is the increase in complexity of some of the angular distributions at the higher energy. The p_0, p_1, p_3 groups are not significantly different in the two cases, but the p_2, d_0, d_1, d_2, and d_3 curves have more maxima and minima at the higher energy. Even when the curves are roughly similar in the two studies, the details concerning the positions of maxima and minima are quite different. This is in contrast to an earlier investigation of the $C^{12}(Li^6,p)O^{17}$ and $C^{12}(Li^6,d)O^{16}$ reactions,[4] where the location of most of the peaks in the angular distributions were fairly constant as the lithium beam energy was varied. The ratios of the various total cross sections are approximately the same in the present work as found earlier at lower energy.[2]

Although these reactions may involve the interactions of deuteron-, triton-, and α-like clusters of nucleons, as suggested earlier,[1,2] the data obtained in the present experiment tend to emphasize the importance of the compound-nucleus mechanism. The asymmetry of the angular distributions about 90°, which could be attributed to the importance of stripping reactions, could also be due to the interference between levels of opposite parity at the relatively high (26 MeV) excitation level of the C^{13} compound nucleus. The shapes of the angular distribution curves observed at 3.50 MeV and their variations from the shapes observed at 2.1 MeV are not particularly indicative of stripping reactions. Finally, the magnitudes of the observed total cross sections are well within the range which could be accounted for by the formation of a compound nucleus. However, since none of these indications are really definitive, the actual reaction mechanism could well be a mixture of processes.

ACKNOWLEDGMENTS

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Semiempirical Formula for Nuclear Rotational Energies

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Using the theory of molecular spectra to obtain an estimate of the higher-order corrections to the energy levels of a nonrigid rotator, we sum the infinite power series in $I(I+1)$ to describe the energy levels in the ground-state rotational bands of deformed even-even nuclei by the expression $E(I)=A(I)I(I+1)$, where

$$A(I)=A[1+(N-1)(B/A)I(I+1)]/[1+N(B/A)I(I+1)],$$

with $N=2.85-0.05I$. The predictions of this two-parameter formula show surprisingly good agreement with the experimentally observed energy levels in even-even nuclei in the rare-earth region, including Os isotopes and $N=90$ nuclei. Comparison with other relatively successful models advanced during the recent years, e.g., the Davydov-Chaban model as adopted by the Berkeley group, the cranking-model extension by Harris, the classical hydrodynamical model used by Moszkowski, the rotator-vibrator model, the asymmetric-rotator model, etc., reveals a distinctly greater success of our description when it is applied to such a wide range of nuclei.

1. INTRODUCTION

ACCORDING to the Bohr-Mottelson hydrodynamical model,[1] the energy levels of deformed nuclei are similar to those of an axially symmetric rotator. In the strong-coupling limit, the energy levels of the even-even nuclei in the lowest rotational band are given by the relation

$$E(I)=AI(I+1), \quad (1)$$

where $A=\hbar^2/2\mathcal{I}$ is the rotational parameter related to the nuclear moment of inertia $\mathcal{I}$, and the total angular momentum (spin) I follows the sequence 0, 2, 4, 6, ···, all with even parity. Equation (1) describes the energy levels for a rigid rotator under the assumption that the deformation, and hence the moment of inertia, are not affected by the rotation. Actually, the deformation increases with the rotational angular momentum because of the centrifugal forces. In principle, one may write the rotational energy as an infinite power series

$$E(I)=AI(I+1)-BI^2(I+1)^2+CI^3(I+1)^3 -DI^4(I+1)^4+\cdots. \quad (2)$$

For small values of I, the first-order correction may be the only significant one. In analogy with the molecular spectra, the coefficient B can be related[1] to the energies

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of the nuclear β and γ vibrations, but it was found[2] that the values of B thus calculated do not agree with those determined empirically.

The inadequacy of the two-parameter description [corresponding to retaining only the first two terms in Eq. (2)] for the transitional nuclei was demonstrated by several investigators, e.g., for Os isotopes by the Brookhaven group,[3] for some borderline nuclei in the actinide region by Stephens *et al.*,[4] for $N=90$ nuclei and some neutron-deficient nuclei by the Copenhagen group.[5] A comprehensive survey of the role of second-order corrections was undertaken by the author and a number of his students[6] over the past few years, and one of the conclusions was that the convergence of the series is very poor (or even breaks down) for high-spin states in transitional and moderately deformed nuclei. Later it has been shown[7] that, in the limit of recently identified high-spin states, the comparison with Eq. (2) is not meaningful since 'such a power series expansion would require almost as many terms as there are points to be fitted if it is to be used for extrapolation.'

In view of the above situation, several models have been proposed during the last few years for predicting the nuclear rotational energies with varying degrees of success. We have arrived at a very simple two-parameter description of the rotational energies by an effective summation of the infinite series guided by the relative magnitudes of the successive terms in the expression for the energy of a rotator vibrator.[8] In the process we obtain a simple analytical expression for the variation of nuclear moment of inertia with angular momentum.

We begin by giving a very brief discussion of the conventional two- and three-parameter fits followed by our semiempirical formulation. The results of our calculations are then compared with the available experimental data, and with the results from other theoretical investigations.

2. CONVENTIONAL TWO- AND THREE-PARAMETER FITS

For a rigid rotator, the energy ratio $E(4^+)/E(2^+)$ has the value 10/3, and the deviations from this value roughly indicate a measure of the corrections to be applied. In our formulation we shall describe all our results as a function of this energy ratio (denoted hereafter by R).

In the conventional two-parameter fit, we have

$$\frac{B}{A}=\frac{10-3R}{200-18R} \tag{3}$$

and the corresponding ratios $E(I)/E(2)$ for levels with spin up to $I=20$ are plotted in Fig. 1 for this case. It is noticed that, particularly with the inclusion of very-high-spin states, even the spin sequence of levels is radically altered from the usual rotational sequence, and the predicted level scheme is quite unlike that observed in deformed nuclei.

The results for the energy ratio of the 10^+ level are shown in Fig. 2 for the conventional one-, two-, and three-parameter fits (labelled A, AB, and ABC, respectively) and the divergence of the series is quite evident. As a specific example the level scheme for the nucleus ^{170}Hf is shown in Fig. 3. It is seen from this figure that although the prediction for the 10^+ level is better from ABC than from A or AB, it is still off by more than 7%. For the 14^+ level, the predicted energy from ABC is twice as bad as from A alone. The divergence of the series expansion can be seen by noting the contributions of the various order terms to the energy of the 16^+ state

$$^{170}\text{Hf}:\ E(16^+)=4617-3935+5965=6647\ \text{keV}. \tag{4}$$

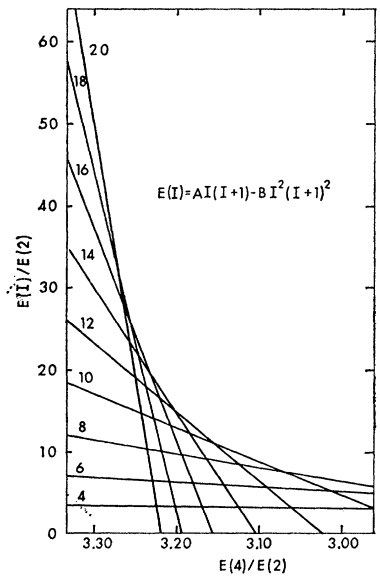

FIG. 1. The energy ratios $E(I)/E(2)$ calculated from the conventional two-parameter formula (given in the figure) plotted as a function of the ratio $E(4)/E(2)$.

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The divergence is already in evidence for the 10^+ level in the transitional nucleus ^{152}Sm.

$$^{152}\text{Sm}: \quad E(10^+)=2353-2347+2709=2715 \text{ keV}. \quad (5)$$

Thus we may conclude that, although one obtains improved agreement with the experimental results for the low-spin levels by including the B and C terms, this is done at the cost of progressively worsening predictions for the high-spin states. If all the high-spin levels were known, the rms deviation with the inclusion of so-called one or two correction terms would be much larger than that obtained with no corrections. In other

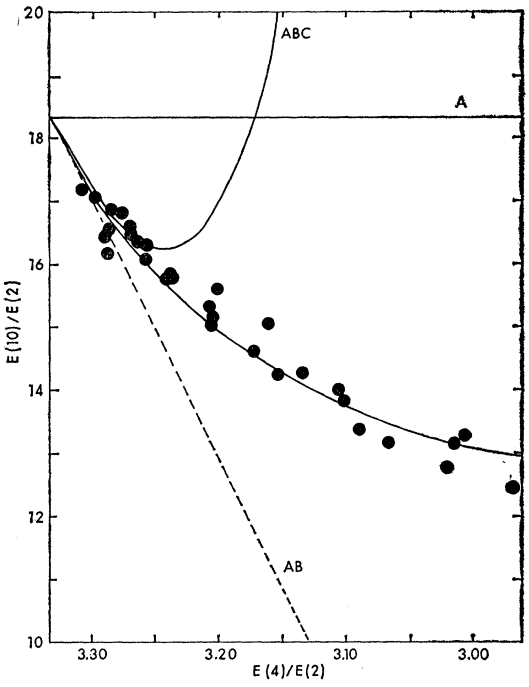

FIG. 2. The energy ratio $E(10)/E(2)$ plotted as a function of the ratio $E(4)/E(2)$. The conventional one-, two-, and three-parameter results are shown by curves labelled A, AB, and ABC, respectively. The solid circles represent the experimental points and the line passing roughly through these points corresponds to our calculated values from Eq. (9).

words, leaving aside a few 'very rigid' nuclei, it is inappropriate to write an expression for the rotational energy as in Eq. (2) with a cutoff at B or C terms. Instead one may write

$$E(I)=A(I)I(I+1), \quad (6)$$

where $A(I)$ expresses the angular-momentum dependence of the nuclear moment of inertia. Unfortunately, no analytical expression for $A(I)$ has been available so far, and hence practically all the reasonable successful recent models do not express the rotational energy with explicit $I(I+1)$ dependence. In the following sections we derive, and apply with rather remarkable success, a very simple analytical expression for $A(I)$ from semiempirical considerations.

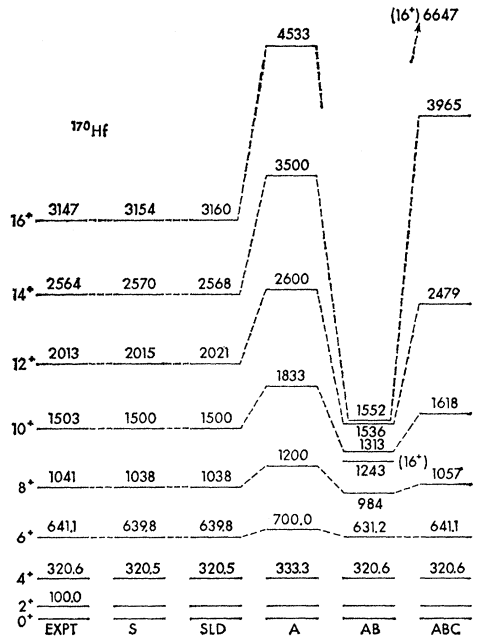

FIG. 3. The observed (the first column on the left) and the predicted energy levels for the nucleus ^{170}Hf. The column labeled S is from our calculations, one labeled SLD is from Stephens *et al.* (Ref. 7), and the following three columns labeled A, AB, and ABC are the results from conventional one-, two-, and three-parameter description.

3. SEMIEMPIRICAL FORMULATION

We may rewrite the series in Eq. (2) as follows:

$$E(I)=AI(I+1)\{1-(B/A)I(I+1)[1-(C/B)I(I+1)+(D/B)I^2(I+1)^2\cdots]\}. \quad (7)$$

The relative order-of-magnitude estimates of the successive coefficients may be obtained from the expression for the energy levels of a nonrigid rotator worked out long ago for molecular spectra. Let us write

$$C/B=N(B/A). \quad (8)$$

Then from molecular spectra theory,[8] we know that N is of the order of 2 to 3, and that (D/B) is of the order of $(C/B)^2$. Following these guidelines, let us assume that the terms within the square brackets in Eq. (7) form an infinite geometric series. This leads to the following relations:

$$\begin{aligned}E(I)&=AI(I+1)\{1-(B/A)I(I+1)[1-N(B/A)I(I+1)+[N(B/A)I(I+1)]^2-[N(B/A)I(I+1)]^3\cdots]\}\\&=AI(I+1)\left[1-\frac{(B/A)I(I+1)}{1+N(B/A)I(I+1)}\right]\\&=A\left[\frac{1+(N-1)(B/A)I(I+1)}{1+N(B/A)I(I+1)}\right]I(I+1)\\&=A(I)I(I+1), \quad (9)\end{aligned}$$

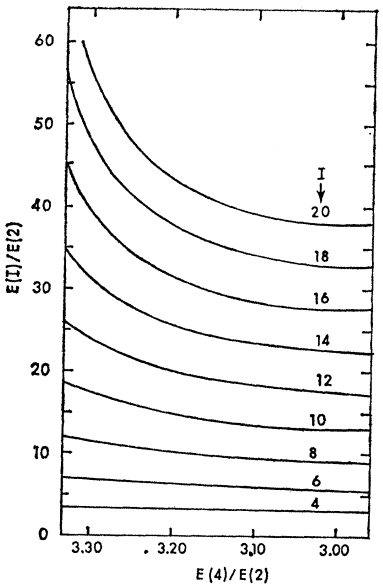

FIG. 4. The energy ratios $E(I)/E(2)$, as predicted from our calculations, plotted as a function of the ratio $E(4)/E(2)$.

which has the same form as Eq. (6), and thus yields the following simple analytical expression for the angular-momentum dependence of the nuclear moment of inertia:

$$\frac{\mathcal{I}(I)}{\mathcal{I}_0}=\frac{1+N(B/A)I(I+1)}{1+(N-1)(B/A)I(I+1)}. \tag{10}$$

The expression for N was obtained from empirical considerations. The experimental energy ratios $E(I)/E(2)$ were plotted for each I as a function of $E(4)/E(2)$ obtaining plots similar to Fig. 2—solid circles—for $I=10$. When the energy ratios are considered, Eq. (9) corresponds to a one-parameter description involving (B/A) which is determined from the observed ratio $E(4)/E(2)$. The procedure used was to assume an expression for N, and then predict the energy ratios for comparison with the experimental data. A constant N for all I did not yield a satisfactory agreement, nor did the $I(I+1)$-dependent form. The best fit so far has been obtained with the choice

$$N=2.85-0.05I. \tag{11}$$

The fit obtained with this choice is shown in Fig. 2 for $I=10$, and the calculated energy ratios $E(I)/E(2)$ for I up to 20 are plotted in Fig. 4 (to be compared with Fig. 1 from the conventional approach). This plot is essentially a one-parameter description of the rotational states for all nuclei.

It may be noted from the spread of the experimental points in Fig. 2 that it is not possible to find a single smooth curve that passes through all the points. This feature is further illustrated in Table I, which lists the

TABLE I. The excitation-energy ratios $E(I)/E(2)$ for a set of nuclei with nearly the same ratio for the excitation energies of the 4^+ and 2^+ states. A one-parameter model (such as ours) will give identical values for each of these nuclei.

	$E(I)/E(2)$			
I	^{158}Dy	^{170}Hf	^{176}W	^{184}Os
4	3.207	3.206	3.206	3.202
6	6.448	6.411	6.434	6.458
8	10.55	10.41	10.49	10.63
10	15.34	15.03	15.16	15.62
12	20.64	20.13	20.29	...
14	...	25.64	25.77	...
16	...	31.47	31.51	...

observed energy ratios for four rare-earth nuclei with approximately the same value of $E(4)/E(2)$. The predicted energy ratios for a given I will be nearly identical for each of the four nuclei, whereas the variation of the experimental value is 0.7% for the 6^+ level and about 4% for the 10^+ level. Thus it may be said that in the one-parameter description of the type given above we cannot expect exact numerical agreement for all nuclei simultaneously.

4. COMPARISON WITH EXPERIMENTAL RESULTS

The excitation energies for the levels in the ground-state rotational bands in even-even nuclei have been calculated using Eqs. (9) and (11) for comparison with the available experimental data; this is presented in Table II. The parameter B/A was calculated from the observed energy ratio $E(4)/E(2)$, and the experimental energy of the 2^+ state was then used to determine A. In order to smooth out the observed spread in energies discussed above, the values of A and B were slightly adjusted, wherever found desirable, to obtain a satisfactory fit to the observed energies.

Keeping in view the inherent limitations of calculations of this type (discussed in the previous section), we find in Table II a surprisingly good agreement with practically all the known levels of the nuclei considered. The agreement with the experiment in each case is within a fraction of 1%—usually within the limits of the experimental uncertainties of the observed energies. Even for the $N=90$ nuclei at one end of the deformation region and for Os isotopes at the other end the agreement is remarkably close, just about as good as for the the nuclei in the middle of the deformation region.

These calculations do not give satisfactory results for very highly neutron-deficient nuclei. Only four such cases have been left out of Table II. They are ^{166}Hf, ^{172}W, ^{178}Os, and ^{180}Os, although the over-all agreement for these nuclei is still better than 2%. Better fit may be obtained by choosing a more rapid decrease of N with I than that of Eq. (11). Preliminary analysis points to the value $N=2.85-0.07I$ for these nuclei. This problem will be discussed in a separate communication.

TABLE II. Excitation energies (in keV) of the states in the ground-state rotational band of even-even nuclei in the rare-earth region. The calculated values correspond to the two-parameter description based on 2^+ and 4^+ state energies as explained in the text.

Nucleus		2^+	4^+	6^+	8^+	10^+	12^+	14^+	16^+
^{152}Sm	Expt[a]	121.8	366.5	707	1125	1608	(2158)		
	Calc	121.8	368.1	706	1122	1609	2156		
^{154}Sm	Expt[b]	82.0	267	552	(934)	(1379)			
	Calc	81.0	267	553	930	1389			
^{154}Gd	Expt[a,b]	123.1	371.2	718	1146	1640	(2189)		
	Calc	123.1	374.2	719	1143	1638	2193		
^{156}Gd	Expt[a]	89.0	288.2	584	965	1417	1924		
	Calc	88.8	288.9	586	966	1413	1916		
^{158}Gd	Expt[b]	79.56	261.9	539	898				
	Calc	79.56	261.7	539	901				
^{160}Gd	Expt[c]	75.3	247	509	(863)				
	Calc	75.0	247	511	859				
^{156}Dy	Expt[b]	138	403	766	1212				
	Calc	136	400	764	1220				
^{158}Dy	Expt[d,b]	98.7	316.5	636	1041	1514	(2037)		
	Calc	98.4	317.2	637	1040	1509	2031		
^{160}Dy	Expt[b,e]	86.7	284	582	972	1442	1977	2602	
	Calc	86.4	283.8	583	974	1443	1980	2572	
^{162}Dy	Expt[c]	80.7	265.9	549	924				
	Calc	80.7	265.9	549	921				
^{164}Dy	Expt[c]	73.4	242.2	501	839				
	Calc	73.5	242.2	500	840				
^{160}Er	Expt[b]	127	394	765	1228	1758			
	Calc	126	394	768	1227	1758			
^{162}Er	Expt[b]	101	327	662	1090	1595			
	Calc	100.7	327	663	1091	1594			
^{164}Er	Expt[d]	91.3	299.2	614	1024	(1534)			
	Calc	91.3	299.5	615	1024	1515			
^{166}Er	Expt[a]	80.6	264.9	545	910	1334			
	Calc	80.6	264.7	544	908	1344			
^{168}Er	Expt[c]	79.8	264	549					
	Calc	79.8	264	549					
^{170}Er	Expt[c]	79	261	542					
	Calc	79	261	542					
^{164}Yb	Expt[f]	122.5	384.0	758	1219	1748			
	Calc	122.1	385.2	758	1217	1747			
^{166}Yb	Expt[f]	101.8	329.7	667	1097	1604	2172		
	Calc	101.5	329.7	668	1098	1604	2172		
^{168}Yb	Expt[d]	87.9	286.9	586	970.4	(1439)			
	Calc	87.8	286.9	586	970.6	1428			
^{170}Yb	Expt[b]	84.2	277.7	572	962	1439	(1986)		
	Calc	84.3	277.8	573	962	1433	1975		
^{172}Yb	Expt[b]	78.7	260.3	540	910	1352			
	Calc	78.7	260.2	540	910	1364			
^{174}Yb	Expt[c]	76.5	252	527	892				
	Calc	76.5	253	527	892				
^{176}	Expt[c]	82.1	270	564	947				
	Calc	81.8	270	561	946				
^{168}Hf	Expt[f]	123.9	385.0	756	1212	1734	2304		
	Calc	123.7	386.7	755	1207	1730	2312		
^{170}Hf	Expt[f]	100.0	320.6	641	1041	1503	2013	2564	3147
	Calc	100.0	320.5	640	1038	1500	2015	2570	3154
^{172}Hf	Expt[f]	94.5	307.9	627	1036	1519	2063	2651	
	Calc	94.5	308.1	627	1036	1519	2063	2656	
^{174}Hf	Expt[d]	91.0	297.6	609	1010	(1502)			
	Calc	90.6	297.6	609	1012	1492			
^{176}Hf	Expt[b]	88.4	290.3	596	998	(1492)			
	Calc	88.4	290.5	597	998	1479			
^{178}Hf	Expt[c]	93.2	306.6	632	1058				
	Calc	93.2	306.7	632	1058				
^{180}Hf	Expt[b]	93.3	308.6	641	1085				
	Calc	93.3	308.8	642	1085				
^{174}W	Expt[f]	111.9	355.0	704	1137	1635	2186		
	Calc	111.9	355.6	704	1135	1634	2188		

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TABLE II. (*Continued*)

Nucleus		2^+	4^+	6^+	8^+	10^+	12^+	14^+	16^+
^{176}W	Expt[f]	108.7	348.5	699	1140	1648	2206		
	Calc	108.7	349.4	699	1138	1647	2214		
^{178}W	Expt[g]	105±5	342	697	1152	1679	2264	2894±30	
	Calc	106	344	697	1145	1671	2261	2901	
^{180}W	Expt[d]	103.6	337.6	688	1139	(1691)			
	Calc	103.3	337.6	689	1142	1680			
^{182}W	Expt[e]	100.1	329.4	680	1137±10	1645±20			
	Calc	100.6	330.0	677	1127	1667			
^{184}W	Expt[e]	111.2	364.0	748					
	Calc	111.0	364.3	748					
^{186}W	Expt[e]	122.5	399	818					
	Calc	122.0	399	817					
^{182}Os	Expt[h]	126.9	400.2	794	1277	1810			
	Calc	126.9	401.2	791	1271	1826			
^{184}Os	Expt[i]	119.8	383.6	774	1274	1871			
	Calc	118.0	383.4	777	1278	1867			
^{186}Os	Expt[h]	137.2	433.9	869	1421	2068			
	Calc	134.5	433.9	872	1424	2067			
^{188}Os	Expt[i]	155.0	477.9	940	1514				
	Calc	152.0	478.8	941	1509				
^{190}Os	Expt[j]	186.7	547.9	1050	1667				
	Calc	186.7	547.8	1047	1674				

5. COMPARISON WITH OTHER CALCULATIONS

The best agreement to date for the high-spin states has been obtained with the semiclassical treatment[7,9] of centrifugal stretching within the framework of the Davydov-Chaban (DC) model.[10] Comparison between their results and our calculations relative to the experimental energies for the nucleus ^{170}Hf is presented in Fig. 3. Also included in this figure are the predictions of the adiabatic Bohr-Mottelson theory with first- and second-order corrections, all the three results being in striking disagreement with the experiment. Our results and those of Stephens *et al.*[7] (SLD) exhibit just about the same degree of fit to the experimental data, ours being somewhat better.

A more sensitive criterion for comparison has been used by SLD wherein they remove the general $I(I+1)$ energy dependence of the levels by defining the transitional rotational constant A_I as follows:

$$A_I=\Delta E(I\rightarrow I-2)/(4I-2), \qquad (12)$$

and the scale parameter is eliminated by taking the ratios of the adjacent rotational constants. These ratios A_{I+2}/A_I are plotted against the intermediate spin I. A comparison of our results with those of Stephens *et al.*[7] (labelled DC in the figure) is shown in Fig. 5 for the transitions in the nucleus ^{172}Hf; the experimental points are shown by solid circles. The better agreement with our predictions is quite evident; whereas the DC curve shows a reversal of slope for higher spins, our curve follows the experimental trend rather well. The advantage of our approach over theirs is also revealed in that they did not consider their calculations suitable for Os isotopes,[11] whereas we find good agreement for these nuclei as well.

Another formulation, which has met with comparable success for the set of nuclei studied by Stephens *et al.*,[7] was suggested by Harris[12] as an extension of the crank-

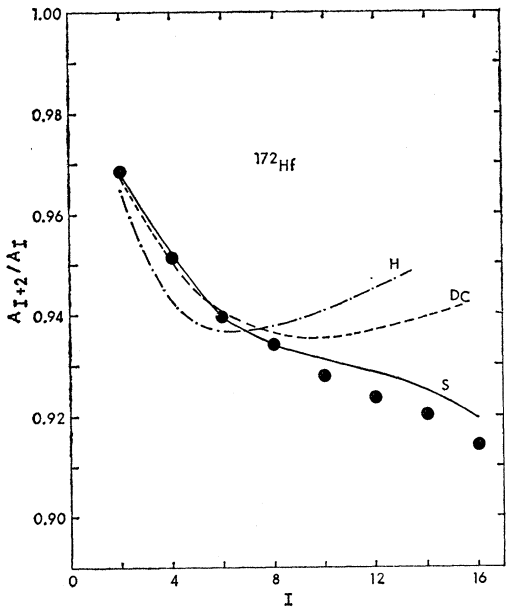

FIG. 5. The solid circles represent the experimental ratios of successive rotational constants A_{I+2}/A_I for the nucleus ^{172}Hf plotted against the intermediate spin I. The solid line, labeled S, is from our calculations, the dashed line labeled DC from Stephens *et al.* (Ref. 7) and the dot-dashed line, labeled H, is from Harris (Ref. 12).

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ing model; he calculates the rotational energies through the equations

$$E=\tfrac{1}{2}\omega^2(\mathcal{I}_0+3C\omega^2+5D\omega^4+\cdots),$$
$$[I(I+1)]^{1/2}=\omega[\mathcal{I}_0+2C\omega^2+3D\omega^4+\cdots]. \quad (13)$$

In principle, the nuclear angular velocity ω may be eliminated from these equations to yield E as a function of I. In the two-parameter description, a least-squares fit to the observed energies is made with Eq. (13) retaining terms in $\mathcal{I}_0$ and C only. In Fig. 5 we have shown the two-parameter results of Harris[12] for comparison with our calculations, and the better fit with the latter is quite obvious.

Moszkowski[13] has suggested a two-parameter description of rotational energies on the basis of classical hydrodynamical model; he also obtains a set of parametric equations

$$2E_0\mathcal{I}_0=DV(1+V)/(1-V)^2,$$
$$I(I+1)=DV/(1-V)^4, \quad (14)$$

where V is a function of nuclear angular velocity ω and may, in principle, be eliminated to give energy as a function of I, in terms of the two parameters $\mathcal{I}_0$ and D. He finds a very good fit to ^{170}Hf and other neighboring nuclei, but elsewhere the predicted values are not very satisfactory. For example, in the case of ^{152}Sm the 10^+ level is predicted at 1492 keV (experimental 1608 keV) and the 12^+ level at 1932 keV(experimental 2158 keV).

Another extension of the Bohr-Mottelson theory[1] is based on the careful inclusion of the rotation vibration interaction in the rotation vibration (RV) model.[14] As mentioned earlier, the deviations from the $I(I+1)$ law are particularly apparent, even for low-spin states, in the transitional nuclei, e.g., Os isotopes. The successes of Davydov and collaborators[15] in the transition regions pointed to the superiority of their asymmetric rotator (AR) model. An extensive comparison of the RV and AR models has been presented by Faessler *et al.*[14] indicating an approximate equivalence of the two theories in explaining the three lowest rotational bands. Our formulation aims at the description of the levels in the ground-state rotational bands only and in Figs. 6 and 7 we present a comparison of our results for two extreme-ends nuclei (in our Table II), ^{152}Sm and ^{190}Os, with the predictions of the AR and RV models.

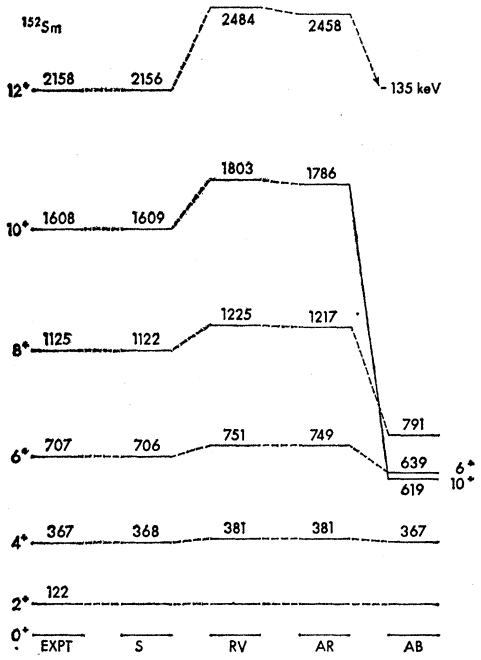

FIG. 6. The observed (the first column on the left) and the predicted energy levels for the nucleus ^{152}Sm (with $N=90$). The column labelled S is from our calculations, the next two labelled RV for rotation-vibration model and AR for asymmetric rotator model are from Faessler *et al.* (Ref. 14) and the last one, labeled AB, corresponds to the results of conventional two-parameter calculations.

In Fig. 6 we have also included the predicted level scheme from the conventional two-parameter AB model. The Os isotopes and $N=90$ nuclei provide a testing ground for models of deformed nuclei, and our simple formula shows a surprisingly good agreement with the experimental data even for these cases.

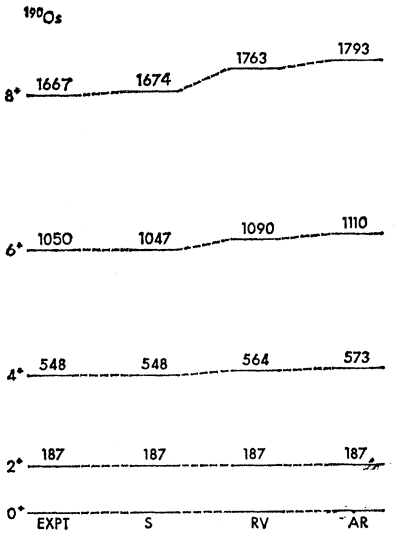

FIG. 7. The plot similar to Fig. 6 for the nucleus ^{190}Os.

[13] S. A. Moszkowski, in *Proceedings of the Nuclear Spin-Parity Assignments*, edited by N. B. Gove and R. L. Robinson (Academic Press Inc., New York, 1966), p. 429; Brookhaven National Laboratory Report No. BNL 948 (C-46), 1965, p. 381 (unpublished).

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6. SUMMARY AND CONCLUSIONS

By summing an infinite power series in $I(I+1)$ under reasonable assumptions about the relative magnitudes (obtained from the molecular spectra theory) of the successive coefficients in the series, we have obtained a very simple analytical expression, Eq. (9), for the energies of the levels in the ground-state rotational bands of even even nuclei. The expression explicitly re-establishes the $I(I+1)$ law for the description of the rotational energies and provides an analytical relation for the variation of the nuclear moment of inertia with the nuclear spin [Eq. (10)].

A detailed comparison of our predictions with the known energy levels of deformed even even nuclei in the rare-earth region, as given in Table II, reveals a surprisingly good agreement in all cases (except for a few highly neutron-deficient nuclei). Comparison has also been given with the results of the calculations of Stephens *et al.*[7] (Davydov-Chaban model), Harris[12] (cranking model with corrections), Moszkowski[13] (classical hydrodynamical model), rotation-vibration model,[14] and the asymmetric rotator model,[15] in addition to the conventional Bohr-Mottelson model.[1] Our model is shown to give a better description of the rotational energies than any of the above models. It is rather remarkable that this simple formula also gives correct energies for $N=90$ nuclei which had not been satisfactorily described by any model proposed so far.

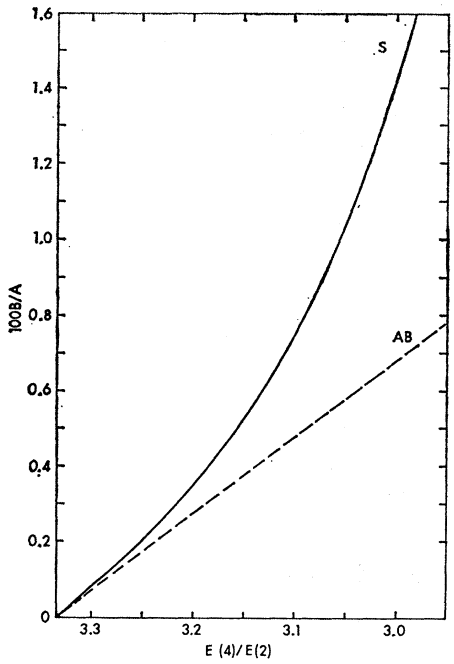

FIG. 8. The rotation-vibration coefficient B plotted versus the energy ratio $E(4)/E(2)$ as a percentage of the rotational parameter A. The dotted line, labeled AB, corresponds to the values derived from the conventional two-parameter description and solid line, labeled S, gives the results of our calculations using Eq. (9).

The expressions developed here are as yet on a semiempirical basis, but provide a few guidelines for developing a theory for rotational energies. Of particular interest is the predicted spin dependence of the nuclear moment of inertia for comparison with an appropriate theory.[16] Also the formulation presented above gives a reliable 'experimental' measure of the rotation vibration coefficient B for comparison with the existing and forthcoming theories[16,17] of this effect. It is seen from Fig. 8 that the value of B obtained from our formulation (solid curve) is appreciably higher than the value obtained from the conventional two-parameter calculations, the change being in the right direction and approximately of the right order to explain earlier anomalies.[2]

It may be remembered that Eq. (9) gives the basic formula of our approach. The expression for N given in Eq. (11) is by no means unique; it is expected that this semiempirical 'constant' will be determined more accurately as more and better data on high-spin states becomes available for a wide range and variety of nuclei.

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