# Quantization of Multispinor Fields*†

SHAU-JIN CHANG‡§
*Physics Department, Harvard University, Cambridge, Massachusetts*
(Received 26 April 1967)

The multispinor fields which describe massive fields with spin $\leq 2$ are constructed. These fields are quantized using Schwinger's action principle. Lorentz invariance and physical positive-definiteness requirements are verified. In particular, those fields with spins 0, 1, and $\frac{1}{2}$ are explored in detail. It is shown that the spin-$\frac{1}{2}$ system described by a third-rank multispinor is not equivalent to the Dirac field in the presence of electromagnetic interaction. It describes a spin-$\frac{1}{2}$ system without intrinsic magnetic moment. The Lorentz invariance of the interacting system is verified. This result will facilitate the physical explanation of the famous $\frac{2}{3}$ ratio of the nucleon magnetic moments.

## 1. INTRODUCTION

It is well known that systems with integer spins can be described by tensor operators, and systems with half-integer spins can be described by tensor-spinors. These field operators satisfy the Fierz-Pauli equations and the Rarita-Schwinger equations, respectively.[1] It has been shown for systems with spin $\leq 2$ that these field equations can be derived from certain Lagrange functions. These systems can be quantized consistently by means of Schwinger's action principle. Lorentz invariance and physical-positiveness requirements have been verified. It is also known that systems with definite spin and mass can be described alternatively by multispinors with definite symmetry properties as well.[2] They satisfy the Bargmann-Wigner equations which have the same mathematical structure for systems with integer and half-integer spins. As free fields, these equations are equivalent to those of Fierz-Pauli and of Rarita-Schwinger. The $SU(6)$ theory of strongly interacting particles suggests that the hadrons should be represented by multispinors of various ranks.[3] The success of $SU(6)$ theory arouses new interests in this formulation. In this paper,[4] an attempt is made to study whether these systems can be quantized consistently according to the techniques of the quantum action principle. We first review the field theory of the second-rank multispinors which is equivalent to those of Kemmer-Duffin fields.[5] Some techniques are learned from the quantization of these simple systems. Lorentz invariance and positive-definiteness requirements are verified in this new formulation. These techniques are then applied to the field theory of multispinors of third and fourth ranks. One finds that the multispinor formulation has its advantages as well as its disadvantages in comparison with the usual formulations. In the last section, the possible applications of our results to strong-interaction physics are discussed.

## 2. SECOND-RANK MULTISPINORS

The second rank multispinors $\phi_{\alpha\beta}$ have two distinct symmetry classes: The antisymmetric multispinor describes a scalar field, and the symmetric multispinor describes a vector field. Since both systems behave quite analogously in this formulation, we shall discuss them collectively. As free fields, both systems can be described by the Lagrange function

$$L=-\tfrac{1}{2}\bar{\phi}(-i\Gamma\partial+m)\phi\,, \tag{1}$$

where $m$ is the field mass, and

$$\bar{\phi}=\phi(\gamma^0)_1(\gamma^0)_2\,,\quad \phi^\dagger=\phi\,,$$
$$\Gamma_\mu=\tfrac{1}{2}[(\gamma_\mu)_1+(\gamma_\mu)_2].$$

The $(A)_i$ is a matrix which operates on $i$th indices only. The field equation follows from the principle of stationary action:

$$(-i\Gamma\partial+m)\phi=0\,, \tag{2}$$

which can be reduced to the Bargmann-Wigner equation,

$$(-i\gamma\partial+m)_i\phi=0\,,\quad i=1,2 \tag{3}$$

regardless of the symmetry of $\phi$.

* Supported in part by the U. S. Air Force Office of Scientific Research under Contract No. AF 49(638)-1380.

† A partial preliminary report of this work was published: S. J. Chang, Phys. Rev. Letters **17**, 597 (1966).

‡ John Parker Fellow.
§ Present address: The Institute for Advanced Study, Princeton, New Jersey.

[1] M. Fierz, Helv. Phys. Acta **12**, 3 (1939); M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939); W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941). A complete list of classical papers can be found in the bibliography of E. M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave-Equations* (Blackie & Son, Ltd., Glasgow, 1953).

[2] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **34**, 211 (1946).

[3] F. Gürsey and L. A. Radicati, Phys. Rev. Letters **13**, 173 (1964); A. Pais, *ibid.* **13**, 175 (1964); F. Gürsey, A. Pais, and L. A. Radicati, *ibid.* **13**, 299 (1964); B. Sakita, *ibid.* **13**, 643 (1964); Phys. Rev. **136**, B1759 (1964); A. Salam, R. Delbourgo, and J. Strathdee, Proc. Roy. Soc. (London) **A284**, 146 (1965).

[4] Throughout this paper we use the following notations: $g_{\mu\nu}=(-1,1,1,1)$. All Greek indices $\mu,\nu,\cdots$, (except $\alpha,\beta,\gamma,\delta$) vary from 0 to 3, and indices $\alpha,\beta,\gamma,\delta$, which represent spinor indices, vary from 1 to 4. All Latin indices $i,j,\cdots$, (except $a,b,c,d$) vary from 1 to 3, and indices $a,b,c,d$, which represent spin indices, vary from 1 to 2. Repeated indices are to be summed over; the $\gamma$ matrices are chosen as $\gamma^0$=antisymmetric and imaginary, $\alpha^0=1$, $\gamma_k=\gamma^0\alpha_k$=symmetric and imaginary, $[\gamma_\mu,\gamma_\nu]_+=2g_{\mu\nu}$, $\sigma_{\mu\nu}=\frac{1}{2}i[\gamma_\mu,\gamma_\nu]$, $\gamma_5=\gamma^0\gamma^1\gamma^2\gamma^3$. All field operators are symmetrically (or antisymmetrically) multiplied.

[5] N. Kemmer, Proc. Roy. Soc. (London) **A173**, 91 (1939); R. J. Duffin, Phys. Rev. **54**, 1114 (1938).

In order to facilitate the quantization procedure, we decompose the field variables into components associated with the positive- and the negative-parity subspaces. This can be done by decomposing the identity operator $\delta_{\alpha\beta}$ into the measurement symbols for different eigenstates of $\gamma^0$ through[6]

$$\delta_{\alpha\beta}=\Lambda^{(+)}{}_{\alpha\beta}+\Lambda^{(-)}{}_{\alpha\beta}, \tag{4}$$

with

$$\Lambda^{(\pm)}=\tfrac{1}{2}(1\pm\gamma^0).$$

Each of the subspaces can be decomposed further according to their particular eigenvalues of spin along a fixed direction, for example, $\sigma'_3$;

$$\Lambda^{(+)}{}_{\alpha\beta}=\sum_{a=1}^{2} M^{(+)+}{}_{\alpha a}M^{(+)}{}_{a\beta},$$
$$\Lambda^{(-)}{}_{\alpha\beta}=\sum_{a=1}^{2} M^{(-)+}{}_{\alpha a}M^{(-)}{}_{a\beta}, \tag{5}$$

where $a=1$, 2 are the spin indices. The $M^{(\pm)}{}_{a\alpha}$ are the measurement symbols which transform a spinor of Dirac index $\alpha$ into a positive (or negative) eigenstate of $\gamma^0$, with $\sigma'_3=a$, i.e.,

$$M^{(\pm)}\gamma^0=\pm M^{(\pm)},$$
$$M^{(\pm)}\sigma_{12}=\sigma'_3M^{(\pm)}.$$

They satisfy the orthonormality conditions:

$$M^{(+)}{}_{a\alpha}M^{(+)\dagger}{}_{\alpha b}=\delta_{ab},$$
$$M^{(-)}{}_{a\alpha}M^{(-)\dagger}{}_{\alpha b}=\delta_{ab}, \tag{6}$$
$$M^{(\pm)}{}_{a\alpha}M^{(\mp)\dagger}{}_{\alpha b}=0.$$

Making use of proper phase conventions, we have

$$(M^{(\pm)})^*=(\mp)\sigma_yM^{(\mp)},$$
$$M^{(\pm)}\gamma_k=i\sigma_kM^{(\mp)}, \tag{7}$$
$$M^{(\pm)}\gamma_5=\pm M^{(\mp)}.$$

Now, we can decompose our field variable $\phi_{\alpha\beta}$ into components associated with various parity subspaces:

$$\begin{aligned}\phi_{\alpha\beta}&=\delta_{\alpha\alpha'}\delta_{\beta\beta'}\phi_{\alpha'\beta'}\\&=[M^{(+)\dagger}M^{(+)}+M^{(-)\dagger}M^{(-)}]_{\alpha\alpha'}\\&\qquad\times[M^{(+)\dagger}M^{(+)}+M^{(-)\dagger}M^{(-)}]_{\beta\beta'}\phi_{\alpha'\beta'}\\&=(M^{(+)\dagger})_1(M^{(+)\dagger})_2\phi^{(+)(+)}+(M^{(+)\dagger})_1(M^{(-)\dagger})_2\phi^{(+)(-)}\\&\quad+(M^{(-)\dagger})_1(M^{(+)\dagger})_2\phi^{(-)(+)}\\&\qquad+(M^{(-)\dagger})_1(M^{(-)\dagger})_2\phi^{(-)(-)},\end{aligned}$$

with

$$\phi^{(+)(+)}=(M^{(+)})_1(M^{(+)})_2\phi,$$
$$\phi^{(+)(-)}=(M^{(+)})_1(M^{(-)})_2\phi,\quad\text{etc.},$$

where $\phi^{(+)(+)}$, etc., are the corresponding field variables associated with various parity subspaces. Note that $\phi^{(+)(+)}{}_{ab}$ and $\phi^{(-)(-)}{}_{ab}$ have the same symmetry properties as $\phi_{\alpha\beta}(x)$. Field equation (2) can be transcribed to

$$\begin{aligned}(-i\partial_0+m)\phi^{(+)(+)}+\tfrac{1}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_1\phi^{(-)(+)}&\\+\tfrac{1}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_2\phi^{(+)(-)}&=0,\\(i\partial_0+m)\phi^{(-)(-)}+\tfrac{1}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_1\phi^{(+)(-)}+\tfrac{1}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_2\phi^{(-)(+)}&=0,\\m\phi^{(+)(-)}+\tfrac{1}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_2\phi^{(+)(+)}+\tfrac{1}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_1\phi^{(-)(-)}&=0,\\m\phi^{(-)(+)}+\tfrac{1}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_1\phi^{(+)(+)}+\tfrac{1}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_2\phi^{(-)(-)}&=0.\end{aligned}$$

It is easy to see that $\phi^{(+)(+)}$ and $\phi^{(-)(-)}$ are independent dynamical variables, while $\phi^{(+)(-)}$ and $\phi^{(-)(+)}$ are dependent variables. There is only one pair of independent variables for an antisymmetric $\phi(x)$, but there are three pairs of them if $\phi(x)$ is symmetric. These justify the fact that the antisymmetric multispinor represents a scalar field, and the symmetric multispinor represents a vector field. The generator,

$$\begin{aligned}G&=\tfrac{1}{2}i\int\bar{\phi}\boldsymbol{\Gamma}^0\delta\phi\,d^3x\\&=\tfrac{1}{2}i\int[\phi^{(-)(-)}(\sigma_y)_1(\sigma_y)_2\delta\phi^{(+)(+)}\\&\qquad-\phi^{(+)(+)}(\sigma_y)_1(\sigma_y)_2\delta\phi^{(-)(-)}]d^3x,\end{aligned}$$

follows also from the action principle, and leads to the following equal-time commutator relations:

$$[\phi_{ab}{}^{(+)(+)}(x),\phi_{a'b'}{}^{(-)(-)}(x')]=[(\sigma_y)_1(\sigma_y)_2]_{\{ab,a'b'\}}\delta(\mathbf{x}-\mathbf{x}'), \tag{8}$$

$$[\phi^{(+)(+)}(x),\phi^{(+)(+)}(x')]=[\phi^{(-)(-)}(x),\phi^{(-)(-)}(x')]=0. \tag{9}$$

The commutators involving the dependent fields can be obtained easily through the constraint equations. All these commutator relations can be put into a covariant form

$$[\phi_{\alpha\beta}(x),\bar{\phi}_{\alpha'\beta'}(x')]=(1/2im)[(m+i\gamma\partial)_1(m+i\gamma\partial)_2]_{\{\alpha\beta,\alpha'\beta'\}}\Delta(x-x'), \tag{10}$$

where the indices $\alpha$, $\beta$ and $\alpha'$, $\beta'$ in the curly brackets are properly symmetrized according to the symmetry of $\phi(x)$. These covariant expressions can be generalized easily to multispinors of arbitrary ranks. This is one of the advantages of using the multispinor formulation.

We now use the action principle to find the stress tensor.[7] Making use of

$$\delta\phi=-\delta x^\lambda\partial_\lambda\phi-\tfrac{1}{4}i(\partial_\alpha\delta x_\beta-\partial_\beta\delta x_\alpha)\Sigma^{\alpha\beta}\phi, \tag{11}$$

with

$$\Sigma_{\alpha\beta}=\tfrac{1}{2}[(\sigma_{\alpha\beta})_1+(\sigma_{\alpha\beta})_2],$$

we have

$$T_{\mu\nu}=Lg_{\mu\nu}-\tfrac{1}{2}i\bar{\phi}\Gamma_{(\mu}\partial_{\nu)}\phi+\tfrac{1}{2}\partial^\lambda[\bar{\phi}\Gamma_{(\mu}\Sigma_{\nu)\lambda}], \tag{12}$$

[6] J. Schwinger, Proc. Natl. Acad. Sci. U. S. **45**, 1542 (1959); **46**, 257 (1960); **46**, 570 (1960).

[7] J. Schwinger, Phys. Rev. **82**, 914 (1951); **91**, 713 (1953).

which can be reduced to

$$T_{\mu\nu}=\frac{1}{2}mg_{\mu\nu}\bar{\phi}\phi+m\bar{\phi}\Gamma_{(\mu}\Gamma_{\nu)}\phi=\frac{1}{2}m\bar{\phi}(\gamma_{(\mu})_1(\gamma_{\nu)})_2\phi. \quad (13)$$

This leads to, in particular,

$$T^{00}=\frac{1}{2}m\phi^\dagger\phi>0,\qquad T^{0k}=m\bar{\phi}\Gamma^{(0}\Gamma^{k)}\phi. \quad (14)$$

Another important consequence is that the Schwinger relation,[8]

$$[T^{00}(x),T^{00}(x')]=-i[T^0{}_k(x)+T^0{}_k(x')]\partial_k\delta(\mathbf{x}-\mathbf{x}'),$$

can be verified easily in this formulation. It can be shown that $P_k$ and $J_{kl}$, which are constructed from $T^0{}_k(x)$, behave correctly as the generators of the 3-dimensional translation and rotation group. This property, combined with the Schwinger relation, asserts the invariance of the theory under proper, orthochronous Lorentz transformation. The Lorentz invariance is not affected even in the presence of the electromagnetic interaction. One can verify this property by means of Schwinger's techniques of extended operators.[9]

The Green's function can be introduced with the aid of external sources as[10]

$$G_{\alpha\beta,\alpha'\beta'}(x,x')=\delta\langle\phi_{\alpha\beta}(x)\rangle/\delta\eta^{\alpha'\beta'}(x')\,|_{\eta=0}. \quad (15)$$

It is easy to show that the Green's function satisfies

$$(\partial^2-m^2)[G_{\alpha\beta,\alpha'\beta'}(x,x')-(1/2m)1_{\{\alpha\beta,\alpha'\beta'\}}\delta(x-x')]$$
$$=-(1/2m)[(m+i\gamma\partial)_1(m+i\gamma\partial)_2]_{\{\alpha\beta,\alpha'\beta'\}}\delta(x-x'), \quad (16)$$

with

$$1_{\{\alpha\beta,\alpha'\beta'\}}=\frac{1}{2}(\delta_{\alpha\alpha'}\delta_{\beta\beta'}\pm\delta_{\alpha\beta'}\delta_{\alpha'\beta}).$$

Note that aside from a contact term, both the covariant commutator and the Green's function have the same differential structure. This serves as another test of our quantization procedures.

## 3. THIRD-RANK MULTISPINORS

Third-rank multispinors are of special interests because they are the natural descriptions of physical baryons in the $SU(6)$ theory of strongly interacting particles. Third-rank multispinors describe systems with either spin $\frac{3}{2}$ or spin $\frac{1}{2}$ according to their symmetry properties. As free fields, they should satisfy Bargmann-Wigner equations, and their Lagrange function have been constructed by Guralnik and Kibble.[11] In this paper, simpler Lagrange functions are presented. They are equivalent to those obtained by Guralnik and Kibble. The quantizations of third-rank multispinors have originally been studied by Chan.[12] Some of our results are essentially a transcription of his derivations into our own language.

### A. Free Spin-½ Field

The Lagrange function for a free spin-$\frac{1}{2}$ field, described by a third-rank multispinor, is of the form (Appendix A)

$$L=-\bar{\psi}(-i\Gamma\partial+m)\psi+\frac{3}{2}m\bar{\Omega}\Omega, \quad (17)$$

where $\psi$ is a non-Hermitian field, and

$$\psi_{\alpha\beta\gamma}=-\psi_{\beta\alpha\gamma},$$
$$\Omega_{\alpha\beta\gamma}=\frac{1}{3}(\psi_{\alpha\beta\gamma}+\psi_{\beta\gamma\alpha}+\psi_{\gamma\alpha\beta}),$$
$$\bar{\psi}=\psi^\dagger(\gamma^0)_1(\gamma^0)_2(\gamma^0)_3.$$

The field equation is given by

$$(-i\Gamma\partial+m)\psi-\frac{3}{2}m\Omega=0. \quad (18)$$

It is straightforward to verify that this equation is equivalent to (see Appendix B)

$$\Omega=0,\qquad (-i\gamma\partial+m)_i\psi=0,\quad i=1,2,3 \quad (19)$$

which are the correct equations to be satisfied. We introduce, in analogy to second-rank multispinors, the field variables in the two-dimensional representation:

$$\psi^{(+)(+)(+)}=(M^{(+)})_1(M^{(+)})_2(M^{(+)})_3\psi=\Psi^{(+)},$$
$$\psi^{(-)(-)(-)}=(M^{(-)})_1(M^{(-)})_2(M^{(-)})_3\psi=\Psi^{(-)},$$
$$\psi^{(+)(+)(-)}=(M^{(+)})_1(M^{(+)})_2(M^{(-)})_3\psi,\quad \text{etc.}$$

It will soon be clear that $\Psi^{(+)}$ and $\Psi^{(-)}$ are independent field variables, and that $\psi^{(+)(+)(-)},\cdots$, are dependent field variables. Making use of the field equations, such as

$$m\psi^{(+)(-)(+)}+\frac{1}{2}(\boldsymbol{\sigma}\cdot\nabla)_1\psi^{(-)(-)(+)}+\frac{1}{2}(\boldsymbol{\sigma}\cdot\nabla)_2\Psi^{(+)}=0,$$
$$m\psi^{(-)(-)(+)}+\frac{1}{2}(\boldsymbol{\sigma}\cdot\nabla)_1\psi^{(+)(-)(+)}+\frac{1}{2}(\boldsymbol{\sigma}\cdot\nabla)_3\Psi^{(-)}=0,$$

we can express these dependent variables in terms of the independent variables, as

$$\psi^{(+)(-)(+)}=D_1[(\boldsymbol{\sigma}\cdot\nabla)_1(\boldsymbol{\sigma}\cdot\nabla)_3\Psi^{(-)}-2m(\boldsymbol{\sigma}\cdot\nabla)_2\Psi^{(+)}],\ \text{etc.}, \quad (20)$$

with

$$(4m^2-\nabla^2)D_1(\mathbf{x}-\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}'). \quad (21)$$

These results are first derived by Chan, and they do not depend on the symmetry properties of the multispinors. The generator can be expressed as

$$G=i\int\bar{\psi}\Gamma^0\delta\psi d^3x$$
$$=i\int[\Psi^{(+)\dagger}\delta\Psi^{(+)}-\psi^{(+)(+)(-)\dagger}\delta\psi^{(+)(+)(-)}+\Psi^{(-)\dagger}\delta\Psi^{(-)}-\psi^{(-)(-)(+)\dagger}\delta\psi^{(-)(-)(+)}]\,d^3x$$
$$=i\int 4m^2[\Psi^{(+)\dagger}D_1\delta\Psi^{(+)}+\Psi^{(-)\dagger}D_1\delta\Psi^{(-)}]\,d^3x. \quad (22)$$

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[8] J. Schwinger, Phys. Rev. **127**, 324 (1962).
[9] J. Schwinger, Nuovo Cimento **30**, 278 (1963); Phys. Rev. **132**, 1317 (1963).
[10] J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452 (1951).
[11] G. S. Guralnik and T. W. B. Kibble, Phys. Rev. **139**, B712 (1965).
[12] L. H. Chan, Ph.D. thesis, Harvard University (unpublished).

This leads to the equal-time anticommutation relations

$$[\Psi^{(+)}{}_{abc}(x),\Psi^{(+)}{}_{a'b'c'}(x')^{\dagger}]_{+} = (1-\nabla^2/4m^2)1_{\{abc,a'b'c'\}}\delta(\mathbf{x}-\mathbf{x}')>0,$$

$$[\Psi^{(-)}{}_{abc}(x),\Psi^{(-)}{}_{a'b'c'}(x')^{\dagger}]_{+} = (1-\nabla^2/4m^2)1_{\{abc,a'b'c'\}}\delta(\mathbf{x}-\mathbf{x}'),$$

$$[\Psi^{(\pm)}(x),\Psi^{(\pm)}(x')]_{+}=[\Psi^{(\pm)}(x)^{\dagger},\Psi^{(\pm)}(x')^{\dagger}]_{+} =[\Psi^{(\pm)}(x),\Psi^{(\mp)}(x')^{\dagger}]_{+}=0. \quad (23)$$

Following Chan, the covariant anticommutator can be written as

$$[\psi_{\alpha\beta\gamma}(x),\bar{\psi}_{\alpha'\beta'\gamma'}(x')]_{+} = (1/4im^2)\prod_i [(m+i\gamma\partial)]_{i,\{\alpha\beta\gamma,\alpha'\beta'\gamma'\}}\Delta(x-x'), \quad (24)$$

where proper symmetrizations over $\alpha$, $\beta$, $\gamma$, and over $\alpha'$, $\beta'$, $\gamma'$, in the curly brackets according to their symmetries are understood.

The physical spin-$\frac{1}{2}$ particles, such as nucleons, are described phenomenologically by Dirac spinors. In order to facilitate physical applications, it is more convenient to introduce some new field variables which have these physical interpretations. Let us introduce

$$\psi_{\alpha\beta\gamma}=(1/2\sqrt{2})[(\gamma_5\gamma^0)_{\alpha\beta}\Psi_\gamma+(i/m)(\gamma_\mu\gamma_5\gamma^0)_{\alpha\beta}(\Psi^\mu)_\gamma +(\gamma^0)_{\alpha\beta}(\gamma_5\chi)_\gamma], \quad (25)$$

where $\Psi$, $\chi$, and $\Psi^\mu$ are spinors and vector-spinor, respectively. They can be expressed in terms of the multispinors $\psi_{\alpha\beta\gamma}$ through

$$\Psi_\gamma=\tfrac{1}{2}\sqrt{2}(\gamma^0\gamma_5)_{\alpha\beta}\psi_{\alpha\beta\gamma}, \quad (26)$$

$$(\Psi^\mu)_\gamma=(im/\sqrt{2})(\gamma^0\gamma_5\gamma^\mu)_{\alpha\beta}\psi_{\alpha\beta\gamma}, \quad (27)$$

and

$$\chi_\gamma=\tfrac{1}{2}\sqrt{2}(\gamma^0)_{\alpha\beta}(\gamma_5)_{\gamma\gamma'}\psi_{\alpha\beta\gamma'}. \quad (28)$$

In terms of these new variables, the Lagrange function can be reduced to

$$L=-(1/2m)\bar{\Psi}^\mu\partial_\mu\Psi+(1/2m)\bar{\Psi}\partial_\mu\Psi^\mu+(1/2m)\bar{\Psi}^\mu\Psi_\mu -\tfrac{1}{2}m\bar{\Psi}\Psi+\tfrac{1}{6}m[\bar{\Psi}+(i/m)\bar{\Psi}\cdot\gamma][\Psi-(i/m)\gamma\cdot\Psi] -\tfrac{3}{8}m[\bar{\chi}+\tfrac{1}{3}(\bar{\Psi}+(i/m)\bar{\Psi}\cdot\gamma)][\chi+\tfrac{1}{3}(\Psi-(i/m)\gamma\cdot\Psi)]. \quad (29)$$

Since $\Psi$, $\Psi_\mu$, and $\chi$ are related directly to the phenomenological fields, this alternative expression is more useful in physical applications.

The field equation (18) can be expressed in terms of $\Psi$, $\Psi_\mu$, and $\chi$ as

$$\chi+\tfrac{1}{3}[\Psi-(i/m)\gamma\cdot\Psi]=0, \quad (30)$$

$$\Psi_\mu-\partial_\mu\Psi+\tfrac{1}{3}im\gamma_\mu[\Psi-(i/m)\gamma\cdot\Psi]=0, \quad (31)$$

$$\partial_\mu\Psi^\mu-m^2\Psi+\tfrac{1}{3}m^2[\Psi-(i/m)\gamma\cdot\Psi]=0, \quad (32)$$

which can be reduced to

$$(m-i\gamma\partial)\Psi=0, \quad (33)$$

$$\Psi_\mu=\partial_\mu\Psi,$$

and

$$\chi=0.$$

These are the correct equations to be satisfied by a Dirac field. The generator can also be reduced to

$$G=\int\left[-\frac{1}{2m}\bar{\Psi}^0\delta\Psi+\frac{1}{2m}\bar{\Psi}\delta\Psi^0\right]d^3x =i\int[\Psi^\dagger\delta\Psi]d^3x, \quad (34)$$

which leads to the usual canonical anticommutation relations between $\Psi$ and $\Psi^\dagger$. The stress tensor can be worked out analogously as

$$T_{\mu\nu}=Lg_{\mu\nu}-i\bar{\psi}\Gamma_{(\mu}\partial_{\nu)}\psi+\tfrac{1}{2}\partial^\lambda[\bar{\psi}\Gamma_{(\mu}\sum_1^3\sigma_{\nu)\lambda}\psi], \quad (35)$$

and it can be expressed in terms of $\Psi$ as

$$T_{\mu\nu}=i\tfrac{1}{2}\partial_{(\mu}\bar{\Psi}\gamma_{\nu)}\Psi-i\tfrac{1}{2}\bar{\Psi}\gamma_{(\mu}\partial_{\nu)}\Psi, \quad (36)$$

which is identical to that of a Dirac field. These observations establish the fact that, as a free field, our formulation is identical to that of a Dirac field. Lorentz invariance and physical positive-definiteness requirements are consequently verified.

The Green's function can be obtained through the use of the external source

$$\eta_{\alpha\beta\gamma}=\tfrac{1}{2}\sqrt{2}[(\gamma_5\gamma^0)_{\alpha\beta}\eta_\gamma-im(\gamma_\mu\gamma_5\gamma^0)_{\alpha\beta}(\eta^\mu)_\gamma+(\gamma^0)_{\alpha\beta}(\gamma_5\zeta)_\gamma]$$

with

$$L_{\text{source}}=\bar{\psi}_{\alpha\beta\gamma}\eta_{\alpha\beta\gamma}+\text{H.c.} =\bar{\Psi}\eta+\bar{\Psi}^\mu\eta_\mu+\bar{\chi}\zeta+\text{H.c.}$$

The Green's function takes a very simple form if $\eta_{\alpha\beta\gamma}$ is chosen to have the mixed symmetry (2,1). Then we have (Appendix B)

$$G_{\alpha\beta\gamma,\alpha'\beta'\gamma'}(x,x') =\left\{\frac{1}{4m^2}\prod_i(m+i\gamma\partial)_i\Delta_+(x-x') +\frac{1}{4m^2}[(i\gamma\partial)_1+(i\gamma\partial)_2-(i\gamma\partial)_3]\delta(x-x') +\frac{3}{4m}\delta(x-x')\right\}_{\{\alpha\beta\gamma,\ \alpha'\beta'\gamma'\}}. \quad (37)$$

This result agrees with those obtained by Guralnik and Kibble from a different approach. The Green's function expressed in terms of $\Psi$, $\Psi_\mu$, and $\chi$ may be obtained either through the transcription of Eq. (37), or directly from the alternative expression Eq. (29)

of the Lagrange function. Then the Green's function can be described through the effective interaction between the external sources as

$$\begin{aligned}L_{\text{eff}}&=\bar{\eta}_{\alpha\beta\gamma}G_{\alpha\beta\gamma,\alpha'\beta'\gamma'}\eta_{\alpha'\beta'\gamma'}\\&=(\bar{\eta}-\partial_\mu\bar{\eta}^\mu)(m-i\gamma\partial-i\epsilon)^{-1}(\eta-\partial_\nu\eta^\nu)\\&\quad+(\bar{\eta}-\partial_\mu\bar{\eta}^\mu)(\zeta/m-i\gamma\eta)\\&\quad+(\bar{\zeta}/m+i\bar{\eta}\cdot\gamma)(m-i\gamma\partial)(\zeta/m-i\gamma\cdot\eta)\\&\quad+(\bar{\zeta}/m+i\bar{\eta}\cdot\gamma)(\eta-\partial_\mu\eta^\mu)-2m\bar{\eta}^\mu\eta_\mu\\&\qquad-2m(\bar{\zeta}/m+i\bar{\eta}\cdot\gamma)(\zeta/m-i\gamma\cdot\eta)+\tfrac{2}{3}\bar{\zeta}\zeta.\end{aligned}\tag{38}$$

This expression is very useful in perturbative calculations.

### B. Spin-$\frac{1}{2}$ Field with Electromagnetic Interaction

It is well known that no consistent theory describing the interaction of a charged spin-$\frac{3}{2}$ field with the electromagnetic field can be constructed by means of the usual field-theory techniques.[13] It is also speculated that such inconsistency may also exist in the theory of spin-$\frac{1}{2}$ field described by a third-rank multispinor with the mixed symmetry.[12] However, we will show in this section that a consistent theory can indeed be constructed. As a charged field, our theory is no longer equivalent to that of a Dirac field. The Lorentz invariance of the interacting system has been verified.

The total Lagrange function of our interacting system is

$$\begin{aligned}L=-\bar{\psi}(-i\Gamma D+m)\psi+\tfrac{3}{2}m\bar{\Omega}\Omega\\-\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu},\end{aligned}\tag{39}$$

with

$$D_\mu=\partial_\mu-ieA_\mu.$$

The field equations which are analogous to Eqs. (18) and (33) are

$$(-i\Gamma D+m)\psi-\tfrac{3}{2}m\Omega=0,\tag{40}$$

$$\begin{aligned}-(-i\gamma D+m)\Psi\\+(i/2m)[D_\mu D^\mu+(\gamma D)(\gamma D)]\Psi=0.\end{aligned}\tag{41}$$

The field equations derived from the variations of the Maxwell fields are

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu,\tag{42}$$

$$\partial_\nu F^{\mu\nu}=j^\mu=(i/2m)(\bar{\Psi}^\mu ie\Psi-\bar{\Psi}ie\Psi^\mu).\tag{43}$$

One verifies easily that the $\Psi$ field describes a charged spin-$\frac{1}{2}$ system without any intrinsic magnetic moment. The generator can be expressed, in the radiation gauge of the electromagnetic field, as

$$\begin{aligned}G&=\int[-(1/2m)\bar{\psi}^0\delta\Psi+(1/2m)\bar{\Psi}\delta\Psi^0-F^{0k}\delta A_k]\,d^3x\\&=\int[i\Psi^\dagger\delta\Psi-F'^{0k^T}\delta A_k{}^T+j'^0\delta\lambda]\,d^3x,\end{aligned}\tag{44}$$

with

$$A_k=A_k{}^T+\partial_k\lambda,\quad \partial_kA_k{}^T=0,$$

$$F_{\mu\nu}'=F_{\mu\nu}+(e/2m)\bar{\Psi}\sigma_{\mu\nu}\Psi,\tag{45}$$

and

$$\partial_\nu F'^{\mu\nu}=j'^\mu=e\bar{\Psi}\gamma^\mu\Psi.$$

Then, the canonical equal-time commutation (anticommutation) relations follow from the generator as

$$\begin{aligned}&[\Psi(x),\Psi^\dagger(x')]_+=1\times\delta(\mathbf{x}-\mathbf{x}')>0,\\&i[A_k{}^T(x),F^0{}_l{}^T(x')]=[\delta_{kl}\delta(\mathbf{x}-\mathbf{x}')]^T\\&\qquad=\delta_{kl}\delta(\mathbf{x}-\mathbf{x}')-\partial_k\partial_l'\mathfrak{D}(\mathbf{x}-\mathbf{x}'),\end{aligned}\tag{46}$$

with

$$\mathfrak{D}(\mathbf{x})=1/(4\pi|\mathbf{x}|),$$

and all other commutators (anticommutators) among these dynamical variables vanish. We would like to emphasize here that it is $F'^{0kT}$ and $j'^0$, rather than $F^{0kT}$ and $j^0$, which appeared in the generator. Therefore, the generator of the gauge transformation on the charged field is given by $j'^0(x)$. The physical meaning of $F_{\mu\nu}$ and $F'_{\mu\nu}$ is simple. The $F_{\mu\nu}$ is related to $E$ and $B$ fields, and the $F'_{\mu\nu}$ is related to $D$ and $H$ fields defined in classical electrodynamics.[14] The stress tensor can be obtained easily from the Lagrange function as

$$\begin{aligned}T_{\mu\nu}=\tfrac{1}{2}iD_{(\mu}\bar{\Psi}\gamma_{\nu)}\Psi-\tfrac{1}{2}i\bar{\Psi}\gamma_{(\mu}D_{\nu)}\Psi\\-\tfrac{1}{4}F^{\lambda\sigma}F_{\lambda\sigma}g_{\mu\nu}+F'_{(\mu\lambda}F_{\nu)}{}^\lambda.\end{aligned}\tag{47}$$

In order to verify the Lorentz invariance, it would be simpler to deal with the Hermitian fields

$$\Psi_1=\tfrac{1}{2}\sqrt{2}(\Psi+\Psi\dagger),$$

$$\Psi_2=i\tfrac{1}{2}\sqrt{2}(\Psi-\Psi\dagger).$$

The stress tensor can then be expressed as[15]

$$T_{\mu\nu}=-\tfrac{1}{2}i\Psi\alpha_{(\mu}D_{\nu)}\Psi-\tfrac{1}{4}F^{\lambda\sigma}F_{\lambda\sigma}g_{\mu\nu}+F'_{(\mu\lambda}F_{\nu)}{}^\lambda,\tag{48}$$

with

$$D_\mu=\partial_\mu-ieqA_\mu,$$

where

$$q=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

is an antisymmetric matrix in the charge space. The momentum density operator is

$$\begin{aligned}T_k{}^0&=\tfrac{1}{2}\Psi\frac{1}{i}D_k\Psi+\partial_l[\tfrac{1}{2}\Psi\tfrac{1}{4}\sigma_{kl}\Psi]+F_{kl}F'^{0l}\\&=\tfrac{1}{2}\Psi\frac{1}{i}\partial_k\Psi+\partial_l[\tfrac{1}{2}\Psi\tfrac{1}{4}\sigma_{kl}\Psi]\\&\qquad+F'^{0m}\partial_kA_m-\partial_m[F'^{0m}A_k].\end{aligned}\tag{49}$$

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[13] K. Johnson and E. C. G. Sudarshan, Ann. Phys. (N. Y.) **13**, 126 (1961).

[14] See, e.g., J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, Inc., New York, 1962). The author wishes to thank S. S. Shei for helpful discussions on this point.

[15] The expression $\psi\,.\,qA\psi$, which is the product of three operators, actually stands for $(\psi\,.\,q\psi)\,.\,A$.

We observe that the charged field and the Maxwell fields are completely decoupled in $T^0{}_k$. This property ensures the correct transformation behavior for $P_k$ and $J_{kl}$, and thereby the validity of the commutation relations for the infinitesimal generators of the 3-dimensional translation-rotation group. In short, it guarantees automatically the 3-dimensional translational and rotational invariance of the theory. This theory will then be invariant under the group of proper, orthochronous Lorentz transformation, if the energy density $T^{00}(x)$ obeys the equal-time commutator relation

$$-i[T^{00}(x),T^{00}(x')]=-(T^0{}_k(x)+T^0{}_k(x'))\partial_k\delta(\mathbf{x}-\mathbf{x}')\,.$$

The energy density operator in our theory is

$$T^{00}(x)=\tfrac{1}{2}\Psi\alpha^k\frac{1}{i}D_k\Psi+\tfrac{1}{2}m\Psi\gamma^0\Psi$$
$$+\tfrac{1}{4}F_{kl}F_{kl}+(e/8m)(\Psi\gamma^0\sigma_{kl}q\Psi)F_{kl}+\tfrac{1}{2}F^{0k}F^{0k}. \quad (50)$$

The Schwinger relation can be verified either by means of the technique of extended operators or simply by direct computation. This completes the verification of the Lorentz invariance of our interacting system.

### C. Spin-$\frac{3}{2}$ Field

A spin-$\frac{3}{2}$ system can be represented by a multispinor $\psi_{\alpha\beta\gamma}$ with the symmetry property

$$\psi_{(\alpha\beta)\gamma}=\psi_{\gamma(\alpha\beta)}\,. \quad (51)$$

The Lagrange function can be written as

$$L=-\bar{\psi}_{(\alpha\beta)\gamma}(-i\Gamma\partial+m)\psi_{(\alpha'\beta')\gamma'}$$
$$+\tfrac{1}{3}\bar{\psi}_{[\alpha\beta]\gamma}(-i\Gamma\partial+3m)\psi_{[\alpha'\beta']\gamma'}-\tfrac{4}{3}m\bar{\Omega}\Omega\,, \quad (52)$$

with

$$\psi_{(\alpha\beta)\gamma}\equiv\tfrac{1}{2}(\psi_{\alpha\beta\gamma}+\psi_{\beta\alpha\gamma})\,,$$
$$\psi_{[\alpha\beta]\gamma}\equiv\tfrac{1}{2}(\psi_{\alpha\beta\gamma}-\psi_{\beta\alpha\gamma})\,,$$
$$\Omega_{\alpha\beta\gamma}\equiv\tfrac{1}{3}(\psi_{[\alpha\beta]\gamma}+\psi_{[\beta\gamma]\alpha}+\psi_{[\gamma\alpha]\beta})\,.$$

It is straightforward, although tedious, to verify that the field equations derived from this Lagrange function can be reduced to (see Appendix B)

$$\psi_{[\alpha\beta]\gamma}=0\,, \quad (53)$$

$$(-i\gamma\partial+m)_i\psi_{\alpha\beta\gamma}=0\,,\quad i=1,\,2,\,3. \quad (54)$$

By combining Eq. (51) with Eq (53), we have

$$\psi_{\alpha\beta\gamma}=\tfrac{1}{3}[\psi_{(\alpha\beta)\gamma}+\psi_{(\beta\gamma)\alpha}+\psi_{(\gamma\alpha)\beta}]\,, \quad (55)$$

which is then totally symmetric. This confirms that our system indeed describes a free, spin-$\frac{3}{2}$ field. We would like to emphasize here that the total symmetry of $\psi$ is derived directly from the action principle rather than added arbitrarily as a further restriction.

The canonical quantization of the spin-$\frac{3}{2}$ system follows exactly from those of the spin-$\frac{1}{2}$ system. Equations (20)–(24) can be derived analogously, and we will not reproduce their derivations here. We only copy down here the covariant anticommutator relations

$$[\psi_{\alpha\beta\gamma}(x),\bar{\psi}_{\alpha'\beta'\gamma'}(x')]_+$$
$$=(1/4im^2)[\prod_{i=1}^{3}(m+i\gamma\partial)_i]_{(\alpha\beta\gamma),(\alpha'\beta,\gamma')}\Delta(x-x')\,, \quad (56)$$

which is identical to Eq. (24). However, a total symmetrization over $\alpha$, $\beta$, $\gamma$, (and over $\alpha'$, $\beta'$, $\gamma'$) according to the total symmetry of $\psi$ is understood.

We will show that as a free field our theory is equivalent to the Rarita-Schwinger formulation. Let us introduce

$$(\Psi^\mu)_\gamma=-\tfrac{1}{2}\sqrt{2}(\gamma^0\gamma^\mu)_{\alpha\beta}\psi_{\alpha\beta\gamma}\,, \quad (57)$$

$$(\Psi^{\mu\nu})_\gamma=\tfrac{1}{2}m\sqrt{2}(\gamma^0\sigma^{\mu\nu})_{\alpha\beta}\psi_{\alpha\beta\gamma}\,. \quad (58)$$

The field equations can be expressed in terms of these field variables through

$$(-i\gamma\partial+m)\Psi^\mu=0\,, \quad (59)$$

$$\gamma\Psi=0\,, \quad (60)$$

and

$$\Psi_{\mu\nu}=\partial_\mu\Psi_\nu-\partial_\nu\Psi_\mu\,. \quad (61)$$

The equal-time anticommutation relations between $\Psi_k$'s can be transcribed easily from Eq. (56), and are

$$[\Psi_k(x),\Psi_l(x')^\dagger]_+$$
$$=[\delta_{kl}+\tfrac{1}{3}\gamma_k\gamma_l+(1/3im)(\gamma_l\partial_k+\gamma_k\partial_l')$$
$$+(2/3m^2)\partial_k\partial_l']\delta(\mathbf{x}-\mathbf{x}')\,. \quad (62)$$

Equations (59), (60), and (62) establish the fact that, as a free field, our system is equivalent to the Rarita-Schwinger formulation. The Green's function can be introduced similarly with the aid of the external source

$$\eta_{\alpha\beta\gamma}=\tfrac{1}{2}\sqrt{2}[(\gamma^\mu\gamma^0)_{\alpha\beta}(\eta_\mu)_\gamma-\tfrac{1}{2}m(\sigma^{\mu\nu}\gamma^0)_{\alpha\beta}(\eta_{\mu\nu})_\gamma]\,.$$

The Green's function $G_{(\alpha\beta\gamma),(\alpha'\beta'\gamma')}(x,x')$, which is chosen to be totally symmetric both in $\alpha$, $\beta$, $\gamma$, and in $\alpha'$, $\beta'$, $\gamma'$, is simple and useful. It is given by (Appendix B)

$$G_{(\alpha\beta\gamma),(\alpha'\beta'\gamma')}(x,x')$$
$$=\delta\langle\psi_{(\alpha\beta\gamma)}(x)\rangle/\delta\eta_{(\alpha'\beta'\gamma')}(x')\,|_{\eta=0}$$
$$=(1/4m^2)[\prod_{i=1}^{3}(m+i\gamma\partial)_i]_{(\alpha\beta\gamma)(\alpha'\beta'\gamma')}\Delta_+(x-x')$$
$$+\{(1/12m^2)[(i\gamma\partial)_1+(i\gamma\partial)_2+(i\gamma\partial)_3]$$
$$+3/4m\}_{(\alpha\beta\gamma),(\alpha'\beta'\gamma')}\delta(x-x')\,. \quad (63)$$

The effective interaction between the external sources can be described by the vacuum probability amplitude as

$$\langle 0_+|0_-\rangle^{\eta}=\exp i\int\big[(\bar{\eta}_\mu+\partial^\nu\bar{\eta}_{\mu\nu})(-i\gamma\partial+m-i\epsilon)^{-1}(\eta^\mu+\partial_\lambda\eta^{\mu\lambda})+(1/m^2)(\partial\bar{\eta})(-i\gamma\partial+m-i\epsilon)^{-1}(\partial\eta)+(1/6m^2)\bar{\eta}^\mu(-2i\gamma\partial+9m)\eta_\mu+(1/3m)(\partial_\lambda\bar{\eta}^{\mu\lambda}\eta_\mu+\bar{\eta}_\mu\partial_\lambda\eta^{\mu\lambda})+\tfrac{1}{12}\bar{\eta}^{\mu\nu}(-4i\gamma\partial-15m)\eta_{\mu\nu}\big]\,d^4x. \quad (64)$$

## 4. MULTISPINORS OF FOURTH RANK—SPIN-2 FIELD

The construction of the Lagrange functions for higher-rank multispinors is very tedious. The spin-2 Lagrange function constructed in this section is much more complicated than the Lagrange functions constructed from the tensor operators. Simplicity in the Lagrange function is definitely not one of the advantages enjoyed by this multispinor formulation. One of the advantages for using the multispinor formulation is that the dynamical variables can be identified explicitly as

$$\phi^{(+)(+)\cdots,(+)} \quad\text{and}\quad \phi^{(-)(-),\cdots,(-)},$$

and the canonical quantization can be carried out easily in terms of these variables.

The spin-2 field is described by a fourth-rank multispinor $\phi_{\alpha\beta,\gamma\delta}$ with the following symmetry properties;

(i) $\phi_{\alpha\beta,\gamma\delta}$ is symmetric between $\alpha$, $\beta$, as well as between $\gamma$, $\delta$;

(ii) $(\gamma^0\sigma^{\mu\nu})_{\alpha\beta}(\gamma^0\gamma^\lambda)_{\gamma\delta}(\phi_{\alpha\beta,\gamma\delta}-\phi_{\gamma\delta,\alpha\beta})=0$.

It is an afterthought to assume that $\phi_{\alpha\beta,\gamma\delta}$ has these symmetry properties. In the actual construction of the Lagrange function, we proceed in the reverse order. Our final goal is to reproduce the correct field equations. In order to accomplish this, we find that these symmetry properties are indispensable. The Lagrange function can be expressed in terms of these variables through

$$L=-\tfrac{1}{2}\bar{\phi}(-i\Gamma\partial+m)\phi+m\bar{\phi}P\phi, \quad (65)$$

with

$$\bar{\phi}=\phi(\gamma^0)_1(\gamma^0)_2(\gamma^0)_3(\gamma^0)_4,\quad \phi^\dagger=\phi$$
$$\Gamma=\tfrac{1}{2}[(\gamma)_1+(\gamma)_2],$$

where

$$P_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'}=\tfrac{1}{32}\{\tfrac{2}{5}(\gamma_\mu\gamma^0)_{\alpha\beta}(\gamma^\mu\gamma^0)_{\gamma\delta}(\gamma^0\gamma_\nu)_{\alpha'\beta'}(\gamma^0\gamma^\nu)_{\gamma'\delta'}-(\sigma_{\mu\nu}\gamma^0)_{\alpha\beta}(\gamma^\nu\gamma^0)_{\gamma\delta}(\gamma^0\sigma^{\mu\lambda})_{\alpha'\beta'}(\gamma^0\gamma_\lambda)_{\gamma'\delta'}+\tfrac{1}{8}[(\sigma_{\mu\nu}\gamma^0)_{\alpha\beta}(\sigma_{\lambda\rho}\gamma^0)_{\gamma\delta}-(\sigma_{\lambda\rho}\gamma^0)_{\alpha\beta}(\sigma_{\mu\nu}\gamma^0)_{\gamma\delta}]\times[(\gamma^0\sigma^{\mu\nu})_{\alpha'\beta'}(\gamma^0\sigma^{\lambda\rho})_{\gamma'\delta'}-(\gamma^0\sigma^{\lambda\rho})_{\alpha'\beta'}(\gamma^0\sigma^{\mu\nu})_{\gamma'\delta'}]\} \quad (66)$$

is a numerical matrix chosen so as to guarantee the vanishing of all auxiliary lower spin fields. One can verify that the field equations derived from this Lagrange function can be reduced to

$$\phi_{\alpha\beta\gamma\delta}=\text{totally symmetric in}\quad \alpha,\beta,\gamma,\delta,$$
$$(-i\gamma\partial+m)_i\phi=0,\quad i=1,2,3,4, \quad (67)$$

which are the correct Bargmann-Wigner equations to be satisfied by a spin-2 field. The independent variables are

$$\Phi^{(+)}=\phi^{(+)(+)(+)(+)}$$

and

$$\Phi^{(-)}=\phi^{(-)(-)(-)(-)},$$

and all other field variables are dependent variables. The dependent variables can be expressed in terms of these independent components through

$$\phi^{(+)(+)(+)(-)}=-[1/4m+\tfrac{1}{2}mD_2](\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_4\Phi^{(+)}-(1/4m)D_2(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_1(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_2(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_3\Phi^{(-)}, \quad (68)$$

$$\phi^{(+)(+)(-)(-)}=\tfrac{1}{2}D_2[(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_1(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_2\Phi^{(-)}+(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_3(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_4\Phi^{(+)}],\quad\text{etc.,} \quad (69)$$

with

$$(2m^2-\nabla^2)D_2(\mathbf{x}-\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}'). \quad (70)$$

The canonical quantization can be carried out analogously. The canonical commutator relations are

$$x^0=x^{0\prime}:\quad [\Phi^{(+)}{}_{abcd}(x),\Phi^{(-)}{}_{a'b'c'd'}(x')]=(1-\nabla^2/2m^2)[(\sigma_y)_1(\sigma_y)_2(\sigma_y)_3(\sigma_y)_4]_{(abcd),(a'b'c'd')}\times\delta(\mathbf{x}-\mathbf{x}'),$$
$$[\Phi^{(+)}(x),\Phi^{(+)}(x')]=[\Phi^{(-)}(x),\Phi^{(-)}(x')]=0, \quad (71)$$

from which the covariant commutator relations can be computed as

$$i[\phi_{\alpha\beta\gamma\delta}(x),\bar{\phi}_{\alpha'\beta'\gamma'\delta'}(x')]=\left(\frac{1}{2m}\right)^3\Big[\prod_{i=1}^{4}(m+i\gamma\partial)_i\Big]_{(\alpha\beta\gamma\delta),(\alpha'\beta'\gamma'\delta')}\Delta(x-x'). \quad (72)$$

These results do not depend on the total symmetry of $\phi_{\alpha\beta\gamma\delta}$, and can be generalized directly to all other fourth-rank multispinors.

The field variables of a spin-2 field expressed in terms of multispinors are related to those expressed in terms of tensor variables through

$$\phi_{\alpha\beta,\gamma\delta}=\tfrac{1}{8}m^{1/2}\{(\gamma_\mu\gamma^0)_{\alpha\beta}(\gamma_\lambda\gamma^0)_{\gamma\delta}h^{\mu\lambda}+(1/2m)[(\sigma_{\mu\nu}\gamma^0)_{\alpha\beta}(\gamma_\lambda\gamma^0)_{\gamma\delta}+(\gamma_\lambda\gamma^0)_{\alpha\beta}(\sigma_{\mu\nu}\gamma^0)_{\gamma\delta}]^\lambda H^{\mu\nu}+(1/4m^2)(\sigma_{\mu\nu}\gamma^0)_{\alpha\beta}(\sigma_{\lambda\rho}\gamma^0)_{\gamma\delta}H^{\mu\nu,\lambda\rho}\}, \quad (73)$$

where $h_{\mu\nu}$ and ${}_\lambda H_{\mu\nu}$ are the field variables introduced in a previous paper,[16] and

$$H_{\mu\nu,\lambda\rho}=\partial_\lambda\,{}_\rho H_{\mu\nu}-\partial_\rho\,{}_\lambda H_{\mu\nu}. \quad (74)$$

Making use of Eq. (73), one can compare the field equations and the commutator relations between different formulations, and show that, as free fields, these two formulations are actually identical.

[16] S. J. Chang, Phys. Rev. 148, 1259 (1966).

In principle, this generalized Lagrange formulation can be applied to multispinor of an arbitrary rank. However, the construction of a Lagrange function in terms of multispinors is so tedious that it becomes less attractive than the usual formulation expressed in terms of tensor variables.

## 5. DISCUSSION

The main purpose of this paper is to verify the internal consistency of multispinor formulations. However, we would like to point out that some of our results are of direct physical applications if the baryons are indeed described by these third-rank multispinors. An important consequence in this theory is that the nucleons should have zero intrinsic magnetic moment. This resolves the paradox previously raised by Bég, Lee, and Pais.[17] Loosely speaking, the paradox is the following: According to $SU(6)$, the ratio of the magnetic moments between those of neutron and of proton is

$$\mu(N)/\mu(p)=-\tfrac{2}{3},$$

which agrees remarkably with the experimental ratio between the total magnetic moments of nucleons $\approx -0.684$. This ratio is a pure number, independent of the strong-interaction coupling constants. If the nucleons are described by Dirac particles and in the limit where the strong interactions are "turned off," we should have $\mu(N)=0$, $\mu(p)=\mu_N$, where $\mu_N=e/2m_N$ is the nucleon magneton. This leads to a ratio different from $-\frac{2}{3}$, and is consequently in contradiction to the previous result. However, there is no such drawback in our theory. The magnetic moments obtained in $SU(6)$ theory are interpreted as the anomalous magnetic moments, and therefore they should be switched off simultaneously with the strong interactions. It is simply because the nucleon has no intrinsic magnetic moment that leads to the famous $-\frac{2}{3}$ ratio. The situation is quite different for the spin-$\frac{3}{2}$ baryon resonance. The intrinsic magnetic moment of a baryon resonance, which is described by a third-rank multispinor given in Sec. 3C, no longer vanishes. It is straightforward to show that the intrinsic magnetic moment of the baryon resonance is $q/2m_D$, where $q$ and $m_D$ are the charge and mass of the baryon resonance, respectively. Moreover, the magnetic moment of the baryon resonance computed from the $SU(6)$ covariant interactions is[18]

$$(q/2m_D)(1+2m_D/m), \tag{75}$$

where $m$ is the mass of the $\rho$ meson. According to the usual $SU(6)$ theory, this magnetic moment is interpreted as the total magnetic moment of the baryon resonance. In our theory, however, this is interpreted as the anomalous magnetic moment produced by the strong interactions. Therefore, the total magnetic moment of the baryon resonance is

$$(q/m_D)(1+m_D/m). \tag{76}$$

For singly charged baryon resonances, such as $\Omega^-$, the magnetic moments computed from these two theories lead to

$$3.1\mu_N \quad \text{and} \quad 3.7\mu_N,$$

respectively. This can serve as an experimental test of our theory.

## ACKNOWLEDGMENTS

I wish to express my sincere gratitude to Professor Julian Schwinger for his guidance and encouragement. It is also my pleasure to thank Professor S. L. Glashow, T. M. Yan, and L. H. Chan for helpful discussions.

## APPENDIX A: CONSTRUCTION OF LAGRANGE FUNCTIONS FOR THE THIRD-RANK MULTISPINOR FIELDS

In the quantization of any system according to the quantum action principle, the Lagrange function must be constructed first. Then, quantizations and Green's functions are carried out on the one hand, Lorentz invariance and other consistency requirements are verified on the other hand. In this paper, however, the constructions of Lagrange functions and Green's functions are omitted from the text in order that the reader follow the development of this paper without going into detailed calculations.

The Lagrange function given in this paper can be constructed analogously as in Ref. 19. The field equations which we want to reproduce are the Bargmann-Wigner equations

$$[(\ )+m]_i\psi=0, \quad i=1,2,3, \tag{A1}$$

where $(\ )_i$ stands for $(-i\gamma\partial)_i$. An important algebraic constraint equation satisfied by $\psi$ is

$$(\ )_1\psi=(\ )_2\psi=(\ )_3\psi. \tag{A2}$$

In order to reproduce (A2) automatically, we introduce the following nonlocal projection operator:

$$\Lambda(\partial)=\tfrac{1}{4}(\partial^2)^{-1}S(\partial), \tag{A3}$$

with

$$S(\partial)=[\partial^2+(\ )_1(\ )_2+(\ )_1(\ )_3+(\ )_2(\ )_3]. \tag{A4}$$

This projection operator $\Lambda(\partial)$ has the following properties:

(i) $\Lambda(\partial)$ is totally symmetric under the interchange of any two indices.

(ii) $$(\ )_1\Lambda(\partial)=(\ )_2\Lambda(\partial)=(\ )_3\Lambda(\partial). \tag{A5}$$

(iii) $$\Lambda(\partial)\Lambda(\partial)=\Lambda(\partial).$$

[17] M. A. B. Bég, B. W. Lee, and A. Pais, Phys. Rev. Letters **13**, 514 (1964).

[18] For the techniques of computing the magnetic moments in the relativistic $SU(6)$ theory, see, e.g., A. Salam *et al.*, Proc. Roy. Soc. (London) **A284**, 146 (1965); B. Sakita and K. C. Wali, Phys. Rev. **139**, B1355 (1965).

[19] S. J. Chang, preceding paper, Phys. Rev. **161**, 1308 (1967).

Property (i) implies that $\psi$ and $\Lambda(\partial)\psi$ have the same symmetry properties under the permutation of their indices. A nonlocal field equation satisfied by $\psi$ with an arbitrary symmetry is

$$[m+(-i\gamma\partial)_i\Lambda(\partial)]\psi=0, \quad \text{any } i.$$

Making use of properties (i)–(iii), one verifies easily that

$$[m+(-i\gamma\partial)]_i\psi=0, \quad i=1,2,3.$$

For a spin-$\frac{1}{2}$ system, $\psi=\phi'_{\alpha\beta\gamma}$ has the symmetry properties

$$\phi'_{\alpha\beta\gamma}=-\phi'_{\beta\alpha\gamma},$$
$$\phi'_{[\alpha\beta\gamma]}=0.$$

The nonlocal equation can be written as

$$[(m-i\Gamma\partial)\phi']_{(2,1)}+\tfrac{1}{6}[(\ )_1+(\ )_2-2(\ )_3]\Omega=0, \quad \text{(A6)}$$

with

$$\Omega=-\tfrac{1}{2}(\partial^2)^{-1}[(\ )_1+(\ )_2+(\ )_3] \times\{[-i\Gamma\partial-(\ )_3]\phi'+\tfrac{1}{2}\sqrt{3}[(\ )_1-(\ )_2]\phi\}$$

$$\phi_{\alpha\beta\gamma}=-\tfrac{1}{3}\sqrt{3}(\phi'_{\beta\gamma\alpha}-\phi'_{\gamma\alpha\beta}),$$

where $[\ ]_{(2,1)}$ stands for the (2,1) part of the expression. Although it is local in appearance, Eq. (A6) is, in fact, nonlocal. In order to transform this equation into a local equation, we have to interprete $\Omega$ as a new auxiliary field variable, rather than the nonlocal expression given above. Then, our problem is reduced to constructing the field equation satisfied by $\Omega$ such that

$$\Omega \quad \text{and} \quad -[i\Gamma\partial+(\ )_3]\phi'+\tfrac{1}{2}\sqrt{3}[(\ )_1-(\ )_2]\phi$$

should vanish identically. Multiplying Eq. (A7) by $-i\Gamma\partial$, and projecting out the totally antisymmetric part, we have

$$\{\tfrac{2}{3}[(\ )_1+(\ )_2+(\ )_3]+m\}\{[\tfrac{1}{2}(\ )_1+\tfrac{1}{2}(\ )_2-(\ )_3]\phi' +\tfrac{1}{2}\sqrt{3}[(\ )_1-(\ )_2]\phi\}+\tfrac{4}{3}[(\ )_1+(\ )_2+(\ )_3]^2\Omega=0 \quad (\mathrm{mod}[\partial^2+(\ )_1(\ )_2+(\ )_1(\ )_3+(\ )_2(\ )_3]\Omega).$$

Therefore, if the equation satisfied by $\Omega$ is chosen as

$$[\tfrac{1}{2}(\ )_1+\tfrac{1}{2}(\ )_2-(\ )_3]\phi'+\tfrac{1}{2}\sqrt{3}[(\ )_1-(\ )_2]\phi +\{2[(\ )_1+(\ )_2+(\ )_3]-3m\}\Omega=0, \quad \text{(A7)}$$

we have

$$\Omega=0 \quad (\mathrm{mod}[\partial^2+(\ )_1(\ )_2+(\ )_1(\ )_3+(\ )_2(\ )_3]\Omega),$$

which implies

$$(\ )_1\Omega=(\ )_2\Omega=(\ )_3\Omega.$$

Since a totally antisymmetric $\Omega$ can not be a simultaneous eigenstate of all three $(\ )_i$, we have

$$\Omega=0=[\tfrac{1}{2}(\ )_1+\tfrac{1}{2}(\ )_2-(\ )_3]\phi'+\tfrac{1}{2}\sqrt{3}[(\ )_1-(\ )_2]\phi.$$

Equations (A6) and (A7) are therefore the required field equations, which can be derived from the Lagrange function

$$L=-(\bar{\phi}'+\bar{\Omega})(-i\Gamma\partial+m)(\phi'+\Omega)+\tfrac{3}{2}m\bar{\Omega}\Omega.$$

After introducing a new field variable through

$$\psi=\phi'+\Omega,$$

we are finally led to the Lagrange function given in Eq. (17). The Lagrange function for a spin-$\frac{3}{2}$ system in the multispinor formulation can be constructed analogously, but the construction is more complicated.

## APPENDIX B: CONSTRUCTION OF THE GREEN'S FUNCTIONS FOR THIRD-RANK MULTISPINOR FIELDS

In this Appendix, we work out the Green's function for third-rank multispinors explicitly. By setting the external sources to zero in these derivations, we show at the same time that the free-field variables satisfy the required Bargmann-Wigner equations as well as the symmetry restrictions.

For a spin-$\frac{1}{2}$ system, the Lagrange function in the presence of an external source is

$$L=-\bar{\psi}(-i\Gamma\partial+m)\psi+\tfrac{3}{2}m\bar{\Omega}\Omega+\bar{\psi}\eta+\bar{\eta}\psi,$$

where the external source $\eta_{\alpha\beta\gamma}$ is assumed to have the mixed symmetry property (2,1). The field equation is

$$(-i\Gamma\partial+m)\psi-\tfrac{3}{2}m\Omega=\eta. \quad \text{(B1)}$$

Multiplying the totally antisymmetric part of Eq. (B1) by $(-i\Gamma\partial)$, we have

$$(-i\Gamma\partial)\{(i\Gamma\partial)\psi+\tfrac{3}{2}m\Omega +\tfrac{1}{2}[-i\Gamma\partial+(\ )_3](\psi-3\Omega)\}=0. \quad \text{(B2)}$$

The expression

$$2(-i\Gamma\partial)[-i\Gamma\partial+(\ )_3] =[\partial^2+(\ )_1(\ )_2+(\ )_1(\ )_3+(\ )_2(\ )_3]=S(\partial),$$

which is related to $\Lambda(\partial)$, satisfies properties (i) and (ii) given in Eq. (A5). Substituting Eq. (B1) into Eq. (B2), we have

$$(-i\Gamma\partial)(m\psi-\eta)=-\tfrac{1}{4}S(\partial)(\psi-3\Omega),$$

and with the help of the properties of $S(\partial)$,

$$(\ )_1(-i\Gamma\partial)(m\psi-\eta)=(\ )_2(-i\Gamma\partial)(m\psi-\eta) =(\ )_3(-i\Gamma\partial)(m\psi-\eta). \quad \text{(B3)}$$

Eliminating $(-i\Gamma\partial)\psi$ from Eqs. (B2) and (B3), we obtain

$$(-i\Gamma\partial)\{[-i\Gamma\partial+(\ )_3]\Omega-m\Omega +\tfrac{1}{3}[-i\Gamma\partial-(\ )_3]\eta/m\}=0,$$

and consequently,

$$[\partial^2-\tfrac{1}{2}m(\ )_i]\Omega'=0, \quad i=1.2.3, \quad \text{(B4)}$$

where

$$\Omega'=\Omega-(1/6m^2)[(\ )_1+(\ )_2-2(\ )_3]\eta -(1/2m^2\sqrt{3})[(\ )_1-(\ )_2]\eta^{(s)} \quad \text{(B5)}$$

is also totally antisymmetric, and

$$\eta^{(s)}{}_{\alpha\beta\gamma}=-\tfrac{1}{3}\sqrt{3}(\eta_{\beta\gamma\alpha}-\eta_{\gamma\alpha\beta}).$$

Since a totally antisymmetric third-rank multispinor can not be a simultaneous eigenstate of all three operators $(-i\gamma\partial)_i$,[11] it follows

$$\Omega'=\Omega-(1/6m^2)[(\ )_1+(\ )_2-2(\ )_3]\eta - (1/2m^2\sqrt{3})[(\ )_1-(\ )_2]\eta^{(s)}=0. \quad \text{(B6)}$$

Multiplying Eq. (B1) by $1\pm(1/2m)[(\ )_1-(\ )_2]$ on the left, we have

$$[(\ )_1+m]\psi=\{1+(1/2m)[(\ )_1-(\ )_2]\}(\eta+\tfrac{3}{2}m\Omega),$$
$$[(\ )_2+m]\psi=\{1+(1/2m)[(\ )_2-(\ )_1]\}(\eta+\tfrac{3}{2}m\Omega), \quad \text{(B7)}$$

which, with the help of Eq. (B6), can be reduced to

$$[(\ )_1+m](\psi-\tfrac{3}{2}\Omega) =(1/4m^2)[m-(\ )_2][m-(\ )_3]\eta+[(\ )_1+m] \times\{-(1/4m^2)[(\ )_1+(\ )_2-(\ )_3]+3/4m\}\eta. \quad \text{(B8)}$$

The Green's function then follows trivially. In the absence of the external source, we have

$$\eta=\Omega=0,$$

and consequently

$$[(-i\gamma\partial)+m]_i\psi=0, \quad i=1,2,3$$

which is exactly Eq. (19).

The computation of the Green's function for a totally symmetric multispinor is somewhat more involved. We assume that the external source $\eta_{\alpha\beta\gamma}$ is totally symmetric. The Lagrange function, in the presence of an external source, is

$$L=-\bar{\psi}_{(\alpha\beta)\gamma}(-i\Gamma\partial+m)\psi_{(\alpha\beta)\gamma} +\tfrac{1}{3}\bar{\psi}_{[\alpha\beta]\gamma}(-i\Gamma\partial+3m)\psi_{[\alpha\beta]\gamma} -\tfrac{4}{3}m\bar{\Omega}\Omega+\bar{\psi}_{\alpha\beta\gamma}\eta_{\alpha\beta\gamma}+\bar{\eta}_{\alpha\beta\gamma}\psi_{\alpha\beta\gamma}, \quad \text{(B9)}$$

where the multispinor $\psi_{\alpha\beta\gamma}$ has the required symmetry property

$$\psi_{(\alpha\beta)\gamma}=\psi_{\gamma(\alpha\beta)}. \quad \text{(B10)}$$

It is known that an arbitrary third-rank multispinor $\psi_{\alpha\beta\gamma}$ can be expanded as

$$\psi_{\alpha\beta\gamma}=\Psi_{\alpha\beta\gamma}+\tfrac{1}{3}\sqrt{3}\phi_{\alpha\beta\gamma}+\phi'_{\alpha\beta\gamma}+\Omega_{\alpha\beta\gamma}, \quad \text{(B11)}$$

where

$\Psi_{\alpha\beta\gamma}$ is totally symmetric,

$\phi_{\alpha\beta\gamma}=\phi_{\beta\alpha\gamma}$ has the mixed symmetry (2,1),

$\phi'_{\alpha\beta\gamma}=-\phi'_{\beta\alpha\gamma}$ has also the mixed symmetry (2,1),

and $\Omega_{\alpha\beta\gamma}$ is totally antisymmetric;
Eq. (B10) implies

$$\phi_{\alpha\beta\gamma}=-\tfrac{1}{3}\sqrt{3}(\phi'_{\beta\gamma\alpha}-\phi'_{\gamma\alpha\beta}),$$

which indicates that $\phi$ and $\phi'$ are related to each other. Under an arbitrary variation on $\psi_{[\alpha\beta]\gamma}$,

$$\delta\psi_{[\alpha\beta]\gamma}=\delta(\Omega+\phi')_{\alpha\beta\gamma},$$

the variation of $\psi_{(\alpha\beta)\gamma}$ is not zero, and is given by

$$\delta\psi_{(\alpha\beta)\gamma}=-\tfrac{1}{3}(\delta\psi_{[\beta\gamma]\alpha}-\delta\psi_{[\gamma\alpha]\beta}).$$

The field equation derived from the variation of $\psi_{[\alpha\beta]\gamma}$ is

$$\{\tfrac{1}{2}[(\ )_\gamma+(\ )_\alpha]+m\}\psi_{(\gamma\alpha)\beta}-\{\tfrac{1}{2}[(\ )_\gamma+(\ )_\beta]+m\}\psi_{(\gamma\beta)\alpha} +(-i\Gamma\partial+3m)\psi_{[\alpha\beta]\gamma}-4m\Omega=0, \quad \text{(B12)}$$

where $(\ )_\alpha=(-i\gamma\partial)_\alpha$ operates on the index $\alpha$, etc. Equation (B12) can be decomposed according to its symmetries as

$$\{\tfrac{1}{3}[(\ )_1+(\ )_2+(\ )_3]-m\}\Omega+\tfrac{1}{12}[(\ )_1+(\ )_2-2(\ )_3]\phi' +\tfrac{1}{12}\sqrt{3}[(\ )_1-(\ )_2]\phi=0, \quad \text{(B13)}$$

and

$$2m\phi'+\tfrac{1}{6}[(\ )_1+(\ )_2-2(\ )_3](\phi'+\Omega) -\tfrac{1}{6}\sqrt{3}[(\ )_1-(\ )_2](\phi-\sqrt{3}\Psi)=0, \quad \text{(B14)}$$

respectively. Similarly, under an independent variation of $\psi_{(\alpha\beta)\gamma}$, we have the field equation

$$-(-i\Gamma\partial+m)\psi_{(\alpha\beta)\gamma}+\tfrac{1}{3}\{\tfrac{1}{2}[(\ )_\gamma+(\ )_\alpha]+3m\}\psi_{(\gamma\alpha)\beta} +\tfrac{1}{3}\{\tfrac{1}{2}[(\ )_\gamma+(\ )_\beta]+3m\}\psi_{[\gamma\beta]\alpha}+\eta_{\alpha\beta\gamma}=0,$$

which can be decomposed into

$$\{\tfrac{1}{3}[(\ )_1+(\ )_2+(\ )_3]+m\}\Psi +(\sqrt{3}/36)[(\ )_1+(\ )_2-2(\ )_3]\phi -\tfrac{1}{12}[(\ )_1-(\ )_2]\phi'-\eta=0, \quad \text{(B15)}$$

and

$$2m\phi-\tfrac{1}{6}[(\ )_1+(\ )_2-2(\ )_3](\sqrt{3}\Psi+\phi) -\tfrac{1}{6}\sqrt{3}[(\ )_1-(\ )_2](\phi'-\Omega)=0. \quad \text{(B16)}$$

Equations (B13)–(B16) form a set of basic equations.

Subtracting twice Eq. (B13) from Eq. (B14), we have

$$2m(\phi'+\Omega)=[-i\Gamma\partial+(\ )_3]\Omega +[(\ )_1-(\ )_2][\tfrac{1}{3}\phi\sqrt{3}-\tfrac{1}{2}\Psi], \quad \text{(B17)}$$

which, when multiplied by $[(\ )_1+(\ )_2]$ on the left, can be reduced to

$$2m[(\ )_1+(\ )_2](\phi+\Omega)=S(\partial)\Omega. \quad \text{(B18)}$$

Multiplying Eq. (B14) by $[(\ )_1+(\ )_2]$ and making use of Eq. (B18), we have

$$[(\ )_1+(\ )_2]\phi'=0. \quad \text{(B19)}$$

Equation (B18) then reduces to

$$[\partial^2-m(\ )_i]\Omega=0, \quad i=1,2,3$$

and consequently,

$$\Omega=0. \quad \text{(B20)}$$

With the help of these results, Eq. (B13) can be simplified to

$$[(\ )_1-(\ )_2]\phi-\tfrac{2}{3}\sqrt{3}(\ )_3\phi'=0. \quad \text{(B21)}$$

The identical technique can be applied to Eqs. (B15) and (B16). Multiplying [Eq. (B15)$+\tfrac{1}{6}\sqrt{3}$ Eq. (B16)] by $[(\ )_1+(\ )_2]$, we have

$$[(\ )_1+(\ )_2][m\Psi+\tfrac{1}{3}m\sqrt{3}\phi-\eta]=-\tfrac{1}{2}S(\partial)\Psi. \quad \text{(B22)}$$

Multiplying Eq. (B16) by $[(\ )_1+(\ )_2]$, and making use of Eq. (B22), we have

$$2m^2[(\ )_1+(\ )_2]\phi = \tfrac{1}{6}\sqrt{3}[(\ )_1+(\ )_2-2(\ )_3][(\ )_1+(\ )_2]\eta\,, \quad \text{(B23)}$$

which, after eliminating $\phi$ from Eq. (B22), can be reduced to

$$[(\ )_1+(\ )_2]\{m\Psi-\eta+(1/3m)[(\ )_1+(\ )_2+(\ )_3]\eta\} = -\tfrac{1}{2}S(\partial)(\Psi-\eta/m)\,,$$

and consequently,

$$(\ )_1\{m\Psi-\eta+(1/3m)[(\ )_1+(\ )_2+(\ )_3]\eta\} = (\ )_2\{m\Psi-\eta+(1/3m)[(\ )_1+(\ )_2+(\ )_3]\eta\} = (\ )_3\{m\Psi-\eta+(1/3m)[(\ )_1+(\ )_2+(\ )_3]\eta\}\,. \quad \text{(B24)}$$

Equation (B17), when combined with Eq. (B24), leads to

$$[(\ )_3-3m]\{\phi'+(1/4m^2)[(\ )_1-(\ )_2]\eta\}=0\,. \quad \text{(B25)}$$

Making use of Eq. (B19), one can show that Eq. (B25) actually implies

$$\phi'=-(1/4m^2)[(\ )_1-(\ )_2]\eta\,, \quad \text{(B26)}$$

and consequently

$$\phi=(1/4m^2\sqrt{3})[(\ )_1+(\ )_2-2(\ )_3]\eta\,. \quad \text{(B27)}$$

Substituting Eqs. (B24), (B26), and (B27) into Eq. (B15), we are finally led to

$$[(\ )_3+m]\Psi = \{(1/4m^2)[(\ )_1-m][(\ )_2-m]\}\eta + [(\ )_3+m]\{-(1/12m^2)[(\ )_1+(\ )_2+(\ )_3] + 3/4m\}\eta\,, \quad \text{(B28)}$$

as well as analogous equations for $[(\ )_1+m]\Psi$ and $[(\ )_2+m]\Psi$. Equation (B28) leads directly to the spin-$\frac{3}{2}$ Green's function.

In the absence of an external source, we have

$$\Omega=\phi'=0=\psi_{[\alpha\beta]\gamma}\,,$$

and consequently

$$[-i\gamma\partial+m]_i\Psi=0\,, \quad i=1,2,3\,,$$

which are the correct equations to be verified.

PHYSICAL REVIEW VOLUME 161, NUMBER 5 25 SEPTEMBER 1967

# Consistency Conditions for Wave Functions Satisfying Dirac or Kemmer Equations

LOH HOOI-TONG AND D. E. DAYKIN
*Department of Mathematics, University of Malaya, Kuala Lumpur, Malaysia*
(Received 1 August 1966, revised manuscript received 7 March 1967)

For both the Dirac and the spin-zero Kemmer equations, conditions on the wave function for the existence of real solutions for the field potentials are obtained. Also the field variables are expressed in terms of the wave function for the Kemmer equations.

## 1. INTRODUCTION

It has been noted by Eliezer[1] that if a wave function $\psi$ satisfies the Dirac equations of an electron moving in an electromagnetic field, then it also satisfies a certain consistency condition. He used a particular set of Dirac matrices $\gamma_\mu$, but we show here that the condition can be obtained for an arbitrary set, and it then takes the form of Eq. (2) below. Moreover, this condition is independent of the particular electromagnetic field in the Dirac equations. Eliezer's results were later reformulated by Rastogi and Vachaspati.[2] Such work is of interest because it raises the question of the existence of a possible simple relationship between the wave function $\psi$ and the field variables $f_{\mu\nu}$. It is hoped that second quantization of such a relationship will lead to formulations of quantum electrodynamics alternative to the usual wave equations, which relate $\psi$ to the potentials $A_\mu$.

In this paper we extend Eliezer's results[1] for the Dirac equations. Then we obtain relationship (10) between $\psi$ and $f_{\mu\nu}$ for Kemmer equations for spin-zero particles, and we derive the equations's consistency condition.

## 2. THE DIRAC EQUATIONS

The Dirac equations[3] for an electron in an electromagnetic field are

$$\partial_\mu\gamma_\mu\psi-\kappa\psi=i(e/\hbar c)A_\mu\gamma_\mu\psi\,, \quad (1)$$

where the wave function $\psi$ is a column matrix $(\psi_1,\psi_2,\psi_3,\psi_4)$ and the $\gamma_\mu$ are $4\times4$ Dirac matrices satisfying the usual anticommutation rules

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}, \quad 1\leqslant\mu,\nu\leqslant4.$$

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