TABLE II. Operators Ω_k for reduced nuclear matrix in Eq. (A1). The spherical Bessel functions $j_L(qr_k)$ are abbreviated by j_L.

Matrix element	Ω_k
$[0\ L\ J]$	$j_L \mathcal{Y}_{0LJ}^M(\hat{r}_k)\delta_{LJ}$
$[1\ L\ J]$	$j_L \mathcal{Y}_{1LJ}^M(\hat{r}_k,\boldsymbol{\sigma}_k)$
$[0\ L\ J\ p]$	$ij_L \mathcal{Y}_{0LJ}^M(\hat{r}_k)\boldsymbol{\sigma}_k\cdot \mathbf{p}_k\delta_{LJ}$
$[1\ L\ J\ p]$	$ij_L \mathcal{Y}_{1LJ}^M(\hat{r}_k,\mathbf{p}_k)$

A similar relation holds for $[S\ L\ J\ p]$. Here $u_{J_i}^{M_i}$ and $u_{J_f}^{M_f}$ are nuclear wave functions of the initial and final states specified by the spin and its projection M. $\tau_-^{(k)}$ and Ω_k are the isospin and the operator for the kth nucleon. S, L, and J in brackets are the resultant spin, the effective orbital angular momentum, and the resultant total angular momentum of the lepton system, respectively. J also specifies the rank of the matrix elements. The symbol p means that the relevant matrix element includes the differential operator $\mathbf{p}$ acting on the nuclear wave function. The parity change is given by $(-)^L$ for $[S\ L\ J]$, and $(-)^{L+1}$ for $[S\ L\ J\ p]$. Nucleon operators Ω_k in Eq. (A1) are summarized in Table II, where the vector harmonics are defined by

$$\mathcal{Y}_{SLJ}^M(\hat{r},\boldsymbol{\sigma})=\sum_m (S\,L\,m\,M-m|J\,M)Y_{L\,M-m}(\theta,\varphi)\times\mathcal{Y}_{Sm}(\boldsymbol{\sigma}),\quad \text{(A2)}$$

with

$$\mathcal{Y}_{00}(\boldsymbol{\sigma})=(1/4\pi)^{1/2},$$

$$\mathcal{Y}_{10}(\boldsymbol{\sigma})=(3/4\pi)^{1/2}\sigma_z,\ \text{etc.}$$

$\mathcal{Y}_{SLJ}^M(\hat{r},\mathbf{p})$ has a similar expression.

PHYSICAL REVIEW VOLUME 161, NUMBER 4 20 SEPTEMBER 1967

Stretch Scheme, a Shell-Model Description of Deformed Nuclei

MICHAEL DANOS AND VINCENT GILLET
Service de Physique Théorique, Centre d'Etudes Nucléaires de Saclay, Gif-sur-Yvett, Seine et Oise, France
and
National Bureau of Standards, Washington, D. C.

(Received 23 March 1967)

A good angular-momentum wave function containing the maximum possible intrinsic angular momenta leads to a microscopic description of the nuclear rotational spectra in terms of spherical shell-model states. The rotational excitation energies arise from the residual two-body force. In the actual model calculations, the only approximation was a partial violation of the exclusion principle. The computed departures from the $I(I+1)$ law are consistent with experiment. Reasons are given for the preference of positive over negative intrinsic deformations.

I. THE STRETCH SCHEME

THE collective model has been extremely successful in giving a phenomenological description of the characteristics of deformed nuclei in terms of macroscopic coordinates associated with the degrees of freedom of an average well.[1] However, it is an outstanding problem of nuclear physics to understand deformed nuclei in terms of the nucleon motion and the two-body force. Up to now, this problem has been attacked in essentially three ways. The first is an attempt to solve the complete nuclear Hamiltonian approximately in as large as possible a configuration space. This method is practicable only for very light nuclei and calculations of this kind have been limited to the $1p$ shell.[2] The second consists in solving exactly an approximate Hamiltonian. The problem here is to obtain a good guess of a simple nuclear Hamiltonian which still is supposed to represent the essential nuclear features. In order to be exactly solvable, this Hamiltonian must be invariant under the symmetry transformation of some groups. So, for example, Elliott's model[3,4] is a nuclear Hamiltonian, invariant under $SU(3)$. The exact solutions are then representations of $SU(3)$. However, to obtain this result, the two-body force must be replaced by a separable, spin- and isospin-independent quadrupole force, and the average field must be an harmonic-oscillator potential. In the third method, the deformed orbital method,[5] one applies the variational principle to a trial wave function which is simple but violates rotational invariance. This way one treats an important part of the nucleon-nucleon interaction, namely the average field effect. Good angular-momentum states then must be obtained by projection. All these methods lead to spectra with rotational features. However, in all these approaches, the

---

[1] A. Faessler, W. Greiner, and R. K. Sheline, Nucl. Phys. **70**, 33 (1965).

[2] D. Kurath, in *Alpha-, Beta-, and Gamma-Ray Spectroscopy*, edited by Kai Siegbahn (North-Holland Publishing Company, Amsterdam, 1965), p. 583.

[3] J. P. Elliott, Proc. Roy. Soc. (London) **245**, 128 (1958); **245**, 562 (1958).

[4] H. J. Lipkin, Nucl. Phys. **26**, 147 (1961).

[5] G. Ripka, The Hartree-Fock Theory and Nuclear Deformations (to be published).

link between the collective rotational motion and the individual particle motion is, at least to us, not very transparent.

In the present paper we propose a wave function, the stretch wave function, which is a single configuration, and has the following properties: (i) It is a good angular-momentum state; (ii) it has a large binding energy and is stable against distortion; (iii) it has an intrinsic quadrupole moment; and (iv) its simplest excitations correspond to a rotational spectrum. The principal merit of the stretch wave function is to give a very simple and direct physical picture of nuclear "rotations" in terms of the nucleon motion and the two-body force. Furthermore it shows that the existence of rotations depends essentially on the proton-neutron correlations.

Two nucleons in a same shell have maximum binding energy when the angular overlap of their wave functions is maximum. For two identical nucleons this is obtained by coupling them to angular momentum $J=0$ (see Fig. 1). Breaking the pair and coupling to $J=2$ requires a large energy as a result of the large loss of overlap. This effect leads to the pairing wave function, which is spherically symmetric. It has the largest possible number of pairs coupled to $J=0$. It is stable against distortion in that it requires a large energy gap for the breaking of a pair. Its "elementary excitations" thus are the two-quasiparticle states. The pairing wave function has proven to be a reasonably good approximation for the ground state of even single closed-shell nuclei.

On the other hand, a proton and a neutron have a large binding energy both when they are paired ($J=0$) and when they are aligned, i.e., coupled to the maximum possible angular momentum $J_{\max}=2j$ (Fig. 1). For like particles, the latter coupling is forbidden by the Pauli principle. Experimentally this is reflected in the spectra of nuclei with two valence nucleons. For like nucleons, e.g., ^{18}O, ^{42}Ca, etc., the lower two-particle spectrum is always 0^+, 2^+, 4^+, etc. On the other hand, for unlike nucleons, e.g., ^{14}N, ^{18}F, ^{42}Sc, etc., there is competition in the spectrum between the $J=0$ and the $J=J_{\max}$ states. This suggests that in nuclei with open shells for both protons and neutrons, a description of the ground state should include the maximum number of aligned proton-neutron pairs. This consideration leads to the stretch wave function.

We shall consider for simplicity (and limited ability on our part) the simplest case of $2N$ protons in a shell with angular momentum j and of $2N$ neutrons in a shell with angular momentum k. Still, this model contains all the relevant physical features. The stretch wave function is then constructed in the following way (see Fig. 2): First, N protons and N neutrons are coupled to the maximum possible angular momentum, say C, which is allowed by the Pauli principle. We shall call this system a "chain." The wave function of a chain is unique. Therefore, as illustrated in Fig. 2, whatever be the apparent coupling order of the nucleons, the chain contains in fact the maximum possible number of aligned proton-neutron pairs. Next, the remaining N protons and N neutrons are coupled to a similar chain. Finally, the two chains are coupled to a state with total angular

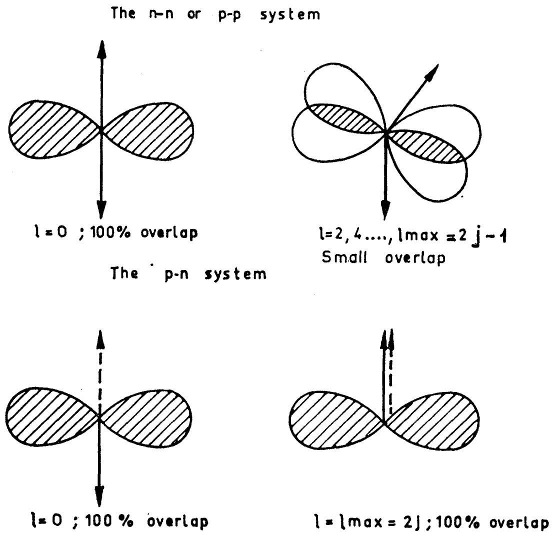

FIG. 1. The overlap of the single-particle wave functions for a two-nucleon system in a same shell. For a short-range force, the binding energy is large only for $I=0$ for identical nucleons. For nonidentical nucleons it is large also for $I=2j$.

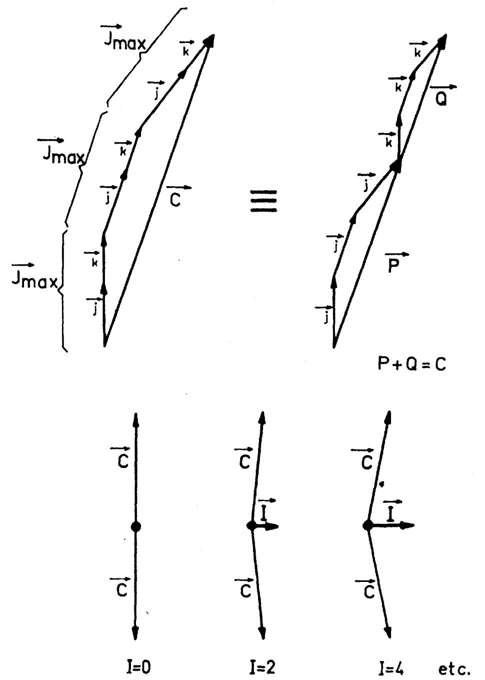

FIG. 2. Top: the structure of a chain. The coupling of aligned pairs to maximum angular momentum C is unique: both coupling schemes shown are identical. Bottom: the stretch state and its rotational excitations. In the rotational states with $I\neq 0$, the effective overlap between the wave functions of the two chains decreases with increasing I in that more and more across-the-chain pairs are broken.

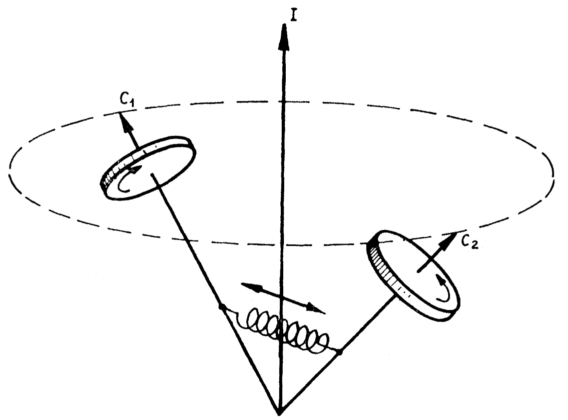

FIG. 3. A classical model for the nuclear rotations. The two chains C behave like two gyroscopes with high angular momentum. The interaction between the two chains which tends to align them is represented by the spring. The torque exerted on the gyroscopes induces a precessional motion of the whole system. The angular velocity of the precession ω_P is proportional to the magnitude of the torque, which, in turn, depends on the deviation from the alignment, hence on I. In a linear system the potential energy of the system, i.e., the compressional energy in the spring V is proportional to I^2. If $I/C \ll 1$, and if V is small as compared to the kinetic energy of the gyroscopes, a "moment of inertia" J_0 can be defined by the equation $E_I = \frac{1}{2} J_0 \omega_P{}^2$ or, equivalently, $E_I = \frac{1}{2} I^2 / J_0$. No "moment of inertia" can be meaningfully defined for nonlinear systems, e.g., if $V \propto I^4$.

momentum $I=0$. This way the overlap between the wave functions of the two chains is maximum. The stretch wave function thus contains correlations between all nucleons.

We now give a more detailed discussion of the properties of the stretch wave function. The already-mentioned stability against distortion has the following meaning. Any other 0^+ state which could be generated from the stretch wave function requires a rather large excitation energy since, to this end, at least two aligned pairs must be broken. By construction, the stretch state is spherically symmetric. The expectation values of all multipole moment operators vanish; still it has large dynamic moments. Thus, the expectation value of the square of, e.g., the quadrupole moment operator $Q_0{}^2$, is large.

Finally, the stretch wave function exhibits two kinds of "elementary excitations," the "rotational" and the "intrinsic" ones. The rotational excitations are obtained by coupling the two chains to angular momenta $I \neq 0$ (see Fig. 2). The intrinsic excitations are obtained by breaking a chain and coupling to an angular momentum $C' < C$. The excitation energy for both these excitations results from the loss of overlap in the single nucleon wave functions.

We now discuss the properties of the rotational excitations. Since the angular momentum arises only from a change in the coupling of the individual particle angular momenta, the kinetic energy of the system remains unchanged; the excitation energy results purely from a change of the potential energy of the system. Still, these excitations deserve the name of rotational states. Namely, in states $I \neq 0$ the angular-momentum vectors $\mathbf{C}_1$ and $\mathbf{C}_2$ of the two chains precess about the direction of $\mathbf{I}$. This precessional motion is in fact equivalent to the motion envisaged in the collective model, since the axis of symmetry of the matter distribution is associated with the direction $\mathbf{C}_1$ and $\mathbf{C}_2$ and rotates about the direction $\mathbf{I}$. The angular velocity of the precession is related to the excitation energies of the rotational states, and increases with increasing I. (This point is illustrated in term of a classical model in Fig. 3.) However, the precession is purely kinematical. No dynamical effects like RPC (Coriolis force) have to be added since the problem is treated in the laboratory system; the treatment is completely rotationally invariant. In order to change the kinetic energy of the system, one has to admix configurations in which some nucleons have been elevated into higher shells, i.e., one has to "break" the shells of the core. This effect has to be expected to take place in real nuclei. Then the kinetic energy of the rotational states can increase with increasing I in that the admixture amplitudes of the different configurations can be I-dependent. However, we are not going to consider configuration mixing in this paper and continue the discussion of the properties of a single stretch configuration.

The wave function of the rotational states are of the form

$$\tilde{\Psi}_0{}^{[I]} = \mathcal{A} \sum_M (C\,C\,M\ {-M} | I\,O) \Psi_M{}^{[C]} \Psi_{-M}{}^{[C]} = \mathcal{A} \Psi_0{}^{[I]}.$$

They must be symmetric under the exchange of the two chains, because of their indistinguishability. Since the phase of the vector coupling coefficients under exchange is $(-)^{2C-I}$, the angular momenta I must be even. The antisymmetrization operator has no bearing on this symmetry.

The energies of the rotational states will be calculated in Sec. III. They can be brought into the form

$$E_I = A + \sum_{\substack{\alpha\beta\gamma \\ \alpha'\beta'\gamma'}} \begin{Bmatrix} C & C & I \\ \alpha & \beta & \gamma \end{Bmatrix} \begin{Bmatrix} C & C & I \\ \alpha' & \beta' & \gamma' \end{Bmatrix} g(\alpha\beta\gamma\alpha'\beta'\gamma').$$

Here A is the contribution to the energy from nucleons in the same chain, while the sum arises from the interaction between nucleons of the two chains. In other words $\frac{1}{2}A$ is the self-energy of each chain, while the second term represents an effective chain-chain interaction energy. Asymptotically for large C and small I, E_I becomes[6]

$$E_I = E_0 + I(I+1)E_R.$$

In this formula higher terms in $[I(I+1)/C(C+1)]$ have been dropped.

If one considers a chain to be representable by an effective particle, then the interaction energy between

---

[6] U. Fano and G. Racah, *Irreducible Tensorial Sets* (Academic Press Inc., New York, 1958), Chap. XVI.

the chains would be of the form

$$V_{12}\approx\sum_k \begin{Bmatrix} C & C & I \\ C & C & k \end{Bmatrix} \langle C\|Y^{(k)}\|C\rangle R_k ,$$

which for large C as compared to I gives again the $I(I+1)$ law.[7] This law is a simple expression of the fact that the interaction energy between the chains must be proportional to the overlap between the chain wave functions, i.e., to $\mathbf{C}_1\cdot\mathbf{C}_2$. In terms of the angle α between C_1 and C_2, the interaction energy is proportional to

$$E\propto \cos\alpha\approx 1-\tfrac{1}{2}\alpha^2\approx 1-I^2/2C^2 .$$

The $I(I+1)$ rule is evidently not exact and, in fact, both the detailed calculations and the experiments show quite large deviations from this rule.

The intrinsic excitations, i.e., those states which result from breaking a single chain to angular momentum $C'<C$, are not unique for a given total angular momentum I. A description of the intrinsic excited states thus requires a treatment of the configuration mixing between these excitations. Let us note that the two chains can again be coupled to $I_0=C-C'$, I_0+1, I_0+2, etc. The previous selection rule limiting I to even numbers does not apply any more since the chains are not identical.

The stretch rotational states are related to the Bohr-Mottelson aligned scheme.[8] This scheme consists in constructing a Slater determinant where the single-particle m states are filled in the order $\pm j$, $\pm(j-1)$, $\pm(j-2)$, etc. leading to a negative intrinsic deformation, or in the order $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{5}{2}$, etc. leading to a positive deformation. The aligned state does not have a good angular momentum. If we construct two such determinants for the $2N$ protons and the $2N$ neutrons, respectively, with the filling order $\pm j$, $\pm(j-1)$, $\pm(j-2)$, etc., we obtain the identity[9]

$$\Phi_{\text{aligned}}(p)\Phi_{\text{aligned}}(n)\approx\Psi_C{}^{[C]}\Psi_{-C}{}^{[C]} = \sum_I (C\,C\,C\,-C|I0)\Psi_0{}^{[I]} ,$$

where the approximate equal sign has to be employed instead of the equal sign because the right-hand side is not antisymmetric between $\Psi_C{}^{[C]}$ and $\Psi_{-C}{}^{[C]}$. Since the particles in these both functions fill different m states, the antisymmetrization is here trivial. The stretch rotational wave functions thus correspond to the angular-momentum-projected products of two aligned-scheme wave functions; they are states with good angular momentum. Furthermore, they incorporate proton-neutron correlations which we believe to be the essential ingredient for the appearance of deformations, in accord with experimental observation. Finally, from the above correspondence with the aligned-scheme wave function one sees that the stretch wave functions built up with holes have positive intrinsic deformations while those built with particles have negative intrinsic deformations.

II. THE STRETCH WAVE FUNCTION

The wave function of a single chain of maximum angular momentum C in its maximum M state, i.e., $M=C$, is given uniquely by a product of two Slater determinants, viz.,

$$\Phi_C{}^{[C]}=\Phi_P{}^{[P]}(p)\Phi_Q{}^{[Q]}(n) . \tag{1}$$

The two Slater determinants, viz.,

$$\Phi_P{}^{[P]}(p)=\frac{1}{(N!)^{1/2}}\begin{vmatrix} \varphi_j(1) & \varphi_j(2) & \cdots & \varphi_j(N) \\ \varphi_{j-1}(1) & \cdots & \cdots & \cdots \\ \vdots & & & \\ \varphi_{j-N+1}(1) & & & \end{vmatrix} \tag{2}$$

and similarly $\Phi_Q{}^{[Q]}$, are good angular-momentum states since P and Q are the maximum possible angular momenta of the N protons in the j shell and of the N neutrons in the k shell. They are given by (see Fig. 2)

$$P=Nj-\tfrac{1}{2}N(N-1),\quad Q=Nk-\tfrac{1}{2}N(N-1) .$$

The total angular momentum C is similarly given by

$$C=P+Q=NJ_{\max}-N(N-1) ,$$
$$J_{\max}=j+k .$$

An arbitrary M state then can be obtained from (1) by applying the M-lowering operator a sufficient number of times. This wave function is then also antisymmetric for protons and neutrons amongst themselves, respectively.

The stretch wave function then would be

$$\tilde{\Psi}^{[I]}=\sum_{LL'}\begin{bmatrix} P & Q & C \\ P & Q & C \\ L & L' & I \end{bmatrix} \times[\mathfrak{A}[\Phi^{[P]}\times\Phi^{[P]}]^{[L]}\times\mathfrak{A}[\Phi^{[Q]}\times\Phi^{[Q]}]^{[L']}]^{[I]} , \tag{3}$$

where we have written for the recoupling coefficient the abbreviation defined in term of the 9-j coefficients

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}=[(2G+1)(2H+1)(2C+1)(2F+1)]^{1/2} \times\begin{Bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{Bmatrix} . \tag{4}$$

The only approximation that we shall make in computing the energies for the wave functions is the neglect

---

[7] This observation is due to U. Fano, who participated in the preliminary discussions in 1958.

[8] B. Mottelson, in *Proceedings of the International School of Physics—"Enrico Fermi" Course XV* (Academic Press Inc., New York, 1960), p. 45.

[9] This relationship was pointed out to us by C. Levinson.

of the antisymmetrization between the two chains. Namely, we shall from now on use the approximate expression

$$\Psi^{(I)}=\sum_{LL'}\begin{Bmatrix}P & Q & C\\ P & Q & C\\ L & L' & I\end{Bmatrix}\times[[\Phi^{[P]}\times\Phi^{[P]}]^{[L]}\times[\Phi^{[Q]}\times\Phi^{[Q]}]^{[L']}]^{[I]}. \quad (5)$$

We shall, however, use antisymmetrized matrix elements in the calculation of the potential energy. Thus two interacting particles from different chains are antisymmetrized. The only error arises from the lack of antisymmetrization of two noninteracting particles from two different chains. Thus, this error is of the same nature as the one committed in the quasiboson approximation. The magnitude of this error can be assessed as follows: When vector coupling the two chains to angular momentum I, beginning with some value M_C, some m values will be occupied by two like particles. For large C, and for N small compared to the degeneracy of the shell, M_C is much smaller than C. More precisely, let us denote by $\Omega_j=2j+1$ and $\Omega_k=2k+1$ the degeneracies of the shells. Let Ω be the smaller of the two. Then errors will appear for values of M_C such that

$$M_C\leq C-\nu, \quad (6a)$$

with

$$\nu=\tfrac{1}{2}\Omega-N+1. \quad (6b)$$

This is evident when considering the Slater determinants Φ. For large M values, only such φ_m participate where the m's have all the same sign. When applying the M-lowering operator ν times to Φ, Eq. (2), in the resulting linear combination of Slater determinants exactly one of the Slater determinants will have one row of $\varphi_{-1/2}$. Similarly, starting from the "mirror" Slater determinant of Eq. (2) with all negative m values, only one among the generated Slater determinants after ν raising operations will have a row of $\varphi_{+1/2}$. Therefore, the states $m=\pm\frac{1}{2}$ will be filled twice in some of the terms of the sum which arises in the process of vector coupling. Such occurrences will increase in number with increasing ν.

Returning now to considering both proton and neutron states, one sees that the maximum number of double fillings will occur at $\nu=\nu_p+\nu_n=C$. The relative importance of these double fillings is smallest for the state $I=0$ and increases with increasing I.

In summary, the order of magnitude of the error will be at most $4N/(\Omega_j+\Omega_k)$, multiplied by a factor less than unity since each chain is antisymmetrized by itself. The exact determination of this factor has to await a more complete treatment.

III. THE STRETCH TWO-BODY MATRIX ELEMENT

In the calculation of the energy two classes of terms appear, viz., terms in which two like particles interact, and terms in which two unlike particles interact. Because of the nonantisymmetrization between the two chains, the first class will split again into two different kinds of terms, viz., those where the two interacting particles are in the same chain and those where they are in different chains.

The expansion of the wave function

$$\Psi^{[I]}=[[[j^N]^{[P]}\times[k^N]^{[Q]}]^{[C]}\times[[j^N]^{[P]}\times[k^N]^{[Q]}]^{[C]}]^{[I]}, \quad (7)$$

thus, written in a very symbolical fashion, is of the form

$$\Psi^{[I]}=\sum[[2]^{[J]}\times[4N-2]^{[T]}]^{[I]}, \quad (8)$$

or, in greater detail,

$$\begin{aligned}\Psi^{[I]}=&\sum_{JRST}[[j^2]^{[J]}\times[[[j^{N-2}]^{[R]}\times[k^N]^{[Q]}]^{[S]}\times[[j^N]^{[P]}\times[k^N]^{[Q]}]^{[C]}]^{[T]}]^{[I]}\\ &\sum_{JR\tilde{R}ST}[[j\times k]^{[J]}\times[[[j^{N-1}]^{[R]}\times[k^{N-1}]^{[\tilde{R}]}]^{[S]}\times[[j^N]^{[P]}\times[k^N]^{[Q]}]^{[C]}]^{[T]}]^{[I]}\\ &+\sum_{JR\tilde{R}S\tilde{S}T}[[j^2]^{[J]}\times[[[j^{N-1}]^{[R]}\times[k^N]^{[Q]}]^{[S]}\times[[j^{N-1}]^{[\tilde{R}]}\times[k^N]^{[Q]}]^{[\tilde{S}]}]^{[T]}]^{[I]}\\ &+\sum_{JR\tilde{R}S\tilde{S}T}[[j\times k]^{[J]}\times[[[j^{N-1}]^{[R]}\times[k^N]^{[Q]}]^{[S]}\times[[j^N]^{[P]}\times[k^{N-1}]^{[\tilde{R}]}]^{[\tilde{S}]}]^{[T]}]^{[I]}\\ &+\sum_{JRST}[[k^2]^{[J]}\times[[[j^N]^{[P]}\times[k^{N-2}]^{[R]}]^{[S]}\times[[j^N]^{[P]}\times[k^N]^{[Q]}]^{[C]}]^{[T]}]^{[I]}\\ &+\sum_{JR\tilde{R}S\tilde{S}T}[[k^2]^{[J]}\times[[[j^N]^{[P]}\times[k^{N-1}]^{[R]}]^{[S]}\times[[j^N]^{[P]}[k^{N-1}]^{[\tilde{R}]}]^{[\tilde{S}]}]^{[T]}]^{[I]}\\ &\equiv\Sigma(1)+\Sigma(2)+\Sigma(3)+\Sigma(4)+\Sigma(5)+\Sigma(6). \quad (9)\end{aligned}$$

Equation (8) is symbolic in that all expansion coefficients, viz., the recoupling and fractional parentage coefficients, as well as the names of the particles and the multiplicities associated with the symmetry between the two chains, have been suppressed.

The matrix element of the two-body force $\langle\Psi^{[I]}|V|$

$\Psi^{[I]}\rangle$ is a sum of terms. Because of the orthogonality of the $4N-2$ part of the wave function, i.e., the noninteracting part, the sum contains only terms with the same $[4N-2]$ part on both sides of the operator. Thus, the matrix element will automatically group into the same partial sums as indicated in Eq. (9), viz.,

$$\langle\Psi^{[I]}|V|\Psi^{[I]}\rangle=\sum_i\langle\Sigma(i)|V|\Sigma(i)\rangle\equiv\sum_i V_i. \quad (10)$$

The single and double fractional parentage coefficients, which we shall abbreviate as SCFP and DCFP, respectively, are defined by

$$\begin{aligned}&|(1,2,\cdots,N)j^NP\}\\&\quad=\sum_{\omega,L}[|(1,2,\cdots,N-1)j^{N-1}\omega L\}\times|(N)j\rangle]^{[P]}\\&\qquad\qquad\times\langle j^{N-1}\omega L;j\rrbracket j^NP\rangle, \quad (11)\end{aligned}$$

$$\begin{aligned}&|(1,2,\cdots,N)j^NP\}\\&\quad=\sum_{\omega,L,J}[|(1,2,\cdots,N-2)j^{N-2}\omega L\}\\&\qquad\times|(N-1,N)j^2J\}]^{[P]}\\&\qquad\qquad\times\langle j^{N-2}\omega L;j^2J\rrbracket\rrbracket j^NP\rangle, \quad (12)\end{aligned}$$

and similar expressions for k. The notation

$$|(1,2,\cdots,N)j^N\omega L\}$$

denotes an antisymmetrized state of N particles coupled to angular momentum L. All the quantum numbers needed in addition to their angular momentum L to specify the states in the expansion are denoted by ω. The computational method of the CFP's is described in Sec. IV.

We now give the complete details of the matrix element, Eq. (10), term by term. First, we consider V_1 and V_6, corresponding to the interaction of two like particles of the same chain. These are the only terms involving DCFP's:

$$\begin{aligned}V&=2\sum_{JRST\omega}\langle(PQ)C,C;I|P,(QC)S;I\rangle^2\\&\quad\times\langle(JR)P,S;I|J,(RS)T;I\rangle^2\\&\quad\times\langle j^{N-2}\omega R;j^2J\rrbracket\rrbracket j^NP\rangle^2\\&\qquad\qquad\times\langle(jj)J\|V\|(jj)J\rangle(2J+1)^{-1/2}, \quad (13)\end{aligned}$$

where the recoupling coefficient is given in terms of the 6-j coefficients as

$$\begin{aligned}&\langle(AB)C,D;I|A,(BD)E;I\rangle\\&\quad=(-)^{A+B+D+I}(2C+1)^{1/2}(2E+1)^{1/2}\\&\qquad\qquad\times\begin{Bmatrix}A&B&C\\D&I&E\end{Bmatrix}, \quad (14)\end{aligned}$$

and where $\langle(jj)J\|V\|(jj)J\rangle$ is the usual reduced antisymmetrized matrix element of the two-body force. The summations over S and T in (13) can be carried out to give unity. Thus, finally, we have

$$\begin{aligned}V_1&=2\sum_{\omega RJ}\langle j^{N-2}\omega R;j^2J\rrbracket\rrbracket j^NP\rangle^2\\&\qquad\qquad\times\langle(jj)J\|V\|(jj)J\rangle(2J+1)^{-1/2} \quad (15)\end{aligned}$$

and a similar equation for V_6, with j and P replaced by k and Q, respectively.

Next, we consider V_2, which describes the interaction of two unlike particles of the same chain,

$$\begin{aligned}V_2&=2\sum_{\omega\tilde{\omega}R\tilde{R}JST}\begin{Bmatrix}R&j&P\\\tilde{R}&k&Q\\S&J&C\end{Bmatrix}^2\langle(JS)C,C;I|J,(SC)T;I\rangle^2\\&\quad\times\langle j^{N-1}\omega R;j\rrbracket j^NP\rangle^2\langle k^{N-1}\tilde{\omega}\tilde{R};k\rrbracket k^NQ\rangle^2\\&\qquad\qquad\times\langle(jk)J\|V\|(jk)J\rangle\times(2J+1)^{-1/2}\\&=2\sum_{\omega\tilde{\omega}R\tilde{R}J}\begin{Bmatrix}R&j&P\\\tilde{R}&k&Q\\S&J&C\end{Bmatrix}^2\langle j^{N-1}\omega R;j\rrbracket j^NP\rangle^2\\&\quad\times\langle k^{N-1}\tilde{\omega}\tilde{R};k\rrbracket k^NQ\rangle^2\\&\qquad\qquad\times\langle(jk)J\|V\|(jk)J\rangle\times(2J+1)^{-1/2}. \quad (16)\end{aligned}$$

Finally we have three terms left (V_3,V_4,V_5) describing the interaction between two particles in different chains. The case of two protons is

$$\begin{aligned}V_3&=\sum_{\omega\tilde{\omega}R\tilde{R}JSTUL}\begin{Bmatrix}P&Q&C\\P&Q&C\\L&U&I\end{Bmatrix}^2\begin{Bmatrix}R&j&P\\\tilde{R}&j&P\\S&J&L\end{Bmatrix}^2\\&\quad\times\langle(JS)L,U;I|J,(SU)T;I\rangle^2\\&\quad\times\langle j^{N-1}\omega R;j\rrbracket j^NP\rangle^2\langle j^{N-1}\tilde{\omega}\tilde{R};j\rrbracket j^NP\rangle^2\\&\qquad\qquad\times\langle(jj)J\|V\|(jj)J\rangle\times(2J+1)^{-1/2}\\&=\sum_{\omega\tilde{\omega}R\tilde{R}JSUL}\begin{Bmatrix}P&Q&C\\P&Q&C\\L&U&I\end{Bmatrix}^2\begin{Bmatrix}R&j&P\\\tilde{R}&j&P\\S&J&L\end{Bmatrix}^2\\&\quad\times\langle j^{N-1}\omega R;j\rrbracket j^NP\rangle^2\langle j^{N-1}\tilde{\omega}\tilde{R};j\rrbracket j^NP\rangle\\&\qquad\qquad\times\langle(jj)J\|V\|(jj)J\rangle\times(2J+1)^{-1/2}. \quad (17)\end{aligned}$$

A similar expression is obtained immediately for the interaction of two neutrons in two different chains, i.e., the expression for V_5, by replacing, in Eq. (17), j by k and by exchanging P and Q. Likewise the interaction between a proton and a neutron in two different chains is given by

$$V_4=2\sum_{\omega\tilde{\omega}R\tilde{R}JSUL}\begin{Bmatrix}P & Q & C\\ Q & P & C\\ L & U & I\end{Bmatrix}^2\begin{Bmatrix}R & j & P\\ \tilde{R} & k & Q\\ S & J & L\end{Bmatrix}^2$$
$$\times\langle j^{N-1}\omega R;j\rrbracket j^NP\rangle^2\langle k^{N-1}\tilde{\omega}\tilde{R};k\rrbracket k^NQ\rangle^2$$
$$\times\langle (jk)J\|V\|(jk)J\rangle\times(2J+1)^{-1/2}. \quad (18)$$

IV. CALCULATION OF THE FRACTIONAL PARENTAGE COEFFICIENTS

The usual method for the computation of CFP's would be in the present case unnecessarily cumbersome, since it is a recurrence procedure which involves the computation of all CFP's for all particle numbers up to N. Only those for N itself are in fact needed. Because of the uniqueness of the stretch wave function, it is possible to compute the CFP's directly for N particles.

We begin with the SCFP's, Eq. (11). This expansion represents the antisymmetric wave function of N particles as a sum of products of antisymmetric wave functions of particles $1,\cdots,N-1$ and of the wave function of particle N. Writing out the coupling explicitly we have

$$\Phi_P{}^{[P]}=|(1,2,\cdots,N)j^NPP\}$$
$$=\sum_{\omega RmM}|(1,2,\cdots,N-1)j^{N-1}\omega RM\}$$
$$\times\varphi_m{}^{[j]}(N)(RjMm|PP)\langle j^{N-1}\omega R;j\rrbracket j^NP\rangle. \quad (19)$$

On the other hand, $\Phi_P{}^{[P]}$ is the single normalized Slater determinant, Eq. (2), which we shall denote by $|N|$. As in Eq. (19), we expand the Slater determinant by factorizing the wave function of particle N,

$$\Phi_P{}^{[P]}=|N|=(-)^{N-1}\sum_{\{m\}}\frac{1}{\sqrt{N}}(-)^{s_m}$$
$$\times|N-1;\{m\}|\,\varphi_m{}^{[j]}(N). \quad (20)$$

The common sign reflects the fact that the determinant was developed according to the last line. The sign $(-)^{s_m}$ is the usual parity of the permutation corresponding to the index m, and here it is simply $s_m=j-m$, and $|N-1;\{m\}|$ is the normalized subdeterminant. Finally, the $(N-1)$-particle wave functions in Eq. (19) can also be expanded in terms of $(N-1)$-dimensional normalized Slater determinants $|N-1;\{m\}|$. In this notation, the $N-1$ denotes the dimensionality of the Slater determinant and $\{m\}$ symbolizes the partition of the m values. Naturally, the Slater determinant $|N-1;\{m\}|$ of Eq. (20) corresponds to the partition $\{j,j-1,\cdots,m+1,m-1,\cdots,j-N+1\}$. Thus

$$|(1,2,\cdots,N-1)j^{N-1}\omega RM\}$$
$$=\sum_{\{m\}}a_{\{m\}}{}^{(\omega RM)}|N-1;\{m\}|. \quad (21)$$

The coefficients $a_{\{m\}}{}^{(\omega RM)}$ will be generated below. Inserting Eq. (21) in Eq. (19) ,and equating the resulting expression with Eq. (20) yields

$$(-)^{N-1}\sum_{\{m\}}\frac{1}{\sqrt{N}}(-)^{s_m}|N-1;\{m\}|\,\varphi_m{}^{[j]}(N)$$
$$=\sum_{\omega RMm}\sum_{\{m'\}}a_{\{m'\}}{}^{(\omega RM)}(RjMm|PP)$$
$$\times\langle j^{N-1}\omega R;j\rrbracket j^NP\rangle|N-1;\{m'\}|\,\varphi_m{}^{[j]}(N). \quad (22)$$

Multiplying both sides by $|N-1;\{m'\}|\,\varphi_m{}^{[j]}(N)$ and integrating over all coordinates, one obtains a set of linear equations for the SCFP's:

$$\frac{(-)^{s_m}}{\sqrt{N}}\delta_{\{m'\},m}=\sum_{\omega RM}a_{\{m'\}}{}^{(\omega RM)}(RjMm|PP)$$
$$\times\langle j^{N-1}\omega R;j\rrbracket j^NP\rangle. \quad (23)$$

The Kronecker symbol $\delta_{\{m'\},m}$ signifies whether the partition $\{m'\}$ is equal to the partition $\{j,j-1,\cdots,m+1,m-1,\cdots,j-N+1\}$. The set of equations (23) in general is redundant. This redundancy can be used to check the numerical calculations. Another possible check is furnished by the fact that the SCFP's must form an orthonormal set.

We now return to the generation of the coefficients $a_{\{m'\}}{}^{(\omega RM)}$. The highest and lowest values of R needed in Eq. (21) are

$$R_>=(N-1)j-(N-1)N/2$$

and

$$R_<=P-j, \quad (24)$$

respectively. Again the wave function for $R_>$ and $M=R_>$ is a single Slater determinant.

Therefore the angular momentum $R_>$ completely specifies the wave function, and there is no need for the quantum number ω. However, we shall in such cases put $\omega=1$. Thus

$$a_{\{j,j-1,\cdots,j-N+2\}}{}^{(1\,R_>\,R_>)}=1. \quad (25)$$

The generation of the other amplitudes proceeds as follows: We apply repeatedly the M-lowering operator J_- to the state $|1\,R_>R_>\}\equiv\{j,j-1,\cdots,j-N+2\}$. This is schematically depicted as

$$\begin{bmatrix}1\\1\\1\\1\\0\\0\\0\end{bmatrix}\rightarrow\begin{bmatrix}1\\1\\1\\0\\1\\0\\0\end{bmatrix}\rightarrow\begin{bmatrix}1\\1\\1\\0\\0\\1\\0\end{bmatrix}+\begin{bmatrix}1\\1\\0\\1\\1\\0\\0\end{bmatrix}\rightarrow\text{etc.}$$

After the first application of J_- one still is left with a single Slater determinant. This state thus has the quantum numbers $(1,R_>,R_>-1)$ and therefore the corresponding amplitude is

$$a_{\{j,j-1,\cdots,j-N+1\}}{}^{(1\,R_>\,R_>-1)}=1. \tag{26}$$

After the second application of J_-, two Slater determinants appear. Thus two linearly independent states can be formed. The one reached by the lowering operation has the quantum numbers $(1, R_>, R_>-2)$ and its two normalized amplitudes result from the action of J_- on the Slater determinants,

$$|1\,R_>\,R_>-2\}=1/[R_>(R_>+1)-M_>(M_>+1)]^{1/2}$$

$$\times\left\{[j(j+1)-m_1(m_1+1)]^{1/2}\begin{pmatrix}1\\1\\1\\0\\0\\1\\0\end{pmatrix}\right.$$

$$\left.+[j(j+1)-m_2(m_2+1)]^{1/2}\begin{pmatrix}1\\1\\0\\1\\1\\0\\0\end{pmatrix}\right\}, \tag{27}$$

with $M_>=R_>-2$, $m_1=j-5$, and $m_2=j-3$ in this example. The other state formed by choosing the orthogonal linear combination has the quantum numbers $(1, R_>-2, R_>-2)$.

When continuing with the lowering operations, one now must apply J_- to both these states. This in general gives three determinants. This way one obtains the states $(1, R_>, R_>-3)$ and $(1, R_>-2, R_>-3)$ and a new state, viz. $(1, R_>-3, R_>-3)$. The amplitudes for the first two states can be obtained from the general recursion relation,

$$a_{\{m\}}{}^{(\omega,R,M)}=\sum_{\{m'\}}\left(\frac{j(j+1)-m'(m'+1)}{R(R+1)-M(M+1)}\right)^{1/2}$$

$$\times a_{\{m'\}}{}^{(\omega,R,M+1)}. \tag{28}$$

Here the partitions $\{m'\}$ are those which can reach the partition $\{m\}$ by one lowering operation, and m' in the square root denotes the m value of the single-particle state which has been filled by the lowering operation. The amplitudes for the new state, viz. $(1, R_>-3, R_>-3)$, again are obtained by requiring that they be orthogonal to the amplitudes of the states $(1, R_>, R_>-3)$ and $(1, R_>-2, R_>-3)$.

The need for the additional quantum number ω arises whenever the number of new partions increases by more than one, thus leading to several new states having the same angular momentum. The amplitudes of these states then are not uniquely determined by the orthogonality requirement. We have used the following procedure: Let the number of partitions in the previous step be n_1, and the number of the new partions be $n_2>n_1$. We abbreviate the notation for the amplitudes to be $a_\alpha{}^{(A)}$. The amplitudes with $|A|\leq n_1$ are obtained from the recursion relation (28). We first consider the amplitudes for the state $B=n_1+1$. They must be orthogonal to all previous vectors. Introducing unnormalized amplitudes

$$x_\alpha=\mathfrak{N}a_\alpha{}^{(B)}, \tag{29}$$

the orthogonality requirement then is

$$\sum_{\alpha=1}^{n_2} a_\alpha{}^{(A)}x_\alpha=0 \quad \text{for all } (A). \tag{30}$$

Since there exist only n_1 equations in the n_2 unknowns x_α, n_2-n_1 conditions still must be imposed. We have chosen

$$x_\alpha=1 \quad \text{for} \quad \alpha>n_1. \tag{31}$$

The set of equations (30) then becomes

$$\sum_{\alpha=1}^{n_1} a_\alpha{}^{(A)}x_\alpha=-\sum_{\alpha=n_1+1}^{n_2} a_\alpha{}^{(A)}. \tag{32}$$

This system yields a unique solution for the x_α's provided that the determinant of the left-hand side of (32) is nonsingular. If it should turn out to be singular, one imposes the conditions (31) on a different set of α's. Finally, the amplitudes $a_\alpha{}^{(B)}$ are obtained from (29) by computing the normalization constant $\mathfrak{N}$. They correspond to the new state with $\omega=1$. The procedure now can be repeated leading to the states with $\omega=2$, etc.

We now turn to the DCFP's, Eq. (12). Again we rewrite explcitly

$$\Phi_P{}^{[P]}=|(1,2,\cdots,N)j^NPP\}$$

$$=\sum_{\omega RJmM}|(1,2,\cdots,N-2)j^{N-2}\omega RM\}$$

$$\times|(N-1,N)j^2JM\}(RJMm|PP)$$

$$\times\langle j^{N-2}\omega R;\,j^2J]\!]\,j^NP\rangle. \tag{33}$$

We again expand the antisymmetric $(N-2)$-particle wave function into normalized Slater determinants

$$|(1,2,\cdots,N-2)j^{N-2}\omega RM\}$$

$$=\sum_{\{m\}} a_{\{m\}}{}^{(\omega RM)}|N-2,\{m\}|. \tag{34}$$

The amplitudes $a_{\{m\}}{}^{(\omega RM)}$ are generated as already explained. The two-particle wave function can also be expanded in terms of 2×2 Slater determinants

$$|(N-1,N)j^2JM\}=\sum_{m_1>m_2}(jjm_1m_2|JM)$$

$$\times\begin{vmatrix}\varphi_{m_1}{}^j(N-1) & \varphi_{m_1}{}^j(N)\\ \varphi_{m_2}{}^j(N-1) & \varphi_{m_2}{}^j(N)\end{vmatrix}. \tag{35}$$

TABLE I. Binding energies of the stretch ground state. The values in italics correspond to the cases where the Pauli principle is strongly violated. The energies are given in MeV.

j,k \ A		4	8	12
$d_{5/2}$	$d_{5/2}$	11.26	*26.84*	*48.41*
$f_{7/2}$	$f_{7/2}$	8.54	19.78	*34.03*
$g_{9/2}$	$g_{9/2}$	7.11	16.23	27.10
$h_{11/2}$	$h_{11/2}$	6.23	14.13	23.23
$g_{9/2}$	$h_{11/2}$	6.21	14.01	23.07
$g_{9/2}$	$i_{13/2}$	4.97	10.95	17.68

On the other hand the unique state $\Phi_P{}^{[P]}$ can be directly expanded into products of two Slater determinants,

$$\Phi_P{}^{[P]}=\sum_{\{m\},\{m'\}}\frac{1}{[2N(N-1)]^{1/2}}(-)^{s_{m_1,m_2}}|N-2;\{m'\}| \times|2;\{m\}|\delta_{\{m\},\{m'\}}, \quad (36)$$

where the partition $\{m\}$ is just $\{m_1,m_2\}$. As previously, we obtain a system of linear equations for the DCFP's, viz.,

$$\frac{(-)^{s_{m_1,m_2}}}{[2N(N-1)]^{1/2}}\delta_{\{m'\},\{m\}}$$

$$\sum_{\omega RJmM} a_{\{m'\}}{}^{(\omega RM)}(jjm_1m_2|JM)(RJMm|PP)\sqrt{2} \times\langle j^{N-2}\omega R;j^2J]\!]j^NP\rangle. \quad (37)$$

Here the Kronecker symbol $\delta_{\{m'\},\{m\}}$ signifies whether the partitions $\{m'\}$ and $\{m\}$ complement themselves to the partition $\{j, j-1, j-2, \cdots, j-N+1\}$.

V. RESULTS

In the numerical calculations of the rotational energies we have used a finite-range Guassian force

$$V=e^{-(r_{12}/\mu)^2}\sum_{ST}V_{ST}P_{ST},$$

with $V_{10}=-40$ MeV, $V_{01}=-24$ MeV, $V_{00}=-24$ MeV, $V_{11}=25$ MeV; P_{ST} is a projection operator. This exchange mixture has been used with good results in particle-hole calculations of ^{12}C, ^{16}O, and ^{208}Pb.[10] It is not necessarily a force which gives good results in particle-particle calculations. However, this is not of any importance in this connection, since only the qualitative features of the solutions are of interest here. The single-particle wave functions were those of a spherically symmetric harmonic-oscillator well. The oscillator-well parameter α and the range μ of the force were chosen such that $\mu\alpha=1$.

TABLE II. The A dependence of the stretch ground-state binding energies. The numbers are the ratios of the binding energies relative to the case $A=4$. The first line gives the number of bonds relative to the case $A=4$. The last line gives the number of quartets.

j,k \ A	4	8	12
$A(A-1)/12$	1	4.67	11
$f_{7/2}-f_{7/2}$	1	2.32	3.96
$g_{9/2}-g_{9/2}$	1	2.28	3.81
$g_{9/2}-h_{11/2}$	1	2.54	4.75
$A/4$	1	2	3

The following checks were included in the calculation. First, the fractional parentage coefficients were tested for orthonormality. Second, the total geometry was verified substituting unity for the two-body matrix elements in the complete program. It then has to yield the number $7N^2-2N$ in place of the energies E_I. This number replaces the number of pairs, viz., $8N^2-2N=\frac{1}{2}A\times(A-1)$, which would result from a completely antisymmetrized wave function. This check is exceedingly strong as it involves the complete calculation. It was performed after the computation of each rotational state, and, as a matter of fact, doubled the computation time. These checks all worked out perfectly and we therefore have a very high degree of confidence in the obtained results. As an aside, the commonly available programs for 6-j and 9-j coefficients were inadequate because of their limited accuracy. A double precision program[11] had to be used to obtain satisfactory checks. The accuracy of the checks turned out to be better than four digits, even for the highest angular momenta and for the largest number of particles.

TABLE III. The excitation energies of the rotational stretch states, in MeV. The Pauli principle is strongly violated in the states $f_{7/2}-f_{7/2}$ with $4N=12$.

	$f_{7/2}-f_{7/2}$			$g_{9/2}-g_{9/2}$		
I	4	8	12	4	8	12
0	0	0	0	0	0	0
2	0.538	0.254	0.068	0.310	0.139	0.080
4	1.426	0.753	0.240	0.866	0.427	0.254
6	2.167	1.317	0.512	1.421	0.790	0.492
8	2.594	1.787	0.801	1.838	1.158	0.757
10	2.756	2.119	0.938	2.113	1.497	1.015
	$h_{11/2}-h_{11/2}$			$g_{9/2}-h_{11/2}$		
I	4	8	12	4	8	12
0	0	0	0	0	0	0
2	0.202	0.084	0.054	0.242	0.103	0.064
4	0.587	0.263	0.174	0.690	0.321	0.203
6	1.013	0.506	0.344	1.163	0.606	0.397
8	1.376	0.777	0.545	1.552	0.914	0.620
10	1.646	1.054	0.761	1.826	1.216	0.871

Calculations were carried through in the $f_{7/2}$, $g_{9/2}$ and $h_{11/2}$ shells, for 4, 8, and 12 particles. The results are given in Tables I–III.

---

[10] V. Gillet, A. M. Green, and E. Sanderson (to be published).

[11] R. Caswell and L. C. Maximon, National Bureau of Standards Technical Note No. 409, 1966 (unpublished). This article gives FORTRAN II and FORTRAN IV programs for the calculation of Wigner 3j, 6j, and 9j coefficients containing individual angular momenta less than or equal to 80. For such angular momenta the errors are of the order of 10^{-6} for 9j and 10^{-7} for 6j coefficients, and the computation times are of the order of 0.001 to 0.01 sec per 6j and 0.1 to 1 sec per 9j.

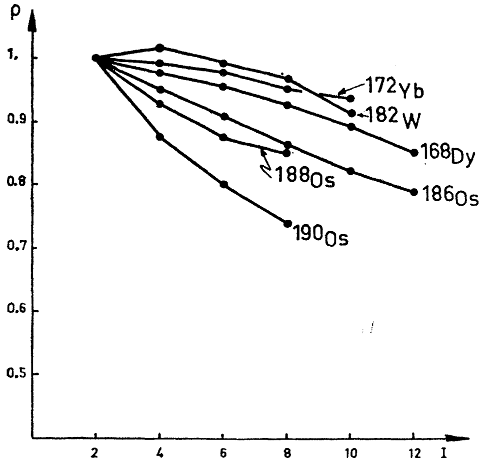

FIG. 4. Departures of some experimental spectra from the $I(I+1)$ law (Ref. 1).

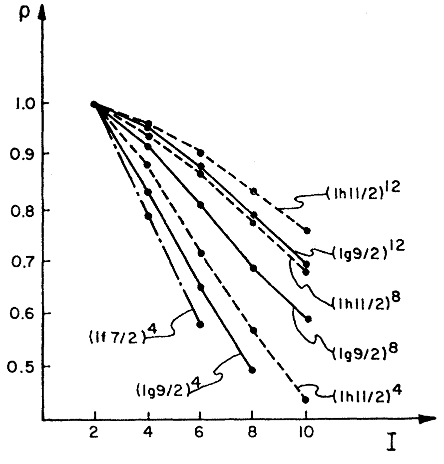

FIG. 5. Departures of the stretch scheme spectra from the $I(I+1)$ law.

The binding energies of the stretch ground state ($I=0$) are given in Table I. Their A dependence is given in Table II. This dependence turns out to be very closely linear with A. It is far from being proportional to the number of bonds $\frac{1}{2}A(A-1)$. As a matter of fact it just barely exceeds a linear dependence. The origin of this behavior lies in the fact that the basic building block of the stretch wave function is a quartet of nucleons consisting of two aligned proton-neutron pairs coupled to angular momentum 0. The correlations between the four nucleons making a quartet are maximal. The interactions between different quartets, however, are weak since the Pauli principle forbids a good overlap between the wave functions of the different quartets. Hence the almost linear dependence of the binding energies on the number of quartets, $\frac{1}{4}A$. Thus the system behaves closely like an independent quartet system. Still, the interaction between the quartets does not vanish and is responsible for the deviations from linearity, and for the over-all alignment of the stretch wave function.

The computed rotational spectra are given in Table III. The values of these energies are of the same order of magnitude as those found in nature. All these spectra have rotational spacings, even though all depart from a pure $I(I+1)$ dependence, However, similar departures are also observed in nature. This is illustrated for a selection of nuclei in Fig. 4, which shows the parameter

$$\rho_I=\frac{(E_I-E_0)/I(I+1)}{\frac{1}{6}(E_2-E_0)},$$

calculated from the experimental energies.[1] This parameter would be $\rho_I=1$, for a pure $I(I+1)$ law. As can be seen the deviations from this law are considerable, even for the most "rotational" nuclei. In Fig. 5 we show the values of ρ calculated from the theoretical spectra (Table III). The departures from the value $\rho=1$ diminish with increasing particle number and grow with increasing I. Both these effects are as expected. The I dependence of ρ in the stretch scheme results from the deviation of the geometrical coupling coefficients from

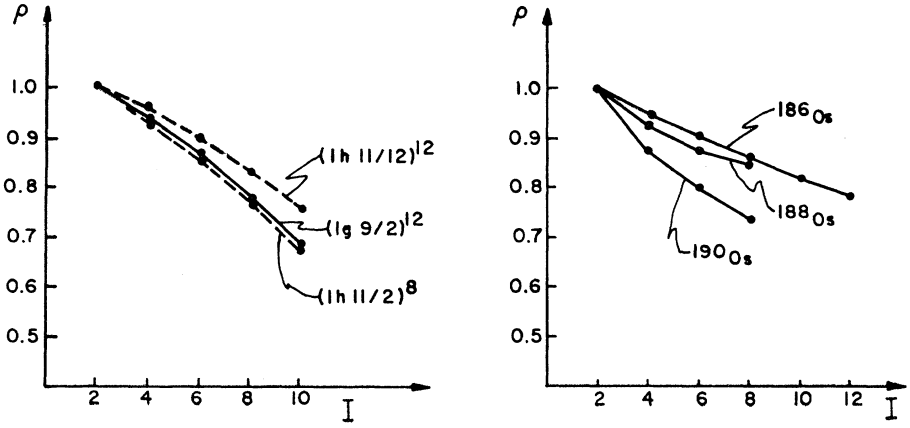

FIG. 6. Comparison of the departures from a purely rotational spectrum in stretch (left part) and in the osmium isotopes which have six proton holes.

their asymptotic form, which was discussed in Sec. I. Also deviations may result from the Pauli correlations neglected in the present treatment.

The similarities between the experimental and the stretch deviations are shown in Fig. 6. In the right part of this figure we have plotted the ρ values for the Os isotopes. These are nuclei with six proton holes and a varying number of neutron holes. We have chosen these nuclei since a stretch wave function corresponding to a positive intrinsic deformation is made of holes, as discussed in Sec. I.

VI. CONCLUSIONS

The stretch wave function owes its simplicity to the uniqueness of the chains. When constructed with particles it has negative deformation. A wave function projected from a product of aligned particle-state determinants with positive intrinsic deformations cannot be represented as a single configuration. It is a linear superposition of configurations with varying chain lengths. Because of configuration interaction, this state is lower in energy than the pure stretch state studied here. This indicates why nuclei have positive rather than negative deformations. A treatment of this situation has not yet been carried through. The stretch scheme is useful in that it gives a description of hole nuclei, and, more importantly, of the basic structure of deformed nuclei.

The most important feature of the stretch scheme which emerged from the calculations is the almost linear dependence of the binding energy on the number of quartets. This is in marked contrast to the presently prevalent view that the binding energy in deformed states should be proportional to the number of pairs, i.e., that it should go quadratically with the number of particles. The physical reason for this behavior is the fact that the internal binding energy of a given quartet is much larger than the interaction energy between two different quartets. The large binding energy of a quartet is the result of the maximization of the overlap of the four-particle wave function. On the other hand, the Pauli principle then limits the possible overlap between two quartets. Thus the correlations between the quartets do not contribute much to the binding energy; they show up only as the deviations from linearity of the dependence of the binding energy on the number of quartets. However, the correlations between the quartets are solely responsible for the alignment of the quartets, and thus for the existence of the nuclear deformation. Parenthetically we note that most of the binding energy thus is associated with an intrinsically spherically symmetrical system, viz., the uncorrelated quartets, while energetically the change of the system to an intrinsically deformed system, viz., the stretch wave function, makes only a small difference.

The stretch scheme, in the form presented here, obviously cannot aim at a quantitative description of actual nuclei. Its great simplicity rests upon two unrealistic features: (i) Only one shell for the protons and one shell for the neutrons are allowed, and (ii) the stretch state has a negative quadrupole moment ("oblate" shape). These limitations are necessary in order to have a unique, energetically nondegenerate, stretch state. They are the price paid for the simple image of the relationship between collective and individual nucleon motions which has been given in this paper. The removal of these two limitations, in the present framework, leads to a set of stretch states all close in energy, requiring a diagonalization of the interaction between the different members of this set. Another approach, now in progress, may turn out to be more fruitful. It uses the above-demonstrated fact that the independent quartet wave function yields a good approximation for the energy of the ground state. The variational principle may be applied to a trial wave function of this type, leading to the energetically best distribution of the quartets in the subshells. The correlations between the quartets which are responsible for the over-all alignment of the nucleon angular momenta may then be introduced by a perturbation treatment.

ACKNOWLEDGMENTS

The authors are grateful to J. Raynal for pointing out some numerical errors in the computations. Raynal also has devised a more concise method for evaluation of the stretch-state energies. This will be the subject of a forthcoming paper. Further, the authors thank U. Fano and C. Levinson for important discussions and L. C. Maximon and R. Caswell for permission to use their geometric codes prior to publication. Last but not least, they thank Nicole Tichit for help in the computations.