Angular Distribution of the Neutrino in Muon Capture and Nuclear Structure*

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As a result of parity nonconservation, emitted neutrinos in muon capture by spin-zero nuclei have an asymmetric angular distribution with respect to the muon-polarization axis. The asymmetry coefficient α is defined by the distribution $1+\alpha P\cos\theta$, and it is given in cases in which the final states have definite spin and parity. In the unique forbidden transitions in which the final states have spin J and parity $(-)^{J+1}$, α is generally dependent on both nuclear structure and the pseudoscalar coupling constant C_P. However, in some cases, it is almost independent of nuclear structure, so that we can study the magnitude of C_P from the asymmetry coefficient. In the nonunique forbidden transitions in which the final states have spin J and parity $(-)^J$, α is independent of C_P, but is strongly dependent on nuclear structure. As extreme cases, α is $+1$ for A-type and -1 for V-type giant dipole excitations, with about 2% estimated error. This holds also for A- and V-type higher multipole excitations. If the final states are 0^- or 0^+, then α is exactly -1, and this case is independent of both nuclear structure and the magnitude of C_P, in contrast to the strong dependence of the capture rate on C_P in the 0^- states. An averaged value of α over four kinds of the giant dipole states is equal to 0.4, in agreement with Primakoff's value in the closure approximation.

I. INTRODUCTION

IN muon capture by complex nuclei, the emitted neutrinos are distributed anisotropically with respect to the muon polarization axis, if a certain amount of polarization is still preserved in the muonic K orbit. Experimentally, the direction of the emitted neutrino can be measured by detecting the recoiled nucleus which goes in the opposite direction to that of the neutrino. The angular distribution of neutrinos is calculated by several authors[1-10] in the form

$$1+\alpha P\cos\theta\,,$$

where θ is the angle of the emitted neutrino with respect to the initial muon polarization, P is the degree of the polarization, and α is the asymmetry coefficient. The standard form of α, which is given by Primakoff,[5] is

$$\alpha=\frac{-G_V^2+G_A^2-G_P^2+2G_AG_P}{G_V^2+3G_A^2+G_P^2-2G_AG_P}\,,$$

with

$$G_V=C_V[1+q/2M]\,,$$
$$G_T=C_A-C_V(1+\mu_p-\mu_n)q/2M\,,$$
$$G_P=[C_P-C_A-C_V(1+\mu_p-\mu_n)]q/2M\,.$$

Here, q is the neutrino energy and M is the nucleon mass. This is given in the closure approximation for estimating nuclear matrix elements, so that the nuclear-structure dependence apparently disappears from the above expression of the asymmetry. The formula gives $\alpha\approx0.4$ for a reasonable assumption of the pseudoscalar coupling constant[11]

$$C_P/C_A\approx8,\quad\text{with}\quad q\approx80\text{ MeV}.$$

It will reach the maximum value $\alpha\approx0.6$ for $C_P/C_A\approx25$ with the same value of q.

In this paper, we study the angular distribution of neutrinos for partial transitions of the muon-capture reaction in which the final states have definite spin and parity. The neutrino (or equivalently, nuclear-recoil) asymmetry is given in the framework of the allowed and forbidden transitions, and it is expressed as a function of weak-coupling constants and nuclear matrix elements. It varies from -1 to $+1$ for different excitations of the final states, without introducing any large pseudoscalar coupling constant or G-parity nonconservation. The asymmetry has also some characteristic values for different modes of the giant dipole excitations. In this connection, the measurements of nuclear recoil will give information on nuclear structure.

In Sec. II, the interaction Hamiltonian, coupling constants, and nuclear matrix elements in muon-capture reactions are summarized. In Secs. III and IV, the formulas for the angular distributions of neutrinos are given in unique and nonunique forbidden transitions, respectively. Here, forbiddenness is defined in the same manner as in the theory of β-decay.[12] The neutrino asymmetries for giant dipole states are discussed in Sec. V, and some remarks are given in Sec. VI.

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In the Appendix, the definition of the reduced nuclear matrix elements is given.

II. INTERACTION HAMILTONIAN, COUPLING CONSTANTS, AND REDUCED NUCLEAR MATRIX ELEMENTS

In muon capture,

$$\mu^- + p \to n + \nu\,,$$

the density of the interaction Hamiltonian is given by[13]

$$H_{\rm int} = (\bar\phi_\nu \gamma_\alpha \psi_\mu)(\bar\psi_n[f_V\gamma_\alpha - ig_V\sigma_{\alpha\beta}p_\beta - ih_V p_\alpha]\psi_p) + (\bar\phi_\nu i\gamma_\alpha\gamma_5\psi_\mu)(\bar\psi_n[if_A\gamma_\alpha\gamma_5 - g_A p_\alpha\gamma_5 + h_A\sigma_{\alpha\beta}\gamma_5 p_\beta]\psi_p)\,, \quad (1)$$

with

$$\phi_\nu = [(1+\gamma_5)/\sqrt{2}]\psi_\nu\,,$$

and

$$\sigma_{\alpha\beta} = \tfrac{1}{2}[\gamma_\alpha,\gamma_\beta]\,.$$

Here p_β is the 4-momentum transfer in the reaction. The six form factors f, g, and h are functions of p^2. They are real if time-reversal invariance holds. In the limit of the low momentum transfer ($p\to 0$), we set

$$f_V(0) = C_V\,, \quad f_A(0) = C_A\,.$$

The G parities of the f_V, g_V, and h_A terms are different from those of the f_A, g_A, and h_V. Assuming G invariance for each nucleon current, we have $h_V = h_A = 0$, since the interactions f and g are well known in experiments.

In the nonrelativistic approximation for the nucleons, $H_{\rm int}$ becomes $H_{\rm int} = u_n^\dagger \mathcal{H} u_p$, with

$$\mathcal{H} = C_V 1\cdot L(1) + C_A\boldsymbol{\sigma}\cdot L(\boldsymbol{\sigma}) + (C_V/2M)[2L(\boldsymbol{\alpha})\cdot\mathbf{p} + \mathbf{p}\cdot L(\boldsymbol{\alpha}) + i\boldsymbol{\sigma}\cdot\mathbf{p}\times L(\boldsymbol{\alpha})] + (C_A/2M)[2L(\gamma_5)(\boldsymbol{\sigma}\cdot\mathbf{p}) + \boldsymbol{\sigma}\cdot\mathbf{p}L(\gamma_5)] - (C_P/2M)\boldsymbol{\sigma}\cdot\mathbf{p}L(\beta\gamma_5) + (\mu_p-\mu_n)(C_V/2M)[i\boldsymbol{\sigma}\cdot\mathbf{p}\times L(\boldsymbol{\alpha})] - C_S 1\cdot L(\beta) - C_T\boldsymbol{\sigma}\cdot L(\boldsymbol{\sigma}) - (C_T/W_0)\boldsymbol{\sigma}\cdot\mathbf{p}L(\gamma_5)\,, \quad (2)$$

where leptonic currents are

$$L(\boldsymbol{\sigma}) = \psi_\nu^\dagger[(1+\gamma_5)\boldsymbol{\sigma}/\sqrt{2}]\psi_\mu,\ \text{etc.}$$

In Eq. (2), the u's are the large components of the relativistic wave functions ψ. We set the following relations between coupling constants C_i and form factors f_i, g_i, h_i:

$$C_V = f_V\,,\quad g_V = (\mu_p-\mu_n)(C_V/2M)\,,\quad C_S = -m_\mu h_V\,,$$
$$C_A = f_A\,,\quad C_P = m_\mu g_A\,,\quad C_T = -h_A W_0\,, \quad (3)$$

where W_0 is the end-point energy in the β decay to the same levels involved in muon capture. M and m_μ are the nucleon mass and the muon mass, and $\mu_p - \mu_n = 3.706$. As is seen in the above relations, the g_V, g_A, h_V, and h_A terms are of the weak magnetism, induced pseudoscalar, induced scalar, and induced tensor interactions, respectively. The relation between g_V and C_V is given by the conserved-vector-current theory, for which $h_V = 0$.

The neutrino angular distribution is calculated in the framework of the theory of allowed and forbidden transitions in muon capture.[12] The nucleon currents are decomposed in the spherical representation, and the reduced nuclear matrix elements are defined as shown in the Appendix. The asymmetry coefficient is expressed in terms of coupling constants in Eqs. (3) and the reduced nuclear matrix elements below.

The reduced nuclear matrix elements are symbolized by $[S\,L\,J]$ or $[S\,L\,J\,p]$. Here, S, L, and J can be understood as the resultant spin, the effective orbital angular momentum, and the resultant total angular momentum of the lepton system, or, respectively, those of the nucleon system. An additional symbol p is used to indicate that the differential operator for the nuclear wave function is involved. The reduced nuclear matrix elements are given under the assumption[13a] that the proton is annihilated in the orbit (n_1,l_1,j_1) and the neutron is created in the orbit (n_2,l_2,j_2). The integration of angular parts is expressed in terms of the 3-j, 6-j, and 9-j coefficients. The radial parts are left for integration. A linear combination of the matrices with a set of appropriate mixing parameters for different values of n_1, l_1, j_1, n_2, l_2, j_2 should be used for the real nuclear wave functions.

$$[S\,L\,J] = (-)^{1+l_1+l_2+J}[(2S+1)/4\pi][2(2L+1)(2l_1+1)(2j_1+1)(2j_2+1)]^{1/2}(l_1\,L\,0\,0|l_2\,0) \times \begin{Bmatrix} j_1 & j_2 & J \\ l_1 & l_2 & L \\ \frac{1}{2} & \frac{1}{2} & S \end{Bmatrix} \int g_{-1}j_L(qr)\phi(n_2l_2)\phi(n_1l_1)r^2dr\,, \quad (4)$$

$$[0\,J\,J\,p] = \sum_{l_1'=l_1\pm1} (-)^{j_1+l_1+1/2}[(2l_1'+1)/4\pi][6(2j_1+1)(2j_2+1)]^{1/2}(l_1'\,J\,0\,0|l_2\,0)W(l_1'j_1l_2j_2,\tfrac{1}{2}\,J) \times W(l_1j_1\,1\,\tfrac{1}{2},\tfrac{1}{2}\,l_1')\int g_{-1}j_J(qr)\phi(n_2l_2)(D_{l_1,l_1'}\phi(n_1l_1))r^2dr\,, \quad (5)$$

$$[1\,L\,J\,p] = \sum_{l_1'=l_1\pm1} (-)^{j_1+l_1+1/2}[(2J+1)(2l_1'+1)/4\pi(2l_2+1)][3(2L+1)(2j_1+1)(2j_2+1)]^{1/2}(l_1'\,L\,0\,0|l_2\,0) \times W(l_1j_1l_2j_2,\tfrac{1}{2}\,J)W(l_1 1 l_2\,L,l_1'\,J)\int g_{-1}j_L(qr)\phi(n_2l_2)(D_{l_1,l_1'}\phi(n_1l_1))r^2dr\,, \quad (6)$$

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[13a] *Note added in proof.* In derivation of Eqs. (4)–(6), it is also assumed that in the initial state the j_1 orbit is completely filled and the j_2 orbit is vacant.

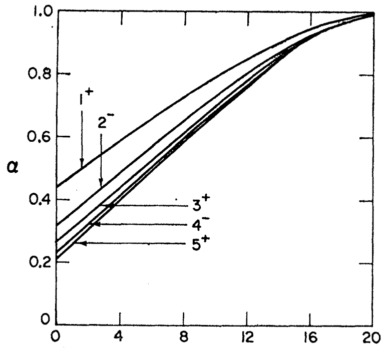

FIG. 1. Asymmetry coefficient α as a function of C_P/C_A in the case of unique forbidden transitions. $q=179$ is assumed in Eq. (17). The spin and parity of the final state are labeled to each curve.

with differential operators

$$D_{l_1,l_1+1}=[(l_1+1)/(2l_1+3)]^{1/2}(d/dr-l_1/r)\,, \tag{7a}$$

$$D_{l_1,l_1-1}=-[l_1/(2l_1-1)]^{1/2}[d/dr+(l_1+1)/r]\,. \tag{7b}$$

For nuclear radial wave functions $\phi(n,l)$, we could use harmonic-oscillator functions. The radial wave function for the neutrino is the spherical Bessel function $j_L(qr)$. The wave function for the muon is assumed to be

$$g_{-1}=2(\alpha Z m_\mu')^{3/2}\exp(-\alpha Z m_\mu' r)\,,\quad\text{and}\quad f_{-1}=0\,, \tag{8}$$

for point nuclear charge of small αZ. The m_μ' is the muon reduced mass

$$m_\mu'=m_\mu[1+(m_\mu/AM)]^{-1}\,, \tag{9}$$

where A is the mass number. In this reaction, the emitted neutrino has an energy of

$$q=(m_\mu-W_0)[1-(m_\mu/2)(m_\mu+AM)^{-1}]\,. \tag{10}$$

All quantities are in the atomic units,

$$\hbar=c=m_e=1\,, \tag{11}$$

where m_e is the electron mass.

The explicit formulas for the angular distribution are given in the following sections.

III. UNIQUE FORBIDDEN TRANSITIONS

This class of transitions is defined by nuclear spin change $J=|J_i-J_f|$ and parity change No for odd and Yes for even values of J. Transitions from the 0^+ to the states 0^-, 1^+, 2^-, 3^+, $\cdots$, are first forbidden, allowed, first forbidden, second forbidden, $\cdots$, respectively. For each transition in muon-capture reactions, there are four matrix elements (except for $0^+\to 0^-$), $[1\,J-1\,J]$, $[1\,J+1\,J]$, $[1\,J\,J\,p]$, and $[0\,J\,J\,p]$. If we keep $[1\,J-1\,J]$, and neglect the other three, the asymmetry coefficient α becomes independent of nuclear structure. It is uniquely determined by the combination of coupling constants in weak interactions, Eq. (3), so that we call these transitions "unique forbidden." In β decay, $[1\,J-1\,J]$ corresponds to $\int\boldsymbol{\sigma}$, $\int B_{ij}$, $\int S_{ijk}$, etc. The spectral shape factor is uniquely determined in this class of transitions. Matrices $[1\,J+1\,J]$, $M^{-1}[0\,J\,J\,p]$, and $M^{-1}[0\,J\,J\,p]$ are negligibly small and they do not appear in the theory of β decay.

The asymmetry coefficient α is given in the unique forbidden transitions as follows:

$$\alpha=N/D\,, \tag{12}$$

with

$$\begin{aligned}\begin{pmatrix}D\\N\end{pmatrix}&=b^2([1\,J-1\,J]^2+[1\,J+1\,J]^2)+(\pm a^2-b^2)\\&\times(2J+1)^{-1}\{J^{1/2}[1\,J-1\,J]+(J+1)^{1/2}[1\,J+1\,J]\}^2\\&+2b(r/M)(2J+1)^{-1/2}[1\,J\,J\,p]\{J^{1/2}[1\,J+1\,J]\\&-(J+1)^{1/2}[1\,J-1\,J]\}\pm2[3/(2J+1)]^{1/2}(a/M)\\&\times[0\,J\,J\,p]\{J^{1/2}[1\,J-1\,J]+(J+1)^{1/2}\\&\times[1\,J+1\,J]\}+(r/M)^2[1\,J\,J\,p]^2\\&\pm3(1/M)^2[0\,J\,J\,p]^2\,,\end{aligned} \tag{13}$$

and

$$a=1-\frac{C_T}{C_A}+\left(1-\frac{C_P}{C_A}-\frac{2MC_T}{W_0C_A}\right)\frac{q}{2M}\,, \tag{14}$$

$$b=1-\frac{C_T}{C_A}-(1+\mu_p-\mu_n)\frac{qC_V}{2MC_A}\,, \tag{15}$$

$$r=\frac{C_V}{C_A}\,. \tag{16}$$

Here the upper sign refers to D and the lower sign refers to N. We can certainly neglect the last two terms in Eq. (13). However, we keep these terms for the positive definiteness of transition probability and for the condition $|\alpha|\leq 1$.

If we can neglect $[1\,J+1\,J]$, $M^{-1}[1\,J\,J\,p]$, and $M^{-1}[0\,J\,J\,p]$ compared with $[1\,J-1\,J]$, the asymmetry coefficient reduces to the following form:

$$\alpha=\frac{(J+1)b^2-Ja^2}{(J+1)b^2+Ja^2}\,. \tag{17}$$

That is, the asymmetry coefficient can be given without knowledge of the details of nuclear structure. This approximation is valid in most of the cases. It does

TABLE I. Asymmetry coefficient α in the case of unique forbidden transitions. Nuclear-structure-independent approximation [Eq. (17)] is assumed, with $C_A/C_V=-1.18$, $C_P/C_A=8$, and $q=179$.

Final states	Forbiddenness	α
0^-	1	-1.00
1^+	0	0.73
2^-	1	0.65
3^+	2	0.62
4^-	3	0.60
5^+	4	0.59

not hold, e.g., in the l-forbidden transitions, where $[1\ J-1\ J]$ is zero, but $[1\ J+1\ J]$ is not.

Figure 1 shows the numerical value of α in Eq. (17) as a function of J and C_P/C_A. We also assume the universal Fermi interaction ($C_T=C_S=0$, $C_A/C_V=-1.18$, and weak magnetism). For $C_P/C_A=8$, α is given in Table I for different values of J. In general, α decreases with J. It increases with C_P in a region of C_P/C_A up to around 20, because the coupling constant a, Eq. (14), decreases with C_P/C_A. α should reach -1 for large values of C_P/C_A, although it is unusual to assume a large value for C_P.

In order to estimate the validity of the neglect of the other matrix elements in Eq. (17), we have computed α with the known values for all four nuclear matrices for the muon capture in C^{12} to the ground state of B^{12}.

$$\begin{aligned} Z_0[1\ 0\ 1] &= -0.138 \quad (-0.134),\\ Z_0[1\ 2\ 1] &= 0.00483 \quad (0.00471),\\ Z_0[1\ 1\ 1\ p] &= 5.55 \quad (5.40),\\ Z_0[0\ 1\ 1\ p] &= 10.6 \quad (10.3), \end{aligned} \tag{18}$$

with

$$Z_0^{-1}=2(\alpha Z m_\mu')^{3/2}, \quad \text{and} \quad Z=6.$$

The numerical values are given for a point nuclear charge[12] and with nuclear finite-size corrections[14] (in parentheses). Both sets of numerical values give an identical value of α, which is curve a in Fig. 2, in comparison with curve b, given by Eq. (17). In particular, for $C_P/C_A=8$,

$$\begin{aligned} \alpha &= 0.84 \quad \text{from Eq. (13)},\\ \alpha &= 0.73 \quad \text{from Eq. (17)}. \end{aligned} \tag{19}$$

The latter value for α has also been obtained by Primakoff.[6]

An exceptional case of the unique forbidden transitions is $0^+\rightarrow 0^-$. The asymmetry coefficient α is derived from Eq. (13) by putting $J=0$,

$$\binom{D}{N}=\pm\{a[1\ 1\ 0]+(3^{1/2}/M)[0\ 1\ 1\ p]\}^2. \tag{20}$$

This gives us

$$\alpha=-1. \tag{21}$$

This result is independent of both nuclear structure and the magnitudes of the coupling constants. It is in contrast to the fact that the muon-capture rate of the $0^+\rightarrow 0^-$ transitions is very sensitive to C_P/C_A.[15]

IV. NONUNIQUE FORBIDDEN TRANSITIONS

This class of transitions is defined by nuclear spin change $J=|J_i-J_f|$ and parity change No for even and Yes for odd values of J. Transitions from the 0^+ state to the states 0^+, 1^-, 2^+, 3^-, $\cdots$, are allowed, first-forbidden, second-forbidden, third-forbidden transitions, $\cdots$, respectively. For each transition in muon-capture reactions, there are four matrix elements, (except for $0^+\rightarrow 0^+$), $[1\ J\ J]$, $[0\ J\ J]$, $[1\ J-1\ J\ p]$, and $[1\ J+1\ J\ p]$, which correspond to $\int \boldsymbol{\sigma}\times\mathbf{r}$, $\int \mathbf{r}$, $\int \boldsymbol{\alpha}$, and $\int(\boldsymbol{\alpha}\cdot\mathbf{r})\mathbf{r}$ of the first-forbidden β decay, respectively. The asymmetry factor α in the nonunique forbidden transitions is given by the formulation described in Ref. 12 as follows:

$$\alpha=N/D \tag{22}$$

with

$$\begin{aligned}\binom{D}{N} =\ & b^2[1\ J\ J]^2\pm 3(rc)^2[0\ J\ J]^2\\ & +2(rb/M)(2J+1)^{-1/2}\{(J+1)^{1/2}[1\ J-1\ J\ p]\\ & -J^{1/2}[1\ J+1\ J\ p]\}[1\ J\ J]\mp 2r^2(c/M)\\ & \times[3/(2J+1)]^{1/2}[0\ J\ J]\{J^{1/2}[1\ J-1\ J\ p]\\ & +(J+1)^{1/2}[1\ J+1\ J\ p]\}\\ & +(r/M)^2([1\ J-1\ J\ p]^2+[1\ J+1\ J\ p]^2)\\ & -\binom{0}{2}(r/M)^2(2J+1)^{-1}\{J^{1/2}[1\ J-1\ J\ p]\\ & +(J+1)^{1/2}[1\ J+1\ J\ p]\}^2, \end{aligned} \tag{23}$$

and

$$c=1-C_S/C_V+q/2M. \tag{24}$$

In Eq. (23) the upper sign refers to D, while the lower sign refers to N.

There is a relation between $[1\ J-1\ J\ p]$ and $[0\ J\ J]$, which can be derived in the conserved-vector-current

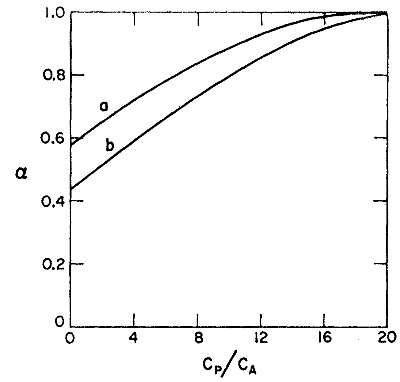

FIG. 2. Asymmetry coefficient α as a function of C_P/C_A in the case of muon capture in C^{12} to the ground state B^{12}, which is 1^+. (a) Eqs. (13) and (18); (b) Eq. (17), nuclear-structure-independent approximation.

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theory,[16]

$$[1\ J-1\ J\ p]\approx -wM[3(2J+1)/J]^{1/2}[0\ J\ J], \quad (25)$$

with

$$w=[W_0+1.15(\alpha Z/R)-2.5]/q.$$

Here W_0 is the total energy of the electron in the β decay between the same levels as in the muon capture, and $\alpha Z/R$ is the Coulomb energy at the nuclear surface, where Z is the atomic number of the parent nuclei in muon capture. If we neglect $[1\ J+1\ J\ p]$ compared with $[1\ J-1\ J\ p]$, and we adopt Eq. (25), then we can express the asymmetry coefficient α by only two matrices, $[1\ J\ J]$ and $[0\ J\ J]$, as follows:

$$\binom{D}{N}=b^2[1\ J\ J]^2-2rbw[3(J+1)/J]^{1/2}[0\ J\ J][1\ J\ J] \pm 3r^2(c^2+2wc+xw^2)[0\ J\ J]^2, \quad (26)$$

with

$$x=1 \quad \text{for } D,$$
$$x=-J^{-1} \quad \text{for } N,$$

and

$$J\geq 1.$$

Since b, c, and r are independent of C_P, we cannot obtain any information on the pseudoscalar coupling constant by measuring nuclear recoil distributions in the nonunique forbidden transitions.

On the other hand, the asymmetry coefficient is strongly dependent on the ratio $[1\ J\ J]/[0\ J\ J]$. This means that the measurement of α gives us information on nuclear structure. For example, we show two extreme cases.

A. V-Type Giant Dipole Excitations

The final states are the 1^- states formed by the isospin mode of collective vibration in the generalized Goldhaber-Teller (GT) model. In this particular case ($J=1$),

$$[0\ J\ J]\neq 0 \quad \text{for vector operator,}$$
and
$$[1\ J\ J]=0 \quad \text{for axial-vector operator.} \quad (27)$$

The asymmetry coefficient derived from Eq. (26) is

$$\alpha=-1, \quad (28)$$

with 2% error. The same value of α is obtained in the V-type excitations with higher multipoles ($J>1$).

B. A-Type Giant Dipole Excitations

The final states are the 1^- states formed by the spin-isospin mode of the collective vibrations in the generalized GT model. In this case ($J=1$),

$$[0\ J\ J]=0 \quad \text{for vector operator,}$$
and
$$[1\ J\ J]\neq 0 \quad \text{for axial-vector operator.} \quad (29)$$

The asymmetry coefficient derived from Eq. (26) is

$$\alpha=1, \quad (30)$$

with 2% error. The same value of α is obtained in the A-type excitations with higher multipoles ($J>1$).

Finally, we give an exceptional case, i.e., the $0^+\to 0^+$ transition. The asymmetry is given by Eq. (23),

$$\binom{D}{N}=\pm\{\sqrt{3}rc[0\ 0\ 0]-(r/M)[1\ 1\ 0\ p]\}^2. \quad (31)$$

This gives us

$$\alpha=-1. \quad (32)$$

This result is independent of both nuclear structure and the magnitudes of the coupling constants.

V. NEUTRINO ASYMMETRIES FOR THE GIANT DIPOLE STATES

We have found in a previous work[12] that the muon-capture rate decreases with increasing forbiddenness, and hence that most of the muon captures take place in the allowed and first-forbidden transitions. (In general, transition rates in the second- and third-forbidden transitions are about 10^{-3} times smaller than those in allowed and first-forbidden transitions, respectively.) The allowed transitions are also suppressed if the parent nuclei have doubly closed shells.

The Wigner supermultiplet theory[17] also gives nonzero transition rates only for the first-forbidden transitions.[18] This is due to the fact that the nuclear matrix elements include an isospin (and spin) operator. Nonzero values occur only for those transitions in which the final states are members of the same supermultiplet of giant dipole states as the parent nucleus. In this connection, collective vibrations of the nucleus in the generalized GT model have been considered.[19–23] The final states of the muon capture are the 1^- state which is excited by the isospin mode (V type) and the 0^-,

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1^-, and 2^- states which are excited by the spin-isospin mode (A type), Fig. 3. The transitions to these states are, therefore, first forbidden.

As was mentioned by Überall,[23] there is a certain indication that the giant dipole states are excited by the absorption of the muon in nuclei such as B^{12}, N^{16}, and K^{40}. This is observed in the neutron energy spectrum from the muon capture. These states have excitation energy 10–25 MeV, and the separation energy among them is about 5 MeV.[24] The spin and parity of these states are (from higher to lower excitation) 0^- (A type), 1^- (A type), 1^- (V type), and 2^- (A type), (the 2^- state is not confirmed in K^{40}). The expected neutrino asymmetries are -1, $+1$, -1, and 0.5—0.6, respectively.

The neutrino asymmetry in the closure approximation can be reproduced from ours in the limiting case where the four kinds of the giant dipole states are degenerate. For simplicity, we neglect all momentum-type matrices $[S\ L\ J\ p]$ in Eqs. (13) and (26).[25] Each transition of the given J should be weighted with a statistical spin factor $(2J+1)$. The matrices satisfy the relations

$$3[0\ 1\ 1]^2=[1\ 1\ 1]^2=[1\ 1\ 0]^2=[1\ 1\ 2]^2. \quad (33)$$

These relations can be proved with the method developed by Überall for the giant dipole states.[26] From Eqs. (13), (26), and (33), the asymmetry becomes

$$\alpha=\frac{-a^2+3(b^2-r^2c^2)+(3b^2-2a^2)}{a^2+3(b^2+r^2c^2)+(3b^2+2a^2)}. \quad (34)$$

The above formula gives

$$\alpha=0.4 \qquad \text{for} \qquad C_P/C_A=8.$$

In fact, Eq. (34) is identical with the formula (given in the Introduction) in the closure approximation,[5] if the coupling constants a, b, c, and r are inserted explicitly.

VI. CONCLUDING REMARKS

The partial muon-capture rate is

$$\mathcal{W}=\tfrac{2}{3}C_A{}^2(2J+1)[1-q(m_\mu+AM)^{-1}]q^2D. \quad (35)$$

Here, D is given by Eqs. (13) and (23), for unique and nonunique forbidden transitions. More accurate formulas for the muon-capture rate with the small components of the muon wave function are given in Refs. 12 and 27.

Until now, we have no direct measurement of nuclear recoil in muon-capture reactions. It is hoped that an experiment of this kind will be possible in the near future. Measurements of the neutron asymmetry which are currently being performed are easier experimentally.[28–32] The neutrons are, however, emitted in the direction of nuclear recoil only if a single neutron carries off most of the available recoil momentum (direct process). Using the Fermi-gas model and the closure approximation, Wolfenstein has shown that the neutron distribution in the direct process is less asymmetric than that of the neutrino if the momentum distribution of the initial proton is taken into account.[10] Furthermore, neutrons from the resonant states are isotropic, since the mechanism of emission is governed by the parity-conserving strong interactions, and the initial polarization is only for the spin-$\frac{1}{2}$ muons.[33] These neutrons make the observable asymmetry smaller. On the other hand, recent experiments show large neutron asymmetries of both signs for high-energy neutrons from muon capture in several nuclei.[30–32] This remains still open to question.

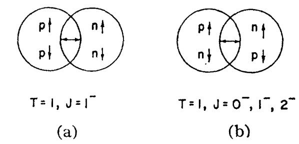

FIG. 3. Collective vibrations in the generalized Goldhaber-Teller model. (a) Isospin mode; (b) spin-isospin mode.

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APPENDIX: DEFINITION OF REDUCED NUCLEAR MATRIX ELEMENTS IN MUON-CAPTURE REACTIONS

We summarize in this Appendix the reduced nuclear matrix elements in muon-capture reactions.[12] The reduced nuclear matrix element $[S\ L\ J]$ is defined by

$$[S\ L\ J](J_i\ J\ M_i\ M\,|\,J_f\ M_f)$$
$$\equiv 2(\alpha Z m_\mu')^{3/2}\int u_{J_f}{}^{M_f\dagger}\sum_{k=1}^{A}\exp(-\alpha Z m_\mu' r_k)$$
$$\times\Omega_k\tau_-^{(k)}u_J{}^{M_i}d_i\mathbf{r}_1 d\mathbf{r}_2\cdots d\mathbf{r}_A. \quad (A1)$$

[24] See, e.g., Figs. 4, 5, and 12 in Ref. 23.

[25] This is correct up to the order of w, since

$$M^{-1}[0\ 0\ 0\ p]\sim w[1\ 1\ 0] \text{ and } M^{-1}[1\ 0\ 1\ p]\sim w[0\ 1\ 1].$$

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TABLE II. Operators Ω_k for reduced nuclear matrix in Eq. (A1). The spherical Bessel functions $j_L(qr_k)$ are abbreviated by j_L.

Matrix element	Ω_k
$[0\ L\ J]$	$j_L \mathcal{Y}_{0LJ}^M(\hat{r}_k)\delta_{LJ}$
$[1\ L\ J]$	$j_L \mathcal{Y}_{1LJ}^M(\hat{r}_k,\boldsymbol{\sigma}_k)$
$[0\ L\ J\ p]$	$ij_L \mathcal{Y}_{0LJ}^M(\hat{r}_k)\boldsymbol{\sigma}_k\cdot\mathbf{p}_k\delta_{LJ}$
$[1\ L\ J\ p]$	$ij_L \mathcal{Y}_{1LJ}^M(\hat{r}_k,\mathbf{p}_k)$

A similar relation holds for $[S\ L\ J\ p]$. Here $u_{J_i}^{M_i}$ and $u_{J_f}^{M_f}$ are nuclear wave functions of the initial and final states specified by the spin and its projection M. $\tau_-^{(k)}$ and Ω_k are the isospin and the operator for the kth nucleon. S, L, and J in brackets are the resultant spin, the effective orbital angular momentum, and the resultant total angular momentum of the lepton system, respectively. J also specifies the rank of the matrix elements. The symbol p means that the relevant matrix element includes the differential operator $\mathbf{p}$ acting on the nuclear wave function. The parity change is given by $(-)^L$ for $[S\ L\ J]$, and $(-)^{L+1}$ for $[S\ L\ J\ p]$. Nucleon operators Ω_k in Eq. (A1) are summarized in Table II, where the vector harmonics are defined by

$$\mathcal{Y}_{SLJ}^M(\hat{r},\boldsymbol{\sigma})=\sum_m (S\ L\ m\ M-m|J\ M)Y_{L\,M-m}(\theta,\varphi)\times\mathcal{Y}_{Sm}(\boldsymbol{\sigma}), \quad \text{(A2)}$$

with

$$\mathcal{Y}_{00}(\boldsymbol{\sigma})=(1/4\pi)^{1/2},$$
$$\mathcal{Y}_{10}(\boldsymbol{\sigma})=(3/4\pi)^{1/2}\sigma_z, \text{ etc.}$$

$\mathcal{Y}_{SLJ}^M(\hat{r},\mathbf{p})$ has a similar expression.

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Stretch Scheme, a Shell-Model Description of Deformed Nuclei

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A good angular-momentum wave function containing the maximum possible intrinsic angular momenta leads to a microscopic description of the nuclear rotational spectra in terms of spherical shell-model states. The rotational excitation energies arise from the residual two-body force. In the actual model calculations, the only approximation was a partial violation of the exclusion principle. The computed departures from the $I(I+1)$ law are consistent with experiment. Reasons are given for the preference of positive over negative intrinsic deformations.

I. THE STRETCH SCHEME

THE collective model has been extremely successful in giving a phenomenological description of the characteristics of deformed nuclei in terms of macroscopic coordinates associated with the degrees of freedom of an average well.[1] However, it is an outstanding problem of nuclear physics to understand deformed nuclei in terms of the nucleon motion and the two-body force. Up to now, this problem has been attacked in essentially three ways. The first is an attempt to solve the complete nuclear Hamiltonian approximately in as large as possible a configuration space. This method is practicable only for very light nuclei and calculations of this kind have been limited to the $1p$ shell.[2] The second consists in solving exactly an approximate Hamiltonian. The problem here is to obtain a good guess of a simple nuclear Hamiltonian which still is supposed to represent the essential nuclear features. In order to be exactly solvable, this Hamiltonian must be invariant under the symmetry transformation of some groups. So, for example, Elliott's model[3,4] is a nuclear Hamiltonian, invariant under $SU(3)$. The exact solutions are then representations of $SU(3)$. However, to obtain this result, the two-body force must be replaced by a separable, spin- and isospin-independent quadrupole force, and the average field must be an harmonic-oscillator potential. In the third method, the deformed orbital method,[5] one applies the variational principle to a trial wave function which is simple but violates rotational invariance. This way one treats an important part of the nucleon-nucleon interaction, namely the average field effect. Good angular-momentum states then must be obtained by projection. All these methods lead to spectra with rotational features. However, in all these approaches, the

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