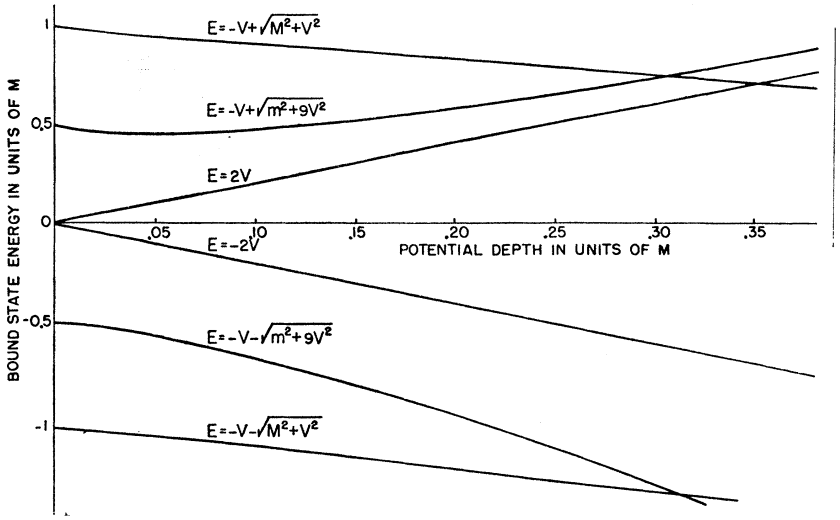

FIG. 3. Locus of poles and zeros of $\alpha^2(E,V)$ for the $^3P_0$ state.

curve $E=-V+(M^2+V^2)^{1/2}$, and, if $m\neq 0$, above the line $E=2V$. This places an absolute minimum on E of $\frac{1}{2}\sqrt{2}M$. However, if $m=0$, the singularity at $E=2V$ in α^2 is cancelled, and the energy of the physical solutions tends asymptotically to zero as $V\to\infty$. We shall call solutions with this property "well behaved."

The limit of equal masses is clearly a strange one. The existence of even a very small splitting completely changes the nature of the solutions. In the mathematical sense the convergence of the differential equation is not uniform as we take the limit $m\to 0$, and although this had no effect for the $^1S_0$ state, we see that, in general, one may lose a lot of information by solving (2.1) only for equal masses.

III. THE TWO-BODY DIRAC EQUATION: THE $J=1$ STATES

A. The $^3S_1+{}^3D_1$ State

For $J\neq 0$ there are two LS states of each parity, which will be mixed by the interaction. The wave function of over-all negative parity (which we identify as the vector meson nonet) is a mixture of $^3S_1$ and $^3D_1$ in its diagonal components, and of $^3P_1$ and $^1P_1$ in its off-diagonal components. We shall refer to it, for convenience, as the $^3S_1$ wave function. It has the form

$$\psi_{-}{}^{1m}(\mathbf{r})=\begin{pmatrix} f_1(r)\chi_{01}{}^{1m}+h_1(r)\chi_{21}{}^{1m} & i(g_{11}(r)\chi_{11}{}^{1m}+g_{21}(r)\chi_{10}{}^{1m}) \\ i(g_{12}(r)\chi_{11}{}^{1m}+g_{22}(r)\chi_{10}{}^{1m}) & f_2(r)\chi_{01}{}^{1m}+h_2(r)\chi_{21}{}^{1m}\end{pmatrix}. \tag{3.1}$$

Equation (2.1) now breaks up into eight equations. Writing these in terms of the linear combinations $f_\pm=f_1\pm f_2$, $g_{1\pm}=g_{11}\pm g_{12}$, $g_{2\pm}=g_{21}\pm g_{22}$, and $h_\pm=h_1\pm h_2$, we have

$$Mf_+-Ef_--\frac{8}{3}\left(\frac{d}{dr}+\frac{2}{r}\right)g_{1-}=2Vf_-, \tag{3.2}$$

$$Mf_--Ef_+-\frac{4}{3}\left(\frac{d}{dr}+\frac{2}{r}\right)g_{2+}=0, \tag{3.3}$$

$$Mg_{1+}-Eg_{1-}+\frac{d}{dr}f_-+2\left(\frac{d}{dr}+\frac{3}{r}\right)h_-=2Vg_{1-}, \tag{3.4}$$

$$Mg_{2-}-Eg_{2+}+\frac{d}{dr}f_+-4\left(\frac{d}{dr}+\frac{3}{r}\right)h_+=4Vg_{2+}, \tag{3.5}$$

$$Mh_+-Eh_--\frac{2}{3}\left(\frac{d}{dr}-\frac{1}{r}\right)g_{1-}=2Vh_-, \tag{3.6}$$

$$Mh_--Eh_+-\frac{2}{3}\left(\frac{d}{dr}-\frac{1}{r}\right)g_{2+}=0, \tag{3.7}$$

$$mg_{1-}-Eg_{1+}=0, \tag{3.8}$$

$$mg_{2+}-Eg_{2-}=-2Vg_{2-}. \tag{3.9}$$

Equations (3.8) and (3.9) may be used to eliminate g_{1+} and g_{2-}. We may then drop the indices 1 and 2 without confusion, defining

$$g_+\equiv g_{2+},\quad g_-\equiv g_{1-}.$$

Again, for a square-well potential, the solutions are spherical Bessel functions:

$$\begin{aligned} &f_\pm=A_\pm j_0(\alpha r),\quad g_\pm=B_\pm j_1(\alpha r),\\ &\qquad\qquad h_\pm=C_\pm j_2(\alpha r),\quad r<r_0\\ &f_\pm=a_\pm h_0{}^{(1)}(\beta r),\quad g_\pm=b_\pm h_1{}^{(1)}(\beta r),\\ &\qquad\qquad h_\pm=c_\pm h_2{}^{(1)}(\beta r),\quad r>r_0. \end{aligned} \tag{3.10}$$

β is still given by Eq. (2.9), but in solving for α we find two solutions:

$$\alpha_1^2=-\frac{1}{4}\frac{[E(E+2V)-M^2][E(E+2V)-m^2]}{E^2}, \tag{3.11}$$

and

$$\alpha_2^2=-\frac{1}{4}\frac{[E(E+2V)-M^2][(E+4V)(E-2V)-m^2]}{E^2-4V^2}. \tag{3.12}$$

Corresponding to these two solutions there are two different sets of coefficients, related to each other by

$$A_+^{(1)}=\frac{M}{E}A_-^{(1)}, \qquad a_+^{(1)}=\frac{M}{E}a_-^{(1)}, \tag{3.13}$$

$$B_+^{(1)}=0, \qquad b_+^{(1)}=0, \tag{3.14}$$

$$B_-^{(1)}=-\frac{3}{2}\frac{\alpha_1}{E+2V-m^2/E}A_-^{(1)},$$

$$b_-^{(1)}=-\frac{3}{2}\frac{\beta}{E-m^2/E}a_-^{(1)}, \tag{3.15}$$

$$C_\pm^{(1)}=-\tfrac{1}{4}A_\pm^{(1)}, \qquad c_\pm^{(1)}=-\tfrac{1}{4}a_\pm^{(1)}, \tag{3.16}$$

for solution 1, and

$$A_-^{(2)}=\frac{M}{E+2V}A_+^{(2)}, \quad a_-^{(2)}=\frac{M}{E}a_+^{(2)}, \tag{3.17}$$

$$B_+^{(2)}=-3\frac{\alpha_2}{E+4V-m^2/(E-2V)}A_+^{(2)},$$

$$b_+^{(2)}=-3\frac{\beta}{E-m^2/E}a_+^{(2)}, \tag{3.18}$$

$$B_-^{(2)}=0, \qquad b_-^{(2)}=0, \tag{3.19}$$

$$C_\pm^{(2)}=\tfrac{1}{2}A_\pm^{(2)}, \qquad c_\pm^{(2)}=\tfrac{1}{2}a_\pm^{(2)}, \tag{3.20}$$

for solution 2.

We see that solution 1 is pure triplet in its off-diagonal components and solution 2 is pure singlet. From the point of view of the spin eigenvalues of its components, solution 2 is identical to the $^3P_0$ state, while solution 1 has no analog for $J=0$. Furthermore, because we are dealing with a constant potential, α depends only on the spin of the components of the wave function. Thus, α_2^2 is the same as the α^2 we had for the $^3P_0$ state, and has the same singularities. α_1^2, which corresponds to a wave function with no singlet part, has no singularities.

The actual eigenstates will be linear combinations of these two solutions, the amount of mixing being determined by the boundary conditions. We see from Eqs. (3.2)–(3.5) that g_+, g_-, f_+-4h_+, and f_-+2h_- are continuous across the boundary. This gives rise to four homogeneous equations in the 24 coefficients $A_\pm^{(1,2)}$, $a_\pm^{(1,2)}$, $B_\pm^{(1,2)}$, $b_\pm^{(1,2)}$, $C_\pm^{(1,2)}$, $c_\pm^{(1,2)}$ which, together with the 20 equations (3.13)–(3.20), enable us to determine them all to within an over-all normalization. After some algebra, we eliminate all the coefficients except $a_-^{(1)}$ and $a_+^{(2)}$, which then are found to satisfy

$$a_-^{(1)}[F_1-E(D-d)]+2a_+^{(2)}M(D-d)=0, \tag{3.21}$$

$$a_+^{(2)}[F_2-2(D(E+2V)-Ed)]+a_-^{(1)}M(D-d)=0, \tag{3.22}$$

where

$$F_i=\tfrac{1}{4}r_0^2\left[\frac{E(E+2V)-m^2}{E}\frac{1}{1-\alpha_i r_0\cot\alpha_i r_0}-\frac{E^2-m^2}{E}\frac{1}{1+i\beta r_0}\right], \quad i=1,2 \tag{3.23}$$

with $D=[E(E+2V)-M^2]^{-1}$, $d=(E^2-M^2)^{-1}$.

In the weak-binding limit, $V\ll M$, $M-E\equiv B\ll M$, we have

$$\alpha_1^2=\alpha_2^2\equiv\alpha^2=2\mu(V-B), \tag{3.24}$$

$$\beta^2=-2\mu B, \tag{3.25}$$

where μ is the usual reduced mass: $\mu=(M^2-m^2)/4M$. The boundary conditions become

$$a_-^{(1)}F=\frac{V}{2B}\frac{1}{V-B}(a_-^{(1)}-2a_+^{(2)}), \tag{3.26}$$

$$-a_+^{(2)}F=\frac{V}{2B}\frac{1}{V-B}(a_-^{(1)}-2a_+^{(2)}), \tag{3.27}$$

where $F\equiv F_1=F_2$. The solutions are

$$F=0, \qquad a_-^{(1)}=2a_+^{(2)}, \quad \text{(pure } {}^3S_1 \text{ state)} \tag{3.28}$$

$$F=\frac{3}{2}\frac{V}{B}\frac{1}{V-B}, \quad a_-^{(1)}=-a_+^{(2)}, \quad \text{(pure } {}^3D_1 \text{ state)} \tag{3.29}$$

as we expect.

If $m\neq 0$, then α_2^2 has a singularity at $E=2V$. The energy will therefore be bounded below by $\frac{1}{2}\sqrt{2}M$, just as in the $^3P_0$ case, unless we can satisfy the boundary conditions with a state consisting of pure solution 1. Setting $a_+^{(2)}=0$ in (3.22), we arrive at a contradiction. The $^3S_1$ state, therefore, behaves like the $^3P_0$; in particular, its energy is never less than $\frac{1}{2}\sqrt{2}M$, if $m\neq 0$. If $m=0$, the solutions are well behaved.

B. The $^3P_1+{}^1P_1$ State

The positive-parity $J=1$ state, which is a mixture of $^3P_1$ and $^1P_1$, satisfies the same equations and boundary conditions as the negative-parity state, with M and m

interchanged. α_1^2 and β^2 remain unchanged, α_2^2 becomes

$$\alpha_2^2=-\frac{1}{4}\frac{[(E+4V)(E-2V)-M^2][E(E+2V)-m^2]}{E^2-4V^2}, \tag{3.30}$$

and the boundary conditions are

$$a_-^{(1)}[F_1-E(D-d)]+2a_+^{(2)}m(D-d)=0, \tag{3.31}$$

$$a_+^{(2)}[F_2-2(D(E+2V)-Ed)]+a_-^{(1)}m(D-d)=0, \tag{3.32}$$

where

$$F_i=\tfrac{1}{4}r_0^2\left[\frac{E(E+2V)-M^2}{E}\frac{1}{1-\alpha_i r_0\cot\alpha_i r_0}-\frac{E^2-M^2}{E}\frac{1}{1+i\beta r_0}\right], \quad i=1,2 \tag{3.33}$$

with $D=[E(E+2V)-m^2]^{-1}$, and $d=(E^2-m^2)^{-1}$.

In the nonrelativistic limit, with $\alpha^2\equiv\alpha_1^2=\alpha_2^2$, $F\equiv F_1=F_2$, these become

$$a_-^{(1)}[F+2V]=0, \tag{3.34}$$

$$a_+^{(2)}[F+4V]=0. \tag{3.35}$$

The two solutions are

$$F=-2V, \quad a_+^{(2)}=0, \quad \text{(pure } {}^3P_1 \text{ state)} \tag{3.36}$$

$$F=-4V, \quad a_-^{(1)}=0, \quad \text{(pure } {}^1P_1 \text{ state)}. \tag{3.37}$$

α_2^2 now corresponds to the ${}^1S_0$ solution, for which we had $E\geqslant\frac{2}{3}\sqrt{2}M$, even when $m=0$. Setting $a_+^{(2)}=0$ in (3.33), we arrive at a contradiction only if $m\neq 0$. If $m=0$, a state which is pure solution 1 must satisfy

$$F_1=E(D-d),$$

which it is certainly possible to do. So the energy of the positive-parity state is well behaved for $m=0$, not because of the cancellation of singularities in that limit, but because the boundary conditions permit the existence of a solution which contains no singlet part, and is inherently well behaved.

The generalization of these results to angular momenta other than 0 and 1 is straightforward. Those states with parity $(-1)^J$ (for $\bar{q}q$ systems) will resemble the ${}^3S_1$ solution, aside from minor changes in the numerical coefficients. The states with parity $(-1)^{J+1}$ will resemble the mixture of P states we have just looked at. In either case, the solutions are well behaved only for the case of equal masses.

IV. OTHER POTENTIALS

A. Other Couplings

A change in the coupling results in a change in the parameter α^2. In Table I we have listed the form of this parameter for all five couplings for the ${}^1S_0$ state. The ${}^3P_0$ state is dealt with, as before, simply by interchanging M and m, except that for scalar, tensor, and pseudoscalar couplings we must also change the sign of V. We shall not discuss the states of higher J. The following observations are easily verified:

(1) All the couplings lead to singularities in α^2 with the exception of T. For this case, the approximation $\alpha^2=0$ leads to

$$E=-3V+(M^2+9V^2)^{1/2} \quad \text{for the } {}^1S_0 \text{ state}, \tag{4.1}$$

$$E=-V+(M^2+V^2)^{1/2} \quad \text{for the } {}^3P_0 \text{ state}. \tag{4.2}$$

The ${}^1S_0$ state lies below the ${}^3P_0$ for all values of V. Both states are well behaved, regardless of whether the masses of the two particles are equal.

(2) For T and A potentials, *either* the ${}^1S_0$ *or* the ${}^3P_0$ state is bound, but not both. An attractive potential for one state is repulsive for the state of opposite parity.

(3) For S coupling, the energy of both states is bounded below by the singularity at $E=V$, if $m\neq 0$. If $m=0$, both solutions are well behaved.

TABLE I. Form of α^2 for the five types of potential. The form given is for the ${}^1S_0$ state.

Type	Coupling	α^2
S	$[\beta^{(1)}\beta^{(2)}]$	$-\frac{1}{4}\frac{[(E+V)^2-M^2][(E-V)^2-m^2]}{(E+V)(E-V)}$
V	$[I^{(1)}I^{(2)}-\boldsymbol{\alpha}^{(1)}\cdot\boldsymbol{\alpha}^{(2)}]$	$-\frac{1}{4}\frac{[(E+4V)(E-2V)-M^2][E(E+2V)-m^2]}{(E+2V)(E-2V)}$
T	$[(\beta\boldsymbol{\sigma})^{(1)}\cdot(\beta\boldsymbol{\sigma})^{(2)}+(\beta\boldsymbol{\alpha})^{(1)}\cdot(\beta\boldsymbol{\alpha})^{(2)}]$	$-\frac{1}{4}\frac{[E^2+6EV-M^2][E^2+2EV-m^2]}{E^2}$
A	$[\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}+(\gamma_5)^{(1)}(\gamma_5)^{(2)}]$	$-\frac{1}{4}\frac{[(E+4V)(E+2V)-M^2][E(E-2V)-m^2]}{(E+2V)(E-2V)}$
P	$[(\beta\gamma_5)^{(1)}(\beta\gamma_5)^{(2)}]$	$-\frac{1}{4}\frac{[E^2-V^2-M^2][E^2-V^2-m^2]}{E^2-V^2}$

(4) For A coupling, the energy of the $^1S_0$ state is bounded by the singularity at $E=2V$, for $m\neq 0$. The $^3P_0$ energy, however, is bounded by the curve $E=3V+(m^2+V^2)^{1/2}$, whether or not $m=0$.

(5) The P coupling admits no bound states, for either sign of the potential. Since this is the only coupling with no nonrelativistic limit, this result is not as surprising as it might appear.

In Tables II and III, the results for the $J=0$ states are summarized.

B. Other Potentials

So far we have dealt only with square-well potentials, and it might fairly be surmised that the unexpected difficulties we have encountered in attempting to create tightly bound states are consequences of this special choice. Unfortunately, even the simplest continuously varying potentials give rise to differential equations which we have been unable to solve, because of the presence in them of an extra singularity, which does not appear, for example, in the single-particle Dirac equation. However, we are able to give qualitative arguments which indicate that the disturbing features of the square-well solutions are present for any continuous, nonsingular potential.

We shall confine our attention to the $^1S_0$ state with $m=0$. The extension to other states and $m\neq 0$ does not involve fundamental complications. We take $V(r)=\lambda f(r)$, where $f(r)$ is a positive, continuous, bounded function, and $\lim_{r\to\infty} f(r)=0$. For simplicity we take $f(r)$ monotonic decreasing. (Our convention is still that a positive potential is attractive.) λ is a positive number which we vary from zero to "turn on" the interaction. Defining $F(r)\equiv rf(r)$, we use Eqs. (2.3)–(2.6) to derive the equation for F:

$$\frac{d^2F}{dr^2}+\alpha^2(r)F=0, \tag{4.3}$$

where

$$\alpha^2=\tfrac{1}{4}E\left[E+4V(r)-\frac{M^2}{E-2V(r)}\right]. \tag{4.4}$$

In order to satisfy the boundary conditions at $r=0$ and $r=\infty$, we must have $\alpha^2>0$ for some finite range in r. (There must be a finite region in which F and its curvature have the opposite sign. When this condition holds, we shall say that F is "closed"; when F and its curvature have the same sign, we shall say that F is "open.") This puts one of two conditions on V:

$$\text{(a)}\quad \tfrac{1}{8}[E-(9E^2-8M^2)^{1/2}]<V<\tfrac{1}{8}[E+(9E^2-8M^2)^{1/2}]$$

or

$$\text{(b)}\quad \tfrac{1}{2}E<V.$$

We shall call the *spatial* regions in which V satisfies conditions (a) and (b) the regions 1 and 2, respectively. (The regions are necessarily disjoint, and the monotonicity of V requires that they be simply connected.) The position and extent of these regions depend, naturally, on E; in particular, region 1 exists only for $E<\frac{2}{3}\sqrt{2}M$.

We vary λ slowly, starting from $\lambda=0$. The physical-solution ground state, with $E\lesssim M$, will appear at some λ, say λ_0. We assume $\lambda_0 f(0)<\frac{1}{2}M$. (It is possible to construct very peculiar potentials for which this is not true. Such potentials do not have physical bound states, and we ignore them.) This condition ensures that there is a range in λ for which the physical solution exists, and is closed in one spatial region only—region 1. Note that the boundedness of the potential is required here. As we increase λ the region 1 moves away from the origin. Although the exact behavior of the energy depends on the detailed shape of the potential, it is certainly bounded below by $\frac{2}{3}\sqrt{2}M$. As $V(0)$ becomes greater than $\frac{1}{8}[E+(9E^2-8M^2)^{1/2}]$, the wave function becomes open at the origin, contrary to expectations based on the single-particle Dirac equation. This condition holds until $V(0)$ reaches $\frac{1}{2}E$.

There is also a spectrum of unphysical states with energies between 0 and $2V(0)$, which correspond to bound states of positive-energy solutions with negative-energy solutions of the single-particle Dirac equation. They are closed and have most of their support in the region 2. As long as $V(0)<\frac{1}{3}\sqrt{2}M$ the two families of solutions are disjoint: Their energy spectra are separated, and the regions where the eigenfunctions are closed do not overlap spatially.

We now increase λ until $V(0)>\frac{1}{3}\sqrt{2}M$, i.e., until there exists a range of energies for which both region 1 and region 2 exist. At this point the permissible spectra of energies for the physical and unphysical states overlap, and the energy levels of the two kinds of states will in general cross. (The physical and unphysical states have orthogonal spin parts, and are not connected by the Hamiltonian, so the Wigner–von Neumann

TABLE II. Characteristics of the $^1S_0$ state for various couplings. (All energies normalized to $M=1$; V_0 is V for maximum binding; $V_{\max}$ is maximum V for which a bound state exists.)

Type	$E_{\min}$	V_0	$V_{\max}$	Remarks
S	0.500	0.500	1.000	only if $m\neq 0$.
V	0.940	0.118	0.250	
T	0	∞	∞	solution well behaved
A	0.408	0.204	0.500	only if $m\neq 0$
P	...	...	...	no bound state exists

TABLE III. Characteristics of the $^3P_0$ state for various couplings. (All energies normalized to $M=1$; V_0 is V for maximum binding; $V_{\max}$ is maximum V for which a bound state exists.)

Type	$E_{\min}$	V_0	$V_{\max}$	Remarks
S	0.500	0.500	1.000	only if $m\neq 0$
V	0.707	0.354	0.500	only if $m\neq 0$
T	0	∞	∞	solution well behaved
A	0.816	0.204	0.250	
P	...	...	...	no bound state exists

theorem[13] does not hold.) It is now possible for the physical states, "taking advantage" of the existence of region 2, to have very large binding energies, though the character of these tightly bound solutions will depend critically on the behavior of the potential near the origin.

The generalization of this analysis to other states, other couplings, and $m\neq 0$ is straightforward. In every case we find that the singularities in α^2, which prevented the physical solutions from becoming tightly bound for square-well potentials, do not do so for continuously varying potentials. However, the physical significance of the singularities, without which no tightly bound solutions can exist, is unclear. We are led to the belief that the use of the two-particle Dirac equation in the tight-binding limit is unjustified, and turn instead to the Bethe-Salpeter equation.

V. THE REDUCED BETHE-SALPETER EQUATION

The Bethe-Salpeter equation,[14] which describes a bound state of two spinors interacting through the exchange of a vector field of mass μ, may be written in the ladder approximation:

$$[\gamma_\mu{}^{(1)}p_1{}^\mu+m_1][\gamma_\nu{}^{(2)}p_2{}^\nu+m_2]\Psi(p)=\int V(p-k)\gamma_\mu{}^{(1)}\gamma^{\mu(2)}\Psi(k)d^4k, \quad (5.1)$$

where $V(p-k)=(2\pi i)^{-1}g^2/[(p-k)^2-\mu^2]$ and p is the 4-momentum about the center of mass: $p\equiv(m_2/M)p_1-(m_1/M)p_2$. We define $P\equiv p_1+p_2$. Then, in the center-of-mass frame, multiplying through by $\gamma_0{}^{(1)}\gamma_0{}^{(2)}$, we obtain

$$\left[H_1(\mathbf{p})-\frac{m_1}{M}E-p_0\right]\left[H_2(\mathbf{p})-\frac{m_2}{M}E+p_0\right]\Psi(\mathbf{p})=\int V(p-k)\Gamma\Psi(k)d^4k, \quad (5.2)$$

in which $H_1(\mathbf{p})=\boldsymbol{\alpha}^{(1)}\cdot\mathbf{p}+\beta_1m_1$, $H_2(\mathbf{p})=-\boldsymbol{\alpha}^{(2)}\cdot\mathbf{p}+\beta_2m_2$, $\Gamma=I^{(1)}I^{(2)}-\boldsymbol{\alpha}^{(1)}\cdot\boldsymbol{\alpha}^{(2)}$, and $E=P_0$.

Equation (5.2) has proven mathematically intractable, and has been solved in closed form only in certain special cases,[15] for which advantage can be taken of its symmetry properties. We resort here to a nonconvariant approximation to (5.2), introduced by Salpeter.[12]

In the nonrelativistic limit, it is a good approximation to neglect the finite speed of propagation of the interaction. This is equivalent to neglecting $(p_0-k_0)^2$ compared to $(\mathbf{p}-\mathbf{k})^2$ in $V(p-k)$. Defining

$$\psi(\mathbf{k})\equiv\int_{-\infty}^{\infty}\Psi(k)dk_0,$$

Salpeter shows that, with the approximation mentioned above, $\psi(\mathbf{k})$ satisfies

$$[H_1(\mathbf{p})+H_2(\mathbf{p})-E]\psi(\mathbf{p})=[\Lambda_+{}^{(1)}(\mathbf{p})\Lambda_+{}^{(2)}(\mathbf{p})-\Lambda_-{}^{(1)}(\mathbf{p})\Lambda_-{}^{(2)}(\mathbf{p})]\times\int V(\mathbf{p}-\mathbf{k})\psi(\mathbf{k})d^3k, \quad (5.3)$$

where the

$$\Lambda_\pm{}^{(i)}(\mathbf{p})\equiv[H_i(\mathbf{p})\pm(p_i{}^2+m_i{}^2)^{1/2}]/[2(p_i{}^2+m_i{}^2)^{1/2}]$$

are the usual energy projection operators.

We remark that these projection operators, which distinguish (5.3) from the two-body Dirac equation, arise from the imposition of the Feynman boundary conditions on the propagator. These require that negative-energy states propagate backward in time. If we had imposed the condition that all states propagate forward in time, which corresponds to the original Dirac theory without antiparticles, we would have recovered Eq. (2.1). The projection operators, moreover, decouple the unphysical states, which are combinations of positive- and negative-energy states, and the minus sign between them effectively changes the sign of the potential when it acts on the terms of order v^2/c^2 in the wave function. This eliminates the singularities which occurred in the Dirac equation.

Below, we use Eq. (5.3) with $m_1=m_2=m_0$ to determine the ground-state energy of the $^1S_0$ state (which we identify with the π), and the $^3S_1+{}^3D_1$ (which we identify with the ρ). The other particles in the pseudoscalar and vector multiplets could then be split off from these lowest-lying ones, for example, by introducing a mass splitting among the quarks. We shall not do this here, however.

We work in the momentum representation. Equation (5.3) then breaks up into four (or eight) coupled integral equations. The solution of these is quite difficult; we shall use an S-wave separable potential, of the type introduced by Yamaguchi.[16] (This is in contrast to the model of Mitra[17] for baryons, in which he uses P-wave forces for the qq interaction.) We write

$$V(\mathbf{p},\mathbf{k})=v(\mathbf{p})v(\mathbf{k}),$$

where

$$v(\mathbf{p})=\frac{g}{\mathbf{p}^2+\mu^2}.$$

Because we have in mind a picture in which the interaction is mediated by the exchange of the $SU(3)$

[13] J. von Neumann and E. P. Wigner, Z. Physik **30**, 467 (1929).
[14] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
[15] R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954); J. Goldstein, *ibid.* **91**, 1516 (1953). Recently, some more general cases have been numerically solved, using a computer. See C. Schwartz and C. Zemach, Phys. Rev. **141**, 1454 (1966).
[16] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).
[17] A. N. Mitra, Phys. Rev. **142**, 1119 (1966).

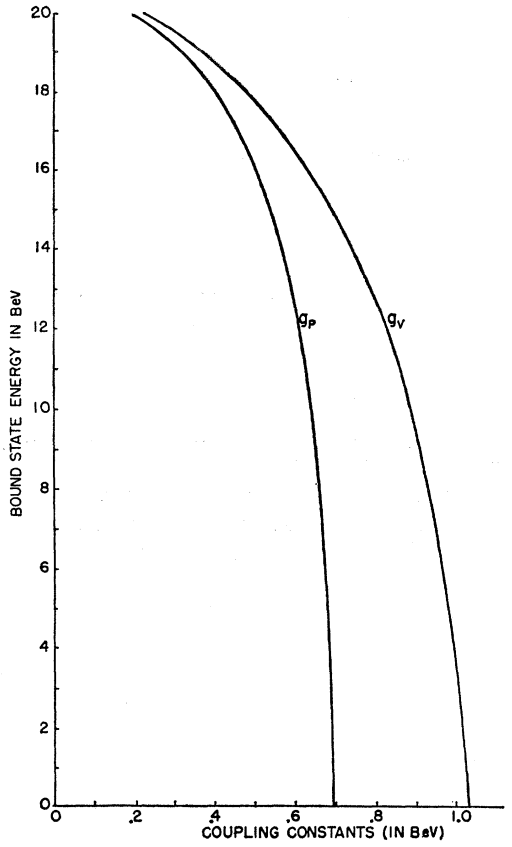

FIG. 4. E versus g_P and g_V for a quark mass of 10 BeV.

scalar member of the vector nonet, we choose μ to be the mass of this particle. Using the ω-ϕ mixing angle given by static $SU(6)$,[18] we find $\mu=862$ MeV. It should be noted that our results are insensitive to the choice of μ, for $\mu<m_0$ $(=m_1=m_2)$.

In comparing results for the $^1S_0$ and $^3S_1$ states, we must recognize that, while the use of a separable potential probably does not affect them differently in the nonrelativistic limit, in the relativistic regime, where the $^3S_1-{}^3D_1$ mixing is appreciable, the effective strength of the interaction is likely to be quite different for the two states. We can, of course, make up for this by allowing for two different coupling constants, but only at the cost of introducing another input parameter.

A. The π Meson

The $^1S_0$ wave function in momentum space has the general form

$$\psi_\pi(\mathbf{p})=\begin{pmatrix}\chi_{00}{}^0 f_1{}^P(p) & \chi_{11}{}^0 g_1{}^P(p)\\ \chi_{11}{}^0 g_2{}^P(p) & \chi_{00}{}^0 f_2{}^P(p)\end{pmatrix}, \tag{5.4}$$

in which we have reverted back to the noncovariant notation $p=|\mathbf{p}|$, $p^2=\mathbf{p}\cdot\mathbf{p}$. Equation (5.3) now gives the four-coupled equations

$$2m_0 f_+{}^P-Ef_-{}^P=4m_0\frac{v(p)}{E_0}f_+{}^0, \tag{5.5}$$

$$2m_0 f_-{}^P-Ef_+{}^P=2pg_+{}^P-2m_0\frac{v(p)}{E_0}f_+{}^0, \tag{5.6}$$

$$Eg_+{}^P=-2pf_+{}^P+4p\frac{v(p)}{E_0}f_+{}^0, \tag{5.7}$$

$$g_-{}^P=0, \tag{5.8}$$

where $f_\pm{}^P=f_1{}^P\pm f_2{}^P$, $g_\pm{}^P=g_1{}^P\pm g_2{}^P$, as before, $E_0=(p^2+m_0{}^2)^{1/2}$, $f_\pm{}^0\equiv\int v(\mathbf{k})f_\pm{}^P(\mathbf{h})d^3k$. The solution to these purely algebraic equations is

$$f_+{}^P=\left[\frac{m_0{}^2(1+2V^0)+2p^2}{p^2+K^2}\right]\frac{v(p)}{E_0}f_+{}^0, \tag{5.9}$$

$$f_-{}^P=\frac{2m_0}{E}\left[\frac{m_0{}^2(1+2V^0)-2K^2}{p^2+K^2}\right]\frac{v(p)}{E_0}f_+{}^0, \tag{5.10}$$

$$g_+{}^P=-\frac{2p}{E}\left[\frac{m_0{}^2(1+2V^0)-2K^2}{p^2+K^2}\right]\frac{v(p)}{E_0}f_+{}^0, \tag{5.11}$$

where

$$K^2=m_0{}^2-\tfrac{1}{4}E^2 \quad\text{and}\quad V^0\equiv\int\frac{v^2(p)}{E_0}\frac{d^3p}{(2\pi)^3}.$$

Multiplying (5.9) by $v(p)$ and integrating over $\mathbf{p}$, we obtain an equation for g_P, the pseudoscalar coupling constant, as a function of m_0 and E.

B. The ρ Meson

For the $^3S_1$ state, ψ has the form

$$\psi_\rho{}^m(p)=\begin{pmatrix}\chi_{01}{}^{1m}f_1{}^V(p)+\chi_{21}{}^{1m}h_1{}^V(p) & \chi_{11}{}^{1m}g_{11}{}^V(p)+\chi_{10}{}^{1m}g_{21}{}^V(p)\\ \chi_{11}{}^{1m}g_{12}{}^V(p)+\chi_{10}{}^{1m}g_{22}{}^V(p) & \chi_{01}{}^{1m}f_2{}^V(p)+\chi_{21}{}^{1m}h_2{}^V(p)\end{pmatrix}. \tag{5.12}$$

Equation (5.3) now yields

$$2m_0 f_+{}^V-Ef_-{}^V=(8/3)pg_{1-}{}^V, \tag{5.13}$$

$$2m_0 f_-{}^V-Ef_+{}^V=\tfrac{4}{3}pg_{2+}{}^V+2m_0\frac{v(p)}{E_0}f_-{}^0, \tag{5.14}$$

$$Eg_{1-}{}^V=p\left[\frac{v(p)}{E_0}f_-{}^0-(f_-{}^V+2h_-{}^V)\right], \tag{5.15}$$

$$Eg_{2+}{}^V=-p(f_+{}^V+4h_+{}^V), \tag{5.16}$$

$$2m_0 h_+{}^V-Eh_-{}^V=\tfrac{2}{3}pg_{1-}{}^V, \tag{5.17}$$

$$2m_0 h_-{}^V-Eh_+{}^V=-\tfrac{2}{3}pg_{2+}{}^V, \tag{5.18}$$

[18] B. Sakita, Phys. Rev. **136**, B1756 (1964).

$$g_{1+}{}^{V}=0, \tag{5.19}$$

$$g_{2-}{}^{V}=0, \tag{5.20}$$

where

$$f_{-}{}^{0}\equiv\int v(p)f_{-}{}^{V}(p)\frac{d^3p}{(2\pi)^3}.$$

In contrast to the situation encountered for the Dirac equation, Eqs. (5.13)–(5.20) have only one solution. This corresponds to the fact that the D wave is not coupled, so that in the nonrelativistic limit only the pure ${}^3S_1$ solution exists. Solving the equations, we obtain

$$f_{+}{}^{V}=\frac{m_0{}^2}{p^2+K^2}\frac{v(p)}{E_0}f_{-}{}^{0}, \tag{5.21}$$

$$f_{-}{}^{V}=\frac{\frac{2}{3}p^2+m_0{}^2}{p^2+K^2}\frac{v(p)}{E_0}f_{-}{}^{0}, \tag{5.22}$$

$$g_{1-}{}^{V}=-\frac{1}{4}\frac{pE}{p^2+K^2}\frac{v(p)}{E_0}f_{-}{}^{0}, \tag{5.23}$$

$$g_{2+}{}^{V}=-\frac{p}{E}\frac{m_0{}^2}{p^2+K^2}\frac{v(p)}{E_0}f_{-}{}^{0}, \tag{5.24}$$

$$h_{+}{}^{V}=0, \tag{5.25}$$

$$h_{-}{}^{V}=-\frac{1}{6}\frac{p^2}{p^2+K^2}\frac{v(p)}{E_0}f_{-}{}^{0}. \tag{5.26}$$

Multiplying (5.22) by $v(p)$ and integrating over p, we obtain an equation for g_V as a function of m_0 and E. In Fig. 4 we have graphed the bound-state energy versus g_P and g_V, for a quark mass of 10 BeV. In Fig. 5 the two coupling constants, determined by the requirements $E_P=m_\pi$, $E_V=m_\rho$, are plotted against m_0. We see that $g_V>g_P$ for all values of m_0. This model therefore gives a π-ρ mass difference which is much too big. This is to be expected, since the separable S-wave potential we have used does not couple the ${}^3D_1$ part of the ρ wave function. g_V would have to be increased to compensate for this, and this would act to bring the calculated energy closer to the observed mass.

VI. APPLICATIONS OF THE MODEL

From the curves of Fig. 4, we see that in the tight-binding limit a small variation in g results in a large variation in E, for both states. This is a welcome feature, in that it predicts that mass splittings which are large on the scale of the low-lying particle masses may nevertheless be produced by small perturbations, on the scale of the quark rest mass. This is in contrast to the situation encountered with the Dirac equation, where the well-behaved solutions showed very little variation of $E(V)$ in the ultrarelativistic limit. We shall not treat $SU(3)$ breaking effects further in this paper, however.

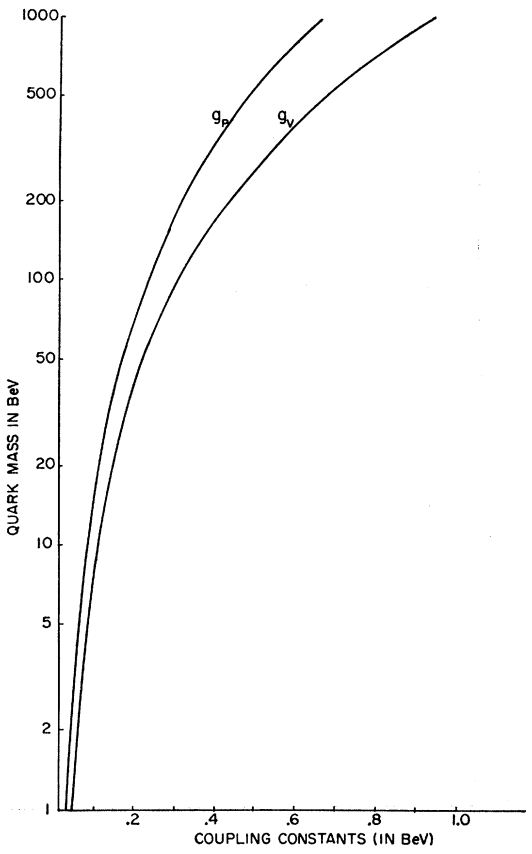

FIG. 5. Pseudoscalar and vector coupling constants determined by the conditions $E_P=m_\pi$, $E_V=m_\rho$.

It is intuitively clear that the pion lifetime is related, in a quark model, to the overlap of the quark spatial wave functions, and is thus, in principle, calculable. We define the pion decay constant f_π by

$$\langle 0|A_\mu|\pi(P)\rangle=i\frac{m_\pi}{(2P_0)^{1/2}}f_\pi(P^2)P_\mu, \tag{6.1}$$

where A_μ is the axial-vector weak current, and P_μ is the momentum of the pion. Experimentally,[19] $f_\pi(m_\pi{}^2)=0.925$. In the quark model, the state $\pi(P)$ is represented by the wave function

$$\left[\gamma_2{}^{(1)}\int\psi_\pi(\mathbf{p})d^3p\right]e^{i\mathbf{P}\cdot\mathbf{R}},$$

in which we have inserted the $\gamma_2{}^{(1)}$ in order to turn ψ_π into a particle-antiparticle state. We adopt the normalization

$$\int\psi_\pi{}^\dagger(\mathbf{p})\psi_\pi(\mathbf{p})\frac{d^3p}{(2\pi)^3}=1, \tag{6.2}$$

[19] R. P. Feynman, in *Proceedings of the International School of Physics "Ettore Majorana"* (Academic Press Inc., New York, 1964), p. 125. His F_π is related to our f_π by $[G/(4\pi)^{1/2}]m_\pi f_\pi=F_\pi$.

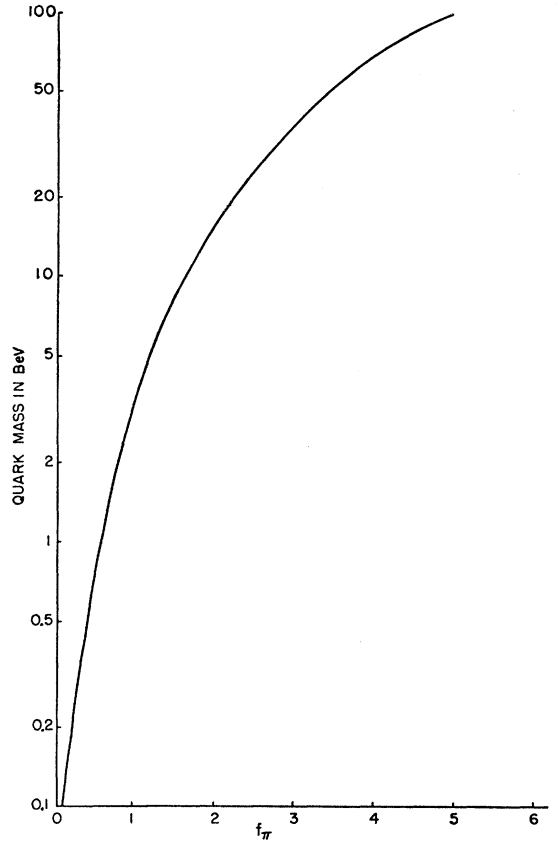

FIG. 6. Pion decay constant as a function of the quark mass.

for a pion at rest, where $\psi_{\alpha\beta}{}^\dagger \equiv \psi_{\beta\alpha}{}^*$. This determines the constant $f_+{}^0$.

The axial-vector current is given by

$$\langle q|A_\mu|q\rangle = \langle 0|\bar{\psi}_q\gamma_\mu\gamma_5\psi_q|0\rangle = \langle 0|\psi_q{}^\dagger\gamma_0\gamma_\mu\gamma_5\psi_q|0\rangle. \quad (6.3)$$

Thus, by crossing symmetry,

$$\langle 0|A_\mu|\pi(P)\rangle = \left[\int \mathrm{Tr}\{\gamma_0\gamma_\mu\gamma_5\gamma_2\psi_\pi(\mathbf{p})\}\frac{d^3p}{(2\pi)^3}\right]e^{i\mathbf{P}\cdot\mathbf{R}}. \quad (6.4)$$

For a pion at rest, only A_0 is nonvanishing, and we have

$$-i\langle 0|A_0|\pi\rangle = \frac{m_\pi{}^{3/2}}{\sqrt{2}}f_\pi = 2\int f_-{}^P(p)\frac{d^3p}{(2\pi)^3}. \quad (6.5)$$

In terms of the Fourier transform $\tilde{f}_-{}^P(\mathbf{r}) \equiv \int f_-{}^P(p) \times e^{-i\mathbf{p}\cdot\mathbf{r}}[d^3p/(2\pi)^3]$,

$$f_\pi = (2/m_\pi)^{3/2}\tilde{f}_-{}^P(0), \quad (6.6)$$

which gives f_π as a function of the overlap. This, in turn, is completely determined by specifying m_0. The results are graphed in Fig. 6; the observed value is obtained for a quark mass of 3.1 BeV.

It is of interest to note that the calculation of the matrix element $\langle 0|P|\pi(P)\rangle$, where P is the pseudoscalar current, $P=\bar{\psi}_q\gamma_5\psi_q$, would have involved the integral

$$\int f_+{}^P(p)\frac{d^3p}{(2\pi)^3}$$

which diverges. This may be due only to our choice of a separable potential, and in any case could be handled, in principle, by the introduction of a cutoff, but calculations of this matrix element are evidently unreliable.

Turning to the ${}^3S_1$ wave function, with g_V chosen so as to make $E_V = m_\rho$, we may calculate the matrix element

$$\langle 0|V_\mu|\rho^k(\mathbf{p})\rangle = \left[\int \mathrm{Tr}\{\gamma_0\gamma_\mu\gamma_2\psi_\rho{}^k(\mathbf{p})\}\frac{d^3p}{(2\pi)^3}\right]e^{i\mathbf{P}\cdot\mathbf{R}}, \quad (6.7)$$

where the superscript k denotes the spin orientation of the ρ. This matrix element appears, for example, in the calculation of the ρ-γ vertex, which is of interest in the calculation of nucleon electromagnetic form factors.[20] For $\mathbf{P}=0$, $P_0=m_\rho$, the zeroth component vanishes, and we find

$$i\sum_{k=1}^{3}\langle 0|V_k|\rho^k\rangle = 2\int f_-{}^V(p)\frac{d^3p}{(2\pi)^3}. \quad (6.8)$$

This integral also diverges. There exists, however, a matrix element analogous to $\langle 0|A_\mu|\pi\rangle$ which converges in this case. In calculating matrix elements of the tensor current, we find

$$\sum_{k=1}^{3}\langle 0|T_{0k}|\rho^k\rangle = 2\int f_+{}^V(p)\frac{d^3p}{(2\pi)^3}, \quad (6.9)$$

which converges.

VII. CONCLUSION

We have seen that the singularities of the two-particle Dirac equation result from the imposition of boundary conditions appropriate to a picture in which the negative-energy states are unfilled, and propagate forward in time. With the boundary conditions corresponding to a filled negative-energy sea, which are those adopted in the second-quantized formalism, the reduction of the Bethe-Salpeter equation gives rise to projection operators which eliminate the singularities and uncouple the unphysical solutions of the two-particle Dirac equation.

We have then used this reduced Bethe-Salpeter equation to construct a model which, though not covariant, incorporates all the kinematical subleties of a realistic one. The pion mass and lifetime then fix the quark mass and the strength of the $q\bar{q}$ interaction. Our

[20] M. Gell-Mann and F. Zachariasen, Phys. Rev. 124, 953 (1961).

wave function for the ρ is intrinsically less reliable than that for the π, because the interaction has been approximated by an S-wave potential. The calculation of the $\rho \to 2\pi$ decay width, which is quite manageable in this model, ought to shed light on this question. We hope to report on this in the future.

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APPENDIX

The bispinors used in this paper were obtained by combining two-component spinors to make an eigenstate of S (which we write as a 2×2 matrix, in an obvious notation), and then adding L to make an eigenstate of J, using the appropriate Clebsch-Gordan coefficients. This defines the bispinors only up to a constant factor, depending on J. We have chosen the set, in coordinate space:

$$\chi_{00}{}^{0}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \tag{A1}$$

$$\chi_{11}{}^{0}=\begin{pmatrix}-\hat{x}_- & \hat{z}\\ \hat{z} & \hat{x}_+\end{pmatrix}, \tag{A2}$$

$$\chi_{01}{}^{11}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\quad \chi_{01}{}^{10}=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \chi_{01}{}^{1-1}=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \tag{A3}$$

$$\chi_{10}{}^{11}=\begin{pmatrix}0 & \hat{x}_+\\ -\hat{x}_+ & 0\end{pmatrix},\quad \chi_{10}{}^{10}=\sqrt{2}\begin{pmatrix}0 & -\hat{z}\\ \hat{z} & 0\end{pmatrix},\quad \chi_{10}{}^{1-1}=\begin{pmatrix}0 & -\hat{x}_-\\ \hat{x}_- & 0\end{pmatrix}, \tag{A4}$$

$$\chi_{11}{}^{11}=\begin{pmatrix}2\hat{z} & \hat{x}_+\\ \hat{x}_+ & 0\end{pmatrix},\quad \chi_{11}{}^{10}=\sqrt{2}\begin{pmatrix}\hat{x}_- & 0\\ 0 & \hat{x}_+\end{pmatrix},\quad \chi_{11}{}^{1-1}=\begin{pmatrix}0 & \hat{x}_-\\ \hat{x}_- & -2\hat{z}\end{pmatrix}, \tag{A5}$$

$$\chi_{21}{}^{11}=\begin{pmatrix}(3\hat{z}^2-1) & 3\hat{z}\hat{x}_+\\ 2\hat{z}\hat{x}_+ & 3\hat{x}_+\end{pmatrix},$$

$$\chi_{21}{}^{10}=\sqrt{2}\begin{pmatrix}3\hat{z}\hat{x}_- & -(3\hat{z}^2-1)\\ -(3\hat{z}^2-1) & -3\hat{z}\hat{x}_+\end{pmatrix}, \tag{A6}$$

$$\chi_{21}{}^{1-1}=\begin{pmatrix}3\hat{x}_-{}^2 & -3\hat{z}\hat{x}_-\\ -3\hat{z}\hat{x}_- & (3\hat{z}^2-1)\end{pmatrix}.$$

These are not normalized consistently. Defining

$$N_{LS}{}^{J}\equiv\sum_{m=-J}^{J}(\chi_{LS}{}^{Jm})^{\dagger}\chi_{LS}{}^{Jm},$$

we obtain

$$C_{00}{}^{0}=C_{11}{}^{0}=I,\quad C_{01}{}^{1}=\tfrac{3}{2}I,\quad C_{10}{}^{1}=2I,\quad C_{11}{}^{1}=4I,\quad C_{21}{}^{1}=12I.$$

Defining the operators $\pi_i\equiv\nabla\cdot\sigma^{(i)}$ $(i=1,2)$, we find

$$\begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix}\chi_{00}{}^{0}=\mp\chi_{11}{}^{0}\frac{d}{dr}, \tag{A7}$$

$$\begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix}\chi_{11}{}^{0}=\mp\chi_{00}{}^{0}\left(\frac{d}{dr}+\frac{2}{r}\right), \tag{A8}$$

$$\begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix}\chi_{01}{}^{1m}=\tfrac{1}{2}[\chi_{11}{}^{1m}\mp\chi_{10}{}^{1m}]\frac{d}{dr}, \tag{A9}$$

$$\begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix}\chi_{11}{}^{1m}=\frac{1}{3}\left[4\chi_{01}{}^{1m}\left(\frac{d}{dr}+\frac{2}{r}\right)+\chi_{21}{}^{1m}\left(\frac{d}{dr}-\frac{1}{r}\right)\right], \tag{A10}$$

$$\begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix}\chi_{10}{}^{1m}=\pm\frac{1}{3}\left[-2\chi_{01}{}^{1m}\left(\frac{d}{dr}+\frac{2}{r}\right)+\chi_{21}{}^{1m}\left(\frac{d}{dr}-\frac{1}{r}\right)\right], \tag{A11}$$

$$\begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix}\chi_{21}{}^{1m}=[\chi_{11}{}^{1m}\pm2\chi_{10}{}^{1m}]\left(\frac{d}{dr}+\frac{3}{r}\right), \tag{A12}$$

which are useful in the decomposition of (2.1) and (5.3). For the bispinors constructed in momentum space, with $-i\nabla$ replaced by p, the same relations hold with all the operators on the right-hand side replaced by p.