Scalar Meson, Pion-Pion Interaction, and Nuclear Forces*

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The σ-meson exchange contribution to nuclear forces is compared with the pion-pion interaction effect and with the two-pion exchange contribution, and the aspects of similarity between these three processes are discussed. The $\sigma+\pi$ exchange contribution to nuclear forces is also obtained.

I. INTRODUCTION

It was suggested[1] before the discovery of the meson resonances that a scalar meson might be necessary to account for some unexplained features of the nucleon-nucleon interaction, and since then several authors[2] have employed one or more scalar mesons in the one-boson exchange model of nuclear forces. On the other hand, experimental evidence about the existence of any scalar meson is uncertain, and conflicting claims have been made with regard to the possible existence and probable mass of the scalar meson.

In view of the above situation, we shall compare the scalar meson-exchange contribution in nuclear forces with the pion-pion interaction effect and with the two-pion exchange contribution. As we shall see, a rough similarity between these three processes indicates that the use of scalar mesons in the one-boson exchange model does not necessarily establish the existence of such mesons. Moreover, because of some interesting differences between the above processes, it should be possible to settle the question of the existence of scalar mesons by a more refined investigation of the nucleon-nucleon interaction.

We would further like to make the following observations:

(1) It would be unreasonable to determine the nucleon-nucleon interaction within the nucleon core by field-theory methods, although it may be possible to reduce the radius of the phenomenological core by calculating the higher-order contributions.

(2) Since some mesons are considerably lighter than others, we should take into account *at least* all those one- and two-boson exchange processes for which the total exchanged mass does not exceed the mass of the heaviest meson included in the investigation. It might, however, be possible to exclude some of these processes by showing that their contributions are negligible for practical purposes.

(3) We should then determine whether the one- and two-boson exchange model with a phenomenological core is able to explain the nucleon-nucleon scattering data by using only the pseudoscalar and vector mesons or whether it is necessary to include scalar mesons and pion-pion interaction.

We shall denote the scalar meson as σ, and take the coupling term for the interaction of σ mesons with nucleons as

$$H_{\sigma,N}=g_\sigma:\bar{\psi}\psi U_0:, \tag{1}$$

while the coupling terms for pion-nucleon and pion-pion interactions are

$$H_{\pi,N}=ig_\pi:\bar{\psi}\gamma_5\tau_i\psi U_i:,\quad H_{\pi,\pi}=(4\pi f/c\hbar):U_iU_iU_jU_j:, \tag{2}$$

where U_0 represents the σ-meson field operator, U_i represents the pion field operator, and all the coupling terms are given as ordered products. Following the procedure and notation of the earlier papers,[3,4] we shall express all contributions to nucleon-nucleon interaction in the form of the W matrix in the center-of-mass system, and for numerical evaluations we shall take the nucleon mass $M=939$ MeV, the pion mass $m_\pi=138$ MeV, and the pion-nucleon coupling constant $g_\pi^2/4\pi c\hbar=14$.

II. COMPARISON OF σ EXCHANGE WITH OTHER EFFECTS

The scattering matrix element for the σ exchange contribution to the nucleon-nucleon interaction is

$$S_\sigma=i(2\pi)^4\delta(p+q-p'-q')(g_\sigma^2/c\hbar)[1/(k^2+\lambda_\sigma^2)]\times[\bar{\psi}^-(\mathbf{p}')\psi^+(\mathbf{p})][\bar{\psi}^-(\mathbf{q}')\psi^+(\mathbf{q})], \tag{3}$$

which yields the W matrix

$$W_\sigma=B_\sigma S+iC_\sigma'\sin\theta(\sigma_n^{(1)}+\sigma_n^{(2)})+\tfrac{1}{2}G_\sigma(\sigma_l^{(1)}\sigma_l^{(2)}+\sigma_m^{(1)}\sigma_m^{(2)})T+N_\sigma\sigma_n^{(1)}\sigma_n^{(2)}T, \tag{4}$$

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[3] S. N. Gupta, R. D. Haracz, and J. Kaskas, Phys. 138, B1500 (1965).

[4] B. M. Barker, S. N. Gupta, and R. D. Haracz, Phys. Rev. 142, 1144 (1966).

with

$$B_\sigma=-\frac{z_\sigma g_\pi^2}{4p_0^2(\mathbf{k}^2+\lambda_\sigma^2)}\left[(\kappa+p_0)^2-2\mathbf{p}^2+\mathbf{k}^2+\frac{\mathbf{p}^4}{(\kappa+p_0)^2}\right],$$

$$C_\sigma'=\frac{z_\sigma g_\pi^2}{4p_0^2(\mathbf{k}^2+\lambda_\sigma^2)}\left[\mathbf{p}^2-\frac{2\mathbf{p}^4-\mathbf{p}^2\mathbf{k}^2}{2(\kappa+p_0)^2}\right],$$

$$G_\sigma=-\frac{z_\sigma g_\pi^2}{4p_0^2(\mathbf{k}^2+\lambda_\sigma^2)}\left[2(\kappa+p_0)^2-4\mathbf{p}^2+2\mathbf{k}^2+\frac{4\mathbf{p}^4-4\mathbf{p}^2\mathbf{k}^2+\mathbf{k}^4}{2(\kappa+p_0)^2}\right], \tag{5}$$

$$N_\sigma=-\frac{z_\sigma g_\pi^2}{4p_0^2(\mathbf{k}^2+\lambda_\sigma^2)}\left[(\kappa+p_0)^2-2\mathbf{p}^2+\mathbf{k}^2+\frac{2\mathbf{p}^4-4\mathbf{p}^2\mathbf{k}^2+\mathbf{k}^4}{2(\kappa+p_0)^2}\right],$$

where

$$z_\sigma=g_\sigma^2/g_\pi^2. \tag{6}$$

On the other hand, the pion-pion interaction effect on nucleon-nucleon scattering is given by the scattering matrix element[5]

$$S=-480i\pi^3(f\kappa^2/\lambda_\pi^4)(g_\pi^2/4\pi c\hbar)^2\delta(p+q-p'-q')\times[\bar{\psi}^-(\mathbf{p}')\psi^+(\mathbf{p})][\bar{\psi}^-(\mathbf{q}')\psi^+(\mathbf{q})][I(|\mathbf{k}|)]^2, \tag{7}$$

with

$$I(|\mathbf{k}|)=\frac{\lambda_\pi^2}{|\mathbf{k}|}\int_0^1 du\frac{1-u}{x}\ln\frac{2x+|\mathbf{k}|u}{2x-|\mathbf{k}|u}, \tag{8}$$

where

$$x=[\kappa^2(1-u)^2+\lambda_\pi^2u+(\mathbf{k}^2u^2/4)]^{1/2}. \tag{9}$$

The W matrix, obtained from (7), is found to be

$$W=BS+iC'\sin\theta(\sigma_n^{(1)}+\sigma_n^{(2)})+\tfrac{1}{2}G(\sigma_l^{(1)}\sigma_l^{(2)}+\sigma_m^{(1)}\sigma_m^{(2)})T+N\sigma_n^{(1)}\sigma_n^{(2)}T, \tag{10}$$

where

$$B=\frac{15fg_\pi^4\kappa^2}{32\pi^3c\hbar\lambda_\pi^4p_0^2}[I(|\mathbf{k}|)]^2\times\left[(\kappa+p_0)^2-2\mathbf{p}^2+\mathbf{k}^2+\frac{\mathbf{p}^4}{(\kappa+p_0)^2}\right], \tag{11}$$

and C', G, and N are given by

$$B/B_\sigma=C'/C'_\sigma=G/G_\sigma=N/N_\sigma. \tag{12}$$

As already pointed out,[5] there exists a resemblance between the σ exchange contribution and the pion-pion interaction effect on nuclear forces, which we shall now discuss more precisely. Because of the relation (12), it is sufficient to compare the coefficients B_σ and B for all values of the scattering angle θ, given by

$$\mathbf{k}^2=2\mathbf{p}^2(1-\cos\theta), \tag{13}$$

and we have carried out this comparison at the incident nucleon energies of 95 and 310 MeV in the laboratory system. We find by using the method of least squares[6] that at 95 MeV the closest agreement between B_σ and B can be brought about by choosing the unknown parameters m_σ, z_σ, and f in (5) and (11) as

$$m_\sigma=513\text{ MeV},\quad z_\sigma/f=-4.56, \tag{14}$$

which results in an agreement between B_σ and B within 1% for all values of θ. Similarly, at 310 MeV the coefficients B_σ and B can be made to agree within 4% for all values of θ if we choose

$$m_\sigma=552\text{ MeV},\quad z_\sigma/f=-5.20. \tag{15}$$

Thus, although the pion-pion interaction effect can be represented as the exchange of a scalar pseudoparticle, the effective mass of such a pseudoparticle increases and becomes more diffused at higher energies of the incident nucleon. It should, therefore, be possible to differentiate between the pion-pion interaction effect and the exchange of a real scalar meson in the nucleon-nucleon interaction.

Further, since the one-boson exchange model neglects the two-pion exchange contribution, it would be interesting to see to what extent the two-pion exchange contribution can be simulated by the σ exchange contribution. For this, we have compared the σ exchange coefficients (5) for various values of θ with the two-pion exchange coefficients[3] in the isotriplet state by using the method of least squares,[6] and the results are as follows:

(1) At 95 MeV the large coefficients $B_{2\pi}$, $G_{2\pi}$ and $N_{2\pi}$ of the two-pion exchange contribution can be made to agree with the σ exchange coefficients B_σ, G_σ, and N_σ within 6% by taking

$$m_\sigma=680\text{ MeV},\quad g_\sigma^2/4\pi c\hbar=14.2. \tag{16}$$

Moreover, although the coefficients $C_{2\pi}'$ and C_σ' do not closely agree with each other, they are both of the same order and quite small.

(2) At 310 MeV the coefficients $B_{2\pi}$, $G_{2\pi}$, and $N_{2\pi}$ can be made to agree with B_σ, G_σ, and N_σ within 38% by taking

$$m_\sigma=900\text{ MeV},\quad g_\sigma^2/4\pi c\hbar=22.8, \tag{17}$$

while $C_{2\pi}'$ and C_σ' agree within 67%.

Thus, it appears that the substitution of the σ exchange for the two-pion exchange process is a reasonable approximation at 95 MeV, but at higher energies there is only a qualitative resemblance between the two contributions.

[5] R. L. Anderson, S. N. Gupta, and J. Huschilt, Phys. Rev. 127, 1377 (1962).

[6] In carrying out the least-squares adjustment we have given equal weight to the values of the W-matrix coefficient for $\theta=0°$, 10°, 20°, 30°, $\cdots$, 180°.

III. $\sigma+\pi$ EXCHANGE CONTRIBUTION

If the σ meson is lighter than the vector mesons, it would also be of interest to evaluate the $\sigma+\pi$ exchange contribution, which can be obtained by following the earlier treatment of the $\eta+\pi$ exchange process.[4]

Thus, we find that the total contribution of the uncrossed and crossed $\sigma+\pi$ exchange diagrams can be expressed in the form of the W matrix as

$$\begin{aligned}W_{\sigma\pi}=B_{\sigma\pi}S+iC_{\sigma\pi}'\sin\theta(\sigma_n{}^{(1)}+\sigma_n{}^{(2)})&\\+\tfrac{1}{2}G_{\sigma\pi}(\sigma_l{}^{(1)}\sigma_l{}^{(2)}+\sigma_m{}^{(1)}\sigma_m{}^{(2)})T&\\+\tfrac{1}{2}H_{\sigma\pi}(\sigma_l{}^{(1)}\sigma_l{}^{(2)}-\sigma_m{}^{(1)}\sigma_m{}^{(2)})T&\\+N_{\sigma\pi}(\sigma_n{}^{(1)}\sigma_n{}^{(2)})T,&\quad(18)\end{aligned}$$

where

$$\begin{aligned}B_{\sigma\pi}&=\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}z_\sigma(g_\pi{}^2/4\pi c\hbar)^2(c\hbar/4p_0{}^2)(U_B+V_B),\\C_{\sigma\pi}'&=\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}z_\sigma(g_\pi{}^2/4\pi c\hbar)^2(c\hbar/4p_0{}^2)(U_C+V_C),\\G_{\sigma\pi}&=\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}z_\sigma(g_\pi{}^2/4\pi c\hbar)^2(c\hbar/4p_0{}^2)(U_G+V_G),\qquad(19)\\H_{\sigma\pi}&=\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}z_\sigma(g_\pi{}^2/4\pi c\hbar)^2(c\hbar/4p_0{}^2)(U_H+V_H),\\N_{\sigma\pi}&=\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}z_\sigma(g_\pi{}^2/4\pi c\hbar)^2(c\hbar/4p_0{}^2)(U_N+V_N).\end{aligned}$$

In the above coefficients the U are given by

$$\begin{aligned}U_B=&\frac{2\mathbf{s}^2}{\kappa^2}I_1+2\left[3-\frac{2\mathbf{p}^2-\mathbf{k}^2}{(\kappa+p_0)^2}+\frac{3\mathbf{p}^4}{(\kappa+p_0)^4}\right]\left(1+\frac{p_0}{\kappa}\right)^2I_2\\&+\left[\frac{\mathbf{s}^2}{2\kappa^2}\left(1+\frac{\mathbf{p}^2}{(\kappa+p_0)^2}\right)^2+\frac{\mathbf{k}^2}{2\kappa^2}\left(1-\frac{\mathbf{p}^2}{(\kappa+p_0)^2}\right)^2\right]\\&\times\left(1+\frac{p_0}{\kappa}\right)^2I_3+\frac{2\mathbf{k}^2}{\kappa^2}\left(1-\frac{\mathbf{p}^2}{(\kappa+p_0)^2}\right)^2\left(1+\frac{p_0}{\kappa}\right)^2I_4,\\U_C=&\frac{2\mathbf{p}^2}{\kappa^2}\left(1-\frac{2\mathbf{p}^2-\mathbf{k}^2}{2(\kappa+p_0)^2}\right)I_2,\\U_G=&-\frac{2\mathbf{s}^2}{\kappa^2}I_1-\left[4+\frac{4\mathbf{p}^4-4\mathbf{p}^2\mathbf{k}^2+\mathbf{k}^4}{(\kappa+p_0)^4}\right]\left(1+\frac{p_0}{\kappa}\right)^2I_2\\&-\left[\frac{\mathbf{s}^2}{2\kappa^2}\left(1+\frac{\mathbf{p}^2}{(\kappa+p_0)^2}\right)^2+\frac{\mathbf{k}^2}{2\kappa^2}\left(1-\frac{\mathbf{p}^2}{(\kappa+p_0)^2}\right)^2\right]\qquad(20)\\&\times\left(1+\frac{p_0}{\kappa}\right)^2I_3-\frac{2\mathbf{k}^2}{\kappa^2}\left(1-\frac{\mathbf{p}^2}{(\kappa+p_0)^2}\right)^2\left(1+\frac{p_0}{\kappa}\right)^2I_4,\\U_H=&\frac{2\mathbf{s}^2}{\kappa^2}I_1+\frac{8\mathbf{p}^2}{\kappa^2}I_2+\left[\frac{\mathbf{s}^2}{2\kappa^2}\left(1+\frac{\mathbf{p}^2}{(\kappa+p_0)^2}\right)^2\right.\\&\left.-\frac{\mathbf{k}^2}{2\kappa^2}\left(1-\frac{\mathbf{p}^2}{(\kappa+p_0)^2}\right)^2\right]\left(1+\frac{p_0}{\kappa}\right)^2I_3\\&-\frac{2\mathbf{k}^2}{\kappa^2}\left(1-\frac{\mathbf{p}^2}{(\kappa+p_0)^2}\right)^2\left(1+\frac{p_0}{\kappa}\right)^2I_4,\\U_N=&-2\left[1-\frac{2\mathbf{p}^2-\mathbf{k}^2}{(\kappa+p_0)^2}+\frac{2\mathbf{p}^4-4\mathbf{p}^2\mathbf{k}^2+\mathbf{k}^4}{2(\kappa+p_0)^4}\right]\left(1+\frac{p_0}{\kappa}\right)^2I_2,\end{aligned}$$

with

$$\begin{aligned}I_1&=\int_0^1du\int_0^u dv\int_0^v dw\left(\frac{\kappa^2}{2A}-\frac{\kappa^2p_0{}^2w^2}{A^2}\right),\\I_2&=\mathrm{Re}\int_0^1du\int_0^u dv\int_0^v dw\frac{\kappa^2}{2A},\\I_3&=\mathrm{Re}\int_0^1du\int_0^u dv\int_0^v dw\frac{\kappa^4v^2}{A^2},\qquad(21)\\I_4&=\mathrm{Re}\int_0^1du\int_0^u dv\int_0^v dw\frac{\kappa^4u(u-v)}{A^2},\end{aligned}$$

and

$$\begin{aligned}A=p_0{}^2w^2+\lambda_\pi{}^2(1-u)+\lambda_\sigma{}^2(u-v)&\\+\mathbf{k}^2(u-v)(1-u)-\mathbf{p}^2v^2.&\quad(22)\end{aligned}$$

Moreover, the V can be obtained from the U on replacing the integrals I_1, I_2, I_3, and I_4 by J_1, J_2, J_3, and J_4, respectively, which are given by

$$\begin{aligned}J_1&=\int_0^1du\int_0^u dv\int_0^v dw\left(\frac{\kappa^2}{2B}-\frac{\kappa^2p_0{}^2v^2}{B^2}\right),\\J_2&=\int_0^1du\int_0^u dv\int_0^v dw\frac{\kappa^2}{2B},\\J_3&=\int_0^1du\int_0^u dv\int_0^v dw\frac{\kappa^4w^2}{B^2},\qquad(23)\\J_4&=\int_0^1du\int_0^u dv\int_0^v dw\frac{\kappa^4[(u-\frac{1}{2}v)^2-\frac{1}{4}w^2]}{B^2},\end{aligned}$$

with

$$\begin{aligned}B=\kappa^2v^2+\lambda_\pi{}^2(1-u)+\lambda_\sigma{}^2(u-v)&\\+\mathbf{k}^2(u-v)(1-u)+\tfrac{1}{4}\mathbf{s}^2(v^2-w^2).&\quad(24)\end{aligned}$$

The integrals (21) and (23) can be expressed in a suitable form for numerical evaluation as indicated in the Appendix.

The estimates of m_σ by various experimental and theoretical groups range from about 400 MeV to considerably higher values. In Table I we have given values of the coefficients (19) for various scattering angles at the incident nucleon energies of 95 and 310 MeV in the laboratory system, where we have taken[7] $m_\sigma=400$ MeV while z_σ is left undetermined. Table I shows that although some of the $\sigma+\pi$ exchange coefficients are quite large in magnitude, their variations with the scattering angle are rather small. Therefore, the $\sigma+\pi$ exchange can be expected to make only a

[7] N. P. Samios, A. H. Bachman, A. M. Lea, T. E. Kalogeropoulos, and W. D. Shephard, Phys. Rev. Letters **9**, 139 (1962); J. Kirz, J. Schwartz, and R. D. Tripp, Phys. Rev. **130**, 2481 (1963).

TABLE I. W-matrix coefficients for the $\sigma+\pi$ exchange interaction in the isotriplet state at 95 and 310 MeV in units of 10^{-36} cm^3 MeV. The coefficients in the isosinglet state can be obtained by multiplying by -3.

	95 MeV					310 MeV				
θ	$B_{\sigma\pi}/z_\sigma$	$C_{\sigma\pi}'/z_\sigma$	$G_{\sigma\pi}/z_\sigma$	$H_{\sigma\pi}/z_\sigma$	$N_{\sigma\pi}/z_\sigma$	$B_{\sigma\pi}/z_\sigma$	$C_{\sigma\pi}'/z_\sigma$	$G_{\sigma\pi}/z_\sigma$	$H_{\sigma\pi}/z_\sigma$	$N_{\sigma\pi}/z_\sigma$
1°	13.34	0.0586	−8.644	−0.746	−4.695	8.390	0.1296	−5.125	−1.405	−3.265
10°	13.35	0.0586	−8.655	−0.751	−4.691	8.404	0.1293	−5.145	−1.399	−3.259
20°	13.37	0.0584	−8.689	−0.764	−4.681	8.446	0.1283	−5.204	−1.380	−3.240
30°	13.41	0.0582	−8.743	−0.784	−4.665	8.511	0.1269	−5.294	−1.350	−3.212
40°	13.46	0.0578	−8.814	−0.810	−4.644	8.593	0.1251	−5.407	−1.308	−3.178
50°	13.52	0.0575	−8.899	−0.840	−4.619	8.685	0.1230	−5.534	−1.258	−3.140
60°	13.59	0.0570	−8.993	−0.870	−4.591	8.784	0.1209	−5.667	−1.203	−3.103
70°	13.66	0.0565	−9.092	−0.900	−4.562	8.885	0.1188	−5.801	−1.145	−3.068
80°	13.73	0.0561	−9.192	−0.928	−4.533	8.984	0.1169	−5.931	−1.087	−3.036
90°	13.80	0.0556	−9.291	−0.954	−4.504	9.081	0.1151	−6.054	−1.031	−3.009
100°	13.87	0.0551	−9.386	−0.976	−4.477	9.172	0.1134	−6.169	−0.980	−2.986
110°	13.93	0.0547	−9.473	−0.995	−4.452	9.256	0.1120	−6.273	−0.934	−2.968
120°	13.98	0.0543	−9.552	−1.011	−4.430	9.332	0.1108	−6.366	−0.893	−2.954
130°	14.03	0.0540	−9.621	−1.023	−4.411	9.399	0.1098	−6.446	−0.860	−2.944
140°	14.07	0.0537	−9.678	−1.033	−4.395	9.456	0.1090	−6.513	−0.832	−2.936
150°	14.11	0.0535	−9.724	−1.040	−4.383	9.501	0.1084	−6.566	−0.811	−2.931
160°	14.13	0.0534	−9.757	−1.045	−4.374	9.533	0.1079	−6.604	−0.796	−2.928
170°	14.15	0.0533	−9.777	−1.048	−4.369	9.553	0.1077	−6.627	−0.787	−2.926
179°	14.15	0.0532	−9.783	−1.049	−4.367	9.560	0.1076	−6.635	−0.785	−2.925

small contribution outside the phenomenological core even when m_σ is only 400 MeV.

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APPENDIX

The integrals appearing in the $\sigma+\pi$ exchange contribution can be reduced to a suitable form for numerical evaluation by following the procedure of the earlier papers.[3,4] Indeed, the integrals I_1, I_2, I_3, J_1, J_2, and J_3, given by (21) and (23), have the same form as those for the $\eta+\pi$ exchange contribution,[4] while I_4 and J_4 can be expressed as follows:

$$I_4=(\kappa^4/p_0^4)(I_{4A}-I_{4B}),$$

where

$$I_{4A}=\int_0^1 du\int_0^u dv\, u(u-v)\times\begin{cases} -\dfrac{1}{2\Lambda^3}\tan^{-1}\left(\dfrac{\Lambda}{v}\right)+\dfrac{v}{2\Lambda^2(v^2+\Lambda^2)} & \text{for } \Lambda^2\geqslant 0,\\[2ex] \dfrac{1}{4|\Lambda|^3}\ln\left(\dfrac{v+|\Lambda|}{v-|\Lambda|}\right)-\dfrac{v}{2|\Lambda|^2(v^2-|\Lambda|^2)} & \text{for } \Lambda^2\leqslant 0,\end{cases}$$

$$I_{4B}=\pi\int_0^{u_0} du\frac{u(c-bu-au^2)^{1/2}}{4ac+b^2}+\pi\int_0^1 du\frac{u(bu-2c)}{2c^{1/2}(4ac+b^2)},$$

with

$$\Lambda=(c-bv-av^2)^{1/2},\quad a=\mathbf{p}^2/p_0^2,\quad b=[\lambda_\sigma^2+\mathbf{k}^2(1-u)]/p_0^2,$$
$$c=[\lambda_\pi^2(1-u)+\lambda_\sigma^2u+\mathbf{k}^2u(1-u)]/p_0^2,\quad u_0=[(1+4\mathbf{p}^2/\lambda_\pi^2)^{1/2}-1](\lambda_\pi^2/2\mathbf{p}^2);$$
$$J_4=J_4'-\tfrac{1}{4}J_3,$$

where

$$J_4'=\frac{1}{2}\int_0^1 du\int_0^u dv\frac{(u-\frac{1}{2}v)^2}{\Delta^2}\left[\frac{v}{\Delta^2-v^2\delta^2}+\frac{1}{2\Delta\delta}\ln\left(\frac{\Delta+v\delta}{\Delta-v\delta}\right)\right],$$

with

$$\Delta=[v^2+(\lambda_\pi^2/\kappa^2)(1-u)+(\lambda_\sigma^2/\kappa^2)(u-v)+(\mathbf{k}^2/\kappa^2)(1-u)(u-v)+(\mathbf{s}^2/4\kappa^2)v^2]^{1/2},\quad \delta=(\mathbf{s}^2/4\kappa^2)^{1/2}.$$