N-Fold Joint Photon Counting Distributions Associated with the Photodetection of Gaussian Light*

GABRIEL BÉDARD
Department of Physics and Astronomy, University of Rochester, Rochester, New York
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Recently, Arecchi, Berné, and Sona reported results of theoretical and experimental investigations on the twofold joint photocount distributions of a stationary Gaussian-Markovian radiation field. In this paper, we generalize their results by deriving the N-fold joint photocount distribution of a Gaussian (thermal) radiation field of arbitrary spectral profile, when the counting-time intervals are short compared to the coherence time of the light. The present analysis provides simple recurrence relations for the N-fold joint photocount distributions and for their generalized factorial moments. These relations are derived with the help of an $N\times N$ generating matrix which is introduced here. The present analysis also indicates to what extent the N-fold joint photocount distributions contain useful information about the higher-order coherence properties of the radiation field.

A GREAT deal of interest has been manifested recently towards the investigations of the statistical behavior of fluctuating light beams by the photon counting techniques.[1–9] Such techniques consist of measuring the statistical distribution of photoelectrons registered, in a fixed time interval, by a single photodetector upon which the light is incident. It is well known that the photocount distribution is related to the probability density for the time-integrated light intensity by a linear transform.[10–12] On the other hand, the possibility of determining the probability density of the light intensity of an optical beam from experimentally obtained photocount distributions has also been investigated.[13,14]

Recently, the twofold joint photocount distribution of a Gaussian-Markovian radiation field has been investigated, theoretically and experimentally, by Arecchi *et al.*[9] In the present paper, we generalize their results by deriving the N-fold joint photocount distributions of a Gaussian (thermal) radiation field of arbitrary spectral profile: namely, the joint probability $p(n_1,T_1;n_2,T_2;\cdots;n_N,T_N)$ that n_1 photoelectrons will be registered in a time interval T_1 by a photodetector located at the space-time point x_1 (more generally $x_l=\mathbf{r}_l,t_l$), n_2 photoelectrons will be registered in a time interval T_2 by a second detector located at x_2, $\cdots$, and n_N photoelectrons will be registered in a time interval T_N by the Nth detector located at x_N.

As will be shown, these joint photocount distributions carry information about the higher-order intensity correlations of the radiation field, and also about the spectral profile of the light. We assume that the optical field is stationary and quasimonochromatic. The counting time intervals T_j $(j=1, 2, \cdots, N)$ are assumed to be much shorter than the coherence time of the light.

The basic formula[10–12]

$$p(n,T)=\int_0^\infty \frac{W^n}{n!}e^{-W}P(W)dW \qquad (1)$$

relates the statistical distribution $p(n,T)$ of photoelectrons, registered with a single detector in the counting time interval T, to the probability density $P(W)$ for the quantity

$$W=\alpha\int_t^{t+T} |V(\mathbf{r},t')|^2dt'. \qquad (2)$$

Here α is a measure of the quantum efficiency of the detector, $\mathbf{r}$ is any point on the sensitive surface of the photodetector, and

$$V(\mathbf{r},t)=\frac{1}{L^{3/2}}\sum_{\mathbf{k},s} v_{\mathbf{k},s}\exp[i(\mathbf{k}\cdot\mathbf{r}-kct)], \qquad (3)$$

where $v_{\mathbf{k},s}$ is the eigenvalue of the photon annihilation operator $a_{\mathbf{k},s}$ for the mode $\mathbf{k}$ and polarization state s. The probability density $P(W)$ is related to the phase-space functional $\Phi(\{v_{\mathbf{k}s}\})$ of the diagonal represen-

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tation[15,16] by an expression of the form[17]

$$P(W')=\int \Phi(\{v_{\mathbf{k},s}\})\delta(W'-W)d^2\{v_{\mathbf{k},s}\}\,, \qquad (4)$$

where the quantity W is obtained from Eqs. (2) and (3). Alternatively, one can write Eq. (1) in the more compact form

$$p(n,T)=\langle (W^n/n!)e^{-W}\rangle\,, \qquad (5)$$

with the understanding that the angular brackets denote the appropriate average with respect to the phase-space functional.

The foregoing results can readily be extended to the multidetector arrangement, in which one measures the N-fold joint photocount distributions $p(n_1,T_1;n_2,T_2;\cdots;n_N,T_N)$. By an analysis quite similar to the one used in deriving Eq. (1), one can show that an expression analogous to Eq. (5) holds for the N-fold joint photocount distribution, namely

$$p(n_1,T_1;n_2,T_2;\cdots;n_N,T_N)=\left\langle \prod_{i=1}^{N}\frac{W_i^{n_i}}{n_i!}e^{-W_i}\right\rangle\,, \qquad (6)$$

where the angular brackets again denote the appropriate average with respect to the phase-space functional $\Phi(\{v_{\mathbf{k}s}\})$. Here the quantities W_i are obtained from Eqs. (2) and (3) by replacing W, α, $\mathbf{r}$, t, and T by W_i, α_i, $\mathbf{r}_i$, t_i, and T_i, respectively. When the counting time intervals are much shorter than the coherence time of the light, the integrand in Eq. (2) is effectively constant, so that the quantities W_i are given by

$$W_i=\alpha_i|V(\mathbf{r}_i,t_i)|^2T_i \qquad (7)$$

$$=\alpha_i|V_i|^2T_i. \qquad (8)$$

In order to simplify the notation, we have set $V_i=V(\mathbf{r}_i,t_i)$.

It follows from Eq. (8) that the appropriate average appearing in Eq. (7) must be taken with respect to the "joint probability density" $p_N(V_1,V_2,\cdots,V_N)$, defined by the relation[18]

$$p_N(V_1,V_2,\cdots,V_N)=\int \Phi(\{v_{\mathbf{k}s}\})\prod_{i=1}^{N}\delta\left(V_i-\frac{1}{L^{3/2}}\sum_{\mathbf{k}s}v_{\mathbf{k}s}\exp[i(\mathbf{k}\cdot\mathbf{r}_i-ckt_i)]\right)d^2\{v_{\mathbf{k}s}\}\,. \qquad (9)$$

Hence, the joint photocount distribution can be expressed in the form

$$p(n_1,T_1;n_2,T_2;\cdots;n_N,T_N)=\int\cdots\int p_N(V_1,V_2,\cdots,V_N)\mathcal{P}(V_1,V_2,\cdots,V_N)\times d^2V_1\cdots d^2V_N\,, \qquad (10)$$

where we have set

$$\mathcal{P}(V_1,V_2,\cdots,V_N)=\prod_{i=1}^{N}\frac{(\alpha_i|V_i|^2T_i)^{n_i}}{n_i!}\times\exp(-\alpha_i|V_i|^2T_i)\,, \qquad (11)$$

and $d^2V_i=d(\mathrm{Re}V_i)d(\mathrm{Im}V_i)$.

The foregoing analysis applies to any optical radiation field. Let us now consider the case of Gaussian (thermal) light. In this case, the phase-space functional $\Phi(\{v_{\mathbf{k}s}\})$ has the form[18]

$$\Phi(\{v_{\mathbf{k}s}\})=\prod_{\mathbf{k}s}\frac{1}{\pi w_{\mathbf{k}s}}\exp(-|v_{\mathbf{k}s}|^2/w_{\mathbf{k}s})\,, \qquad (12)$$

and $w_{\mathbf{k},s}$ is the average photon occupation number in the mode $\mathbf{k}$, s of the radiation field. Upon substitution of Eq. (12) into Eq. (9), one obtains the following expression:

$$p_N(V_1,V_2,\cdots,V_N)=\frac{|A|}{(2\pi)^N}\exp(-\tfrac{1}{2}V^\dagger AV)\,, \qquad (13)$$

with the column matrix

$$V=\begin{bmatrix}V_1\\V_2\\\vdots\\V_N\end{bmatrix}\,, \qquad (14)$$

and its Hermitian conjugate

$$V^\dagger=(V_1^*V_2^*\cdots V_N^*)\,. \qquad (15)$$

Here, $|A|$ stands for the determinant of the Hermitian positive definite matrix A, defined in terms of the mutual coherence function[18]

$$\Gamma_{ij}=\langle V_i^*V_j\rangle\,, \qquad (16)$$

through the relation

$$2(A^{-1})_{ji}=\Gamma_{ij}. \qquad (17)$$

Hence, the matrix A, through its relation to the mutual coherence function, carries information about the spectral profile of the light. The probability density $p_N(V_1,V_2,\cdots,V_N)$ is therefore a multivariate Gaussian distribution in the N complex variables V_i, with zero mean.[18]

Let us now introduce the multidimensional generating functions $G(s_1,s_2,\cdots,s_N)$, defined by the relation

$$G(s_1,s_2,\cdots,s_N)=\sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\left[\prod_{i=1}^{N}(1-s_i)^{n_i}\right]\times p(n_1,T_1;\cdots;n_N,T_N)\,, \qquad (18)$$

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where the expansion coefficients

$$p(n_1,T_1;\cdots;n_N,T_N)=\left\{\prod_{i=1}^{N}\frac{(-1)^{n_i}}{n_i!}\frac{\partial^{n_i}}{\partial s_i^{n_i}}\right\}\times G(s_1,s_2,\cdots,s_N)|_{s_1=1,\cdots,s_N=1} \quad (19)$$

constitute the joint photocount distribution. The generalized factorial moments, defined as

$$\langle n_1^{[l_1]}\cdots n_N^{[l_N]}\rangle=\sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\left[\prod_{i=1}^{N}\frac{n_i!}{(n_i-l_i)!}\right]\times p(n_1,T_1;\cdots;n_N,T_N), \quad (20)$$

can be obtained from the generating function $G(s_1,s_2,\cdots,s_N)$ according to the following rule:

$$\langle n_1^{[l_1]}\cdots n_N^{[l_N]}\rangle=\left\{\prod_{i=1}^{N}(-1)^{n_i}\frac{\partial^{n_i}}{\partial s_i^{n_i}}\right\}\times G(s_1,s_2,\cdots,s_N)|_{s_1=0,\cdots,s_N=0}. \quad (21)$$

Alternatively, one can write the generating function as

$$G(s_1,s_2,\cdots,s_N)=\langle\prod_{i=1}^{N}e^{-s_iW_i}\rangle, \quad (22)$$

or in terms of the joint probability density $p_N(V_1, V_2,\cdots V_N)$ as

$$G(s_1,s_2,\cdots s_N)=\int\cdots\int p_N(V_1,V_2,\cdots,V_N)\times[\prod_{i=1}^{N}\exp(-\alpha_i|V_i|^2T_is_i)d^2V_i]. \quad (23)$$

Upon substitution of Eq. (13) into Eq. (23), one obtains

$$G(s_1,s_2,\cdots s_N)=\frac{|A|}{(2\pi)^N}\int\cdots\int\exp(-\tfrac{1}{2}V^\dagger BV)\times d^2V_1\cdots d^2V_N, \quad (24)$$

where B is the $N\times N$ matrix whose elements B_{ij} are defined as

$$B_{ij}=A_{ij}+2\alpha_iT_is_i\delta_{ij}. \quad (25)$$

The evaluation of Eq. (24) is quite straightforward and yields the following expression for the generating function:

$$G(s_1,s_2,\cdots,s_N)=[|A^{-1}B|]^{-1}, \quad (26)$$

where $|A^{-1}B|$ stands for the determinant of the product matrix $A^{-1}B$. It follows from Eqs. (17), (25), and (26) that

$$G(s_1,s_2,\cdots,s_N)=|\Delta_N|^{-1}, \quad (27)$$

where $|\Delta_N|$ is the determinant of the $N\times N$ matrix Δ_N whose elements are given by

$$(\Delta_N)_{ij}=\delta_{ij}+\langle n_i\rangle s_i\gamma_{ij}^*\mu_{ij} \quad (28)$$

with

$$\mu_{ij}=\left[\frac{\langle n_i\rangle}{\langle n_j\rangle}\frac{T_j}{T_i}\right]^{1/2}. \quad (29)$$

Here γ_{ij} is the second-order complex degree of coherence for the radiation field at the ith detector located at the space time point $\mathbf{r}_i$, t_i at the jth detector located at $\mathbf{r}_j$, t_j. The average number of photocounts at the lth detector is given by $\langle n_l\rangle=\alpha_l\langle I_l\rangle T_l$. We shall refer to Δ_N as the generating matrix of the N-fold photocount distribution for the thermal radiation field.

Rewriting Eq. (27) in the form

$$G(s_1,s_2,\cdots s_N)|\Delta_N|=1, \quad (30)$$

and performing partial differentiation with respect to the variables s_i, we obtain

$$\left\{\prod_{i=1}^{N}\frac{\partial^{n_i}}{\partial s_i^{n_i}}\right\}G(s_1,s_2,\cdots s_N)|\Delta_N|=0. \quad (31)$$

This result holds as long as the sum of the n_i is nonzero. Applying Leibnitz's differentiation rule to the product $G|\Delta_N|$, Eq. (31) can be expressed in the following form:

$$\sum_{r_1=0}^{n_1}\cdots\sum_{r_N=0}^{n_N}\left\{\left[\prod_{i=1}^{N}\binom{n_i}{r_i}\frac{\partial^{n_i-r_i}}{\partial s_i^{n_i-r_i}}\right]G\right\}\times\left[\prod_{i=1}^{N}\frac{\partial^{r_i}}{\partial s_i^{r_i}}\right]|\Delta_N|=0. \quad (32)$$

It follows from Eq. (28) that the only nonvanishing partial derivatives of the determinant $|\Delta_N|$ are those for which r_i is either 0 or 1.

At this point, if one evaluates Eq. (32) first at $s_j=1$ $(j=1, 2, \cdots N)$ and then at $s_j=0$ $(j=1, 2, \cdots N)$, one obtains, respectively, the following recurrence relations for the joint photocount distribution:

$$\sum_{r_1=0}^{1}\cdots\sum_{r_N=0}^{1}(-1)^{r_1+r_2+\cdots+r_N}A_{r_1r_2\cdots r_N}\times p(n_1-r_1, T_1;\cdots;n_N-r_N, T_N)=0; \quad (33)$$

and for its generalized factorial moments:

$$\sum_{r_1=0}^{1}\cdots\sum_{r_N=0}^{1}(-1)^{r_1+r_2+\cdots+r_N}B_{r_1r_2\cdots r_N}\times\langle n_1^{[l_1-r_1]}\cdots n_N^{[l_N-r_N]}\rangle=0. \quad (34)$$

Here we have set

$$A_{r_1r_2\cdots r_N}=\left[\left\{\prod_{i=1}^{N}\frac{\partial^{r_i}}{\partial s_i^{r_i}}\right\}|\Delta_N|\right]_{s_1=1,\cdots,s_N=1}, \quad (35)$$

and

$$B_{r_1r_2\cdots r_N}=\left[\left\{\prod_{i=1}^{N}\frac{1}{(l_i-r_i)!}\frac{\partial^{r_i}}{\partial s_i^{r_i}}\right\}|\Delta_N|\right]_{s_1=0,\cdots,s_N=0}. \quad (36)$$

To illustrate the previous results, let us consider the first few joint photocount distributions of the thermal radiation field, when the counting time interval is much shorter than the coherence time of the light.

Example 1. The single-detector case ($N=1$): In such a case, the determinant of the generating matrix Δ_1 reduces to one term, namely

$$|\Delta_1|=[1+\langle n\rangle s],$$

as is easily seen from Eq. (28), and the corresponding generating function is

$$G(s)=[1+\langle n\rangle s]^{-1}.$$

It readily follows from the recurrence relation (33) that

$$\sum_{r=0}^{1}(-1)^r A_r p(n-r,T)=0, \quad (37)$$

with

$$A_0=1+\langle n\rangle,\quad A_1=\langle n\rangle,$$

and that the photocount distribution is given by the relation

$$p(n,T)=\frac{\langle n\rangle}{\langle n\rangle+1}p(n-1,T). \quad (38)$$

If one now uses the initial value

$$p(0,T)=G(1)=[1+\langle n\rangle]^{-1},$$

one finds that the function $G(s)$ generates the well-known Bose-Einstein distribution

$$p(n,T)=\frac{\langle n\rangle^n}{[\langle n\rangle+1]^{n+1}}, \quad (39)$$

tor the single-detector photocount distribution $p(n)$.

Example 2. The two-detector arrangement ($N=2$): The determinant of the 2×2 generating matrix

$$\Delta_2=\begin{bmatrix}\langle n_1\rangle s_1+1 & \langle n_1\rangle s_1\gamma_{12}^*\mu_{12}\\ \langle n_2\rangle s_2\gamma_{12}\mu_{21} & \langle n_2\rangle s_2+1\end{bmatrix}$$

is of the form

$$|\Delta_2|=(\langle n_1\rangle s_1+1)(\langle n_2\rangle s_2+1)-\langle n_1\rangle\langle n_2\rangle s_1s_2|\gamma_{12}|^2.$$

The corresponding generating functing $G(s_1,s_2)$ can be expressed as

$$G(s_1,s_2)=[1+\langle n_1\rangle s_1+\langle n_2\rangle s_2+\langle n_1\rangle\langle n_2\rangle s_1s_2(1-|\gamma_{12}|^2)]^{-1},$$

and the recurrence relation (33) now reads

$$\sum_{r_1=0}^{1}\sum_{r_2=0}^{1}(-1)^{r_1+r_2}A_{r_1r_2}p(n_1-r_1,T_1;n_2-r_2,T_2)=0. \quad (40)$$

For convenience, the coefficients A_{km} are expressed here in a matrix form:

$$A=\begin{bmatrix}A_{00} & A_{10}\\ A_{01} & A_{11}\end{bmatrix}=\begin{bmatrix}1+\langle n_1\rangle+\langle n_2\rangle+\langle n_1\rangle\langle n_2\rangle\beta & \langle n_1\rangle(1+\langle n_2\rangle)\beta\\ \langle n_2\rangle(1+\langle n_1\rangle)\beta & \langle n_1\rangle\langle n_2\rangle\beta\end{bmatrix}, \quad (41)$$

with

$$\beta=1-|\gamma_{12}|^2.$$

Hence, Eqs. (40) and (41), along with the boundary values

$$p(k,T_1;0,T_2)=A_{10}^kA_{00}^{-k-1},$$
$$p(0,T_1;k,T_2)=A_{01}^kA_{00}^{-k-1},$$

completely characterize the twofold joint photocount distribution $p(n_1,T_1;n_2,T_2)$ of a thermal light beam of arbitrary spectral profile, when the counting time intervals T_1 and T_2 are short compared to the coherence time of the light. Information about the spectra profile of the light is contained in the coefficients A_{km} through the parameter β.

Example 3. The three-detector arrangement ($N=3$): The generating matrix Δ_3 is of the form

$$\Delta_3=\begin{pmatrix}\langle n_1\rangle s_1+1 & \langle n_1\rangle\mu_{12}\gamma_{12}^*s_1 & \langle n_1\rangle\mu_{13}\gamma_{13}^*s_1\\ \langle n_2\rangle\mu_{21}\gamma_{12}s_2 & \langle n_2\rangle s_2+1 & \langle n_2\rangle\mu_{23}\gamma_{23}^*s_2\\ \langle n_3\rangle\mu_{31}\gamma_{13}s_3 & \langle n_3\rangle\mu_{23}\gamma_{23}s_3 & \langle n_3\rangle s_3+1\end{pmatrix}.$$

The corresponding generating function

$$G(s_1,s_2,s_3)=|\Delta_3|^{-1}$$

takes the form

$$G(s_1,s_2,s_3)=[1+\langle n_1\rangle s_1+\langle n_2\rangle s_2+\langle n_3\rangle s_3+|g_{12}|\langle n_1\rangle\langle n_2\rangle s_1s_2+|g_{13}|\langle n_1\rangle\langle n_3\rangle s_1s_3+|g_{23}|\langle n_2\rangle\langle n_3\rangle s_2s_3+|g_{123}|\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle s_1s_2s_3]^{-1},$$

where $|g_{ij}|$ is the determinant of the submatrix

$$g_{ij}=\begin{bmatrix}1 & \gamma_{ij}\\ \gamma_{ij}^* & 1\end{bmatrix},$$

and $|g_{123}|$ is the determinant of the matrix

$$g_{123}=\begin{pmatrix}1 & \gamma_{12} & \gamma_{13}\\ \gamma_{12}^* & 1 & \gamma_{23}\\ \gamma_{13}^* & \gamma_{23}^* & 1\end{pmatrix},$$

namely

$$|g_{123}|=1-|\gamma_{12}|^2-|\gamma_{13}|^2-|\gamma_{23}|^2+2\,\mathrm{Re}\gamma_{12}\gamma_{23}\gamma_{13}^*.$$

Clearly the generating function will contain information about the phases of the γ_{ij} in view of the presence of the term $\gamma_{12}\gamma_{23}\gamma_{13}^*$. Hence, information about the spectral profile of the light can be obtained from the threefold joint photocount distribution.

The present analysis provides simple recurrence relations for the N-fold joint photocount distributions, and their generalized factorial moments, for a thermal radiation field of arbitrary spectral profile, when the counting time intervals are much shorter than the coherence time of the light. As mentioned previously, these generalized factorial moments are closely related to the high-order intensity correlation functions of the radiation field, and therefore contain information about the phase of the second-order complex degree of coherence γ_{ij} for the light field at two space-time points. As is well known, the knowledge of the phase of the degree of coherence is essential in determining the spectral profile of the light beam. Clearly, the N-fold joint photocount distributions provide useful and interesting information about the higher-order coherence properties of the radiation field.

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Lagrange Formulation for Systems with Higher Spin*

Shau-Jin Chang†‡
Department of Physics, Harvard University, Cambridge, Massachusetts
(Received 27 April 1967)

Prescriptions for constructing the generalized Lagrange function for a system with an arbitrary spin S are presented. By the use of the spin projection operators introduced by Fronsdal, nonlocal field equations are constructed to describe these higher-spin systems. Then, auxiliary fields are introduced systematically to remove the nonlocalities appearing in these field equations. Lagrange functions describing systems with $S\leq 4$ are constructed explicitly according to this new prescription. For $S=0, \frac{1}{2}, 1$, they agree with the well-known local Lagrange functions. For $S=\frac{3}{2}$ and 2, they are equivalent to the results previously obtained by Rarita and Schwinger, and by Fierz and Pauli. With the help of the quantum action principle, canonical quantizations are carried out and Green's functions are constructed. Some physical positiveness requirements are also verified.

I. INTRODUCTION

THE problem of quantization for systems with higher spins has been studied extensively ever since the earlier development of quantum field theory.[1] The recent discovery of many higher-spin resonances arouses new interest in this problem. Roughly speaking, there are two different approaches to describe the field theory of higher spins. The first approach emphasizes the transformation properties of field variables under the homogeneous Lorentz group.[2] The physical interacting field operators are considered as the asymptotic field operators—the field operators before and after the interactions are taken place. This approach has the advantage that these asymptotic field variables satisfy very simple field equations, and that no complicated Lagrange function is required to describe them. These asymptotic field variables can be quantized easily by expanding them in terms of creation and annihilation operators, and their corresponding propagation functions can then be determined. This approach is very successful in perturbative applications as well as in the S-matrix theory. The simplicity in this approach originates from the fact that we have bypassed the detailed structures of the interactions. This advantage turns out to be its disadvantage when we try to describe the interaction. Neither the canonical commutator relations nor the stress tensor can be obtained without solving the full dynamics. The canonical quantization conditions as well as further consistency requirements in the presence of interactions are consequently ignored. The second approach follows that of Pauli and Fierz,[1] and demands that all field equations and subsidiary conditions should be derived from a generalized action principle. This classical approach has the advantages that the interaction can be introduced explicitly, and that the Green's function can be computed. The canonical quantization relations and the stress tensor of a system can be obtained directly from the action principle, even in the presence of interaction. The validity of all these consequences is not limited by perturbations. However, this approach has at least one defect. For a system with spin ≥ 2, even the construction of a free Lagrange function is very tedious and, in some sense, rather ambiguous. The introduction of auxiliary field variables is by itself quite arbitrary. In

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† John Parker Fellow.
‡ Present address: The Institute for Advanced Study, Princeton, New Jersey.
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