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One-Pole Approximation in the Elastic Scattering of Composite Particles*

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We consider deuteron-alpha scattering, since it is a physical system in which the small separation between elastic and inelastic (breakup) thresholds can lead to observable consequences. Taking into account the three-body channel, we derive an approximate dispersion relation for the elastic amplitude and solve it in the one-pole approximation. We test its internal consistency by fitting the pole parameters to the Li^6 binding energy and the d-α scattering length in the $J^\pi=1^+$ channel, and to the position and width of the Li^6 resonance in the 3^+ channel. We then calculated the corresponding phase shifts and compared them with the experimental values, obtaining good agreement. The good agreement of this fit gives us confidence in our subsequent assessment of the qualitative influence of the nearby three-particle inelastic channel on the (two-body) elastic scattering amplitude.

THREE-BODY intermediate states occur in many problems in nuclear and particle physics. The proverbial difficulty of the three-body problem makes it hard to assess the qualitative importance of three-particle channels in two-body elastic scattering. A recent model calculation by Bronzan[1] indicates that in some circumstances a closed three-body channel can produce bound states and resonances even when the Born term is repulsive. Although the possibility of closed inelastic channels dominating the first-order interaction has long been known, few specific examples of this behavior have been discussed in the literature.

The case of deuteron-alpha scattering is interesting because of the small separation, $\epsilon=2.225$ MeV, between elastic and breakup thresholds. Furthermore, the next inelastic threshold (the He^3+H^3 channel) lies well above the breakup threshold, so that this system offers the possibility of observing the effect of three-body inelastic in a relatively unambiguous way. Finally, the d-α scattering amplitude is empirically known with some precision,[2] so comparison with experiment is possible.

We study the effect of the three-particle threshold on the analytic behavior of the elastic d-α scattering amplitude by means of a simple model. We begin with the unitarity relation for the elastic (on-shell) amplitude, which has the form[3,4] (for convenience, all particles are taken to be spinless)

$$\begin{aligned}\operatorname{Im}\langle\hat{K}'|A^+(E)|\hat{K}\rangle &= -\tfrac{1}{2}\pi(2M)^{3/2}(E+\epsilon)^{1/2}\theta(E+\epsilon)\\ &\times\int d\hat{K}''\langle\hat{K}'|A^-(E)|\hat{K}''\rangle\langle\hat{K}''|A^+(E)|\hat{K}\rangle\\ &-\pi\int d\mathbf{K}''\int d\mathbf{k}''\langle\hat{K}'|B^-(E)|\mathbf{K}''\mathbf{k}''\rangle\\ &\times\delta(E-k''^2/2\mu-K''^2/2M)\langle\mathbf{K}''\mathbf{k}''|B^+(E)|\hat{K}\rangle.\end{aligned}\quad(1)$$

In Eq. (1), $\mathbf{k}''$ and $\mathbf{K}''$ are the relative n-p momentum and the n-p total momentum relative to α, respectively, in the intermediate states. The reduced masses are

$$M=M_dM_\alpha/(M_d+M_\alpha)\,,\quad(2a)$$

$$\mu=M_nM_p/(M_n+M_p)\,.\quad(2b)$$

$\langle\hat{K}'|A^+(E)|\hat{K}\rangle$ is the elastic (d-α) on-shell scatterimg amplitude, and $\langle\mathbf{K}'\mathbf{k}'|B^\pm(E)|\hat{K}\rangle$ are, respectively, the normal or time-reversed amplitudes for the breakup reaction, in which the particle d, incident on α with momentum $\mathbf{K}$ and energy $E=K^2/2M-\epsilon$, breaks up into $n+p$, with relative momentum $\mathbf{k}'$ and total momentum $\mathbf{K}'$ (relative to α).[5] We approximate the 5-dimensional integral in Eq. (1) by assuming that the major contribution to it comes from states with low relative n-p momentum, i.e., k'' near zero. In order to avoid a divergent integral, we first make the transformation

$$|\mathbf{K}''|=(2ME'')^{1/2}\cos\lambda'',\quad |\mathbf{k}''|=(2\mu E'')^{1/2}\sin\lambda''.$$

The 5-dimensional integral is then given by

$$\begin{aligned}2\pi(4\mu M)^{3/2}E^2\theta(E)\int d\hat{K}''\int_0^{\pi/2}d\lambda''\sin^2\lambda''\cos^2\lambda''\\ \times\langle\hat{K}'|B^-(E)|\hat{K}''(2ME)^{1/2}\cos\lambda'',0\rangle\\ \times\langle\hat{K}''(2ME)^{1/2}\cos\lambda'',0|B^+(E)|\hat{K}\rangle.\end{aligned}\quad(3)$$

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[1] J. B. Bronzan, Phys. Rev. **154**, 1545 (1967).

[2] L. C. McIntyre and W. Haeberli, Nucl. Phys. **A91**, 382 (1967).

[3] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, New York, 1964), p. 201.

[4] J. V. Noble, Ph.D. thesis, Princeton University, 1966 (unpublished); J. V. Noble, Phys. Rev. **157**, 939 (1967).

[5] Our notation is identical with that of Goldberger and Watson (Ref. 3) throughout. The energy is measured from the three-body threshold in the barycentric system, and ϵ is the binding energy of d relative to $n+p$.

[This approximation is really quite reasonable: for small E, $|\mathbf{k}''|$ must be small anyway; for large E, there is experimental evidence that the protons from the $H^2(\alpha,\alpha')np$ reaction come off preferentially with small momentum relative to the neutron.[6]] We now approximate the breakup amplitude with $|\mathbf{k}''|=0$ by the off-shell elastic amplitude

$$\langle \mathbf{K}',0|B^{\pm}(E)|\mathbf{K}\rangle \simeq \xi\langle \mathbf{K}'|A^{\pm}(E)|\mathbf{K}\rangle\,, \qquad (4)$$

where ξ is a constant. This approximation follows from the exact operator relation[7]

$$\mathcal{B}(W)=\mathcal{A}(W)+t_{np}(W)G_0(W)\mathcal{A}(W)\,, \qquad (5)$$

where $\mathcal{B}(W)$ and $\mathcal{A}(W)$ are the formal scattering operators for breakup and elastic d-α scattering, $t_{np}(W)$ is the n-p scattering matrix, and $G_0(W)$ the free-particle Green's function.[4] Taking the appropriate matrix elements, and using a separable approximation[4] $t_{np}(W)\simeq -|v\rangle\hat{\tau}_d(W)\langle v|$, we find that $(E'=K'^2/2M)$

$$\begin{aligned}\langle \mathbf{K}',0|B^{+}(E')|\mathbf{K}\rangle &\equiv \lim_{k'\to 0}\int d\mathbf{k}\langle \mathbf{K}',\mathbf{k}'|\mathcal{B}(E'+i\eta)|\mathbf{K}\mathbf{k}\rangle\varphi_d(\mathbf{k})\\ &\simeq \int \langle \mathbf{K}',0|\mathcal{A}(E'+i\eta)|\mathbf{K},\mathbf{k}\rangle\varphi_d(\mathbf{k})d\mathbf{k}\\ &\quad -\langle 0|v\rangle\tau_d(0)\int d\mathbf{k}'\int d\mathbf{k}\;\varphi_d(\mathbf{k}')\left(\frac{k'^2+\kappa_d^2}{k'^2}\right)\\ &\qquad\times\langle \mathbf{K}',\mathbf{k}'|\mathcal{A}(E'+i\eta)|\mathbf{K},\mathbf{k}\rangle\varphi_d(\mathbf{k})\,. \end{aligned} \qquad (6)$$

Because κ_d^2 $(=R^{-2}$, where R is the deuteron radius) is small, $k^2\varphi_d(k)$ is peaked at small k (because of the diffuseness of the deuteron), and so we have

$$\int \langle \mathbf{K}',0|\mathcal{A}(E'+i\eta)|\mathbf{K},\mathbf{k}\rangle\varphi_d(\mathbf{k})d\mathbf{k} \simeq \langle \mathbf{K}'|A^{+}(E')|\mathbf{K}\rangle\left[\int d\mathbf{k}'\varphi_d^*(\mathbf{k}')\right]^{-1},$$

and hence

$$\langle \mathbf{K}',0|B^{+}(E')|\mathbf{K}\rangle \simeq \left\{\left[\int d\mathbf{k}\;\varphi_d^*(\mathbf{k})\right]^{-1}-\langle 0|v\rangle\tau_d(0)\right\}\times\langle \mathbf{K}'|A^{+}(E')|\mathbf{K}\rangle\,. \qquad (7)$$

[The contents of the braces are what we have called ξ in Eq. (4).] Putting expression (4) into expression (3) and taking partial-wave matrix elements

$$\langle \hat{K}'|A^{+}(E)|\hat{K}\rangle =\sum_l [(2l+1)/4\pi]K^{2l}A_l^{+}(E)P_l(\hat{K}'\cdot\hat{K})\,, \qquad (8)$$

one can write a partial-wave dispersion relation for the function $A_l^+(E)$. Writing $A_l^+(E)=N_l(E)/D_l(E)$, we obtain the usual relation

$$D_l(E)=1+\int_{-\epsilon}^{\infty} ds\;\rho_l(s)N_l(s)(s-E-i\eta)^{-1}\,, \qquad (9)$$

where

$$\rho_l(E)=M(2M)^{l+1/2}\left[(E+\epsilon)^{l+1/2}\theta(E+\epsilon)+\alpha_l E^{l+2}\theta(E)\right] \qquad (10)$$

and

$$\alpha_l=4\pi(2M)^{1/2}(2\mu)^{3/2}|\xi|^2\int_0^{\pi/2} d\theta\,\sin^2\theta(\cos\theta)^{2l+2}\,. \qquad (11)$$

Using the fact that $N_l(s)$ is analytic for $\mathrm{Re}(s)\geqslant -\epsilon$, we may write[8]

$$D_l(E)=R_l(E)+i\pi\rho_l(E)N_l(E) -M(2M)^{l+1/2}\alpha_l E^{l+2}N_l(E)\ln|E|\,, \qquad (12)$$

where $R_l(E)$ has no singularities for $\mathrm{Re}(E)\geqslant -\epsilon$. [In fact, if the left-hand cut is approximated by poles, the function $p_l(E)=R_l(E)/N_l(E)$ turns out to be a polynomial in E.] It is clear that $p_l(E)$ is meromorphic in the E plane excluding the left-hand cut. The radius of analyticity of $N_l(E)p_l(E)$ at $E=-\epsilon$ will be the distance between the left-hand branch point and the elastic threshold, which is typically larger than ϵ. (For d-α scattering it is around 6 MeV.) Furthermore, the analytic properties of $R_l(E)$ turn out to be independent of the number of subtractions needed to make the dispersion integral converge, so this result is general. What we have found is completely analogous with the situation in potential scattering, in which partial-wave dispersion relations lead to the construction of a function of the energy analytic at the elastic threshold, and whose nearest singularity is the left-hand branch point. This function is what is usually written $k^{2l+1}\cot\delta_l(k)^3$; the expansion of this function in a power series (in energy) is the usual effective-range expansion. Application of the two-body effective range formalism to the d-α problem is limited, because the radius of convergence of the effective-range expansion is only ϵ. In our model, the partial-wave amplitude may be written $[E=(K^2/2M)-\epsilon$, and we have let $K^{2l}A_l^+(E)\equiv a_l^+(E)]$

$$a_l^+(E)=K^{2l}\left[p_l(E)-M(2M)^{l+1/2}\alpha_l E^{l+2}\ln|E| +i\pi\rho_l(E)\right]^{-1}. \qquad (13)$$

The generalized effective-range expansion is obtained by expanding $p_l(E)$ about $E=-\epsilon$ and keeping only the two lowest-order terms:

$$p_l(E)\simeq \pi M\left[1/a_c(l,\xi)+\tfrac{1}{2}r_e(l,\xi)K^2\right]. \qquad (14)$$

This approximation may be useful, as it exactly satisfies

[6] K. Nagatani, T. A. Tombrello, and D. A. Bromley, Phys. Rev. **140**, B824 (1965). Examination of the proton spectra given by these authors, and comparison with the phase space normalized to the same area indicates this correlation effect. There are also good theoretical reasons why this must be so. See Ref. 7 for details.
[7] P. M. Fishbane and J. V. Noble (unpublished).

[8] This result follows straightforwardly from the assumptions; we shall include the details in our forthcoming article (Ref. 7).

TABLE I. Comparison of s-wave phase shift derived from Eq. (13) using one-pole calculation of $p_0(E)$, with experimental $\delta_0^1(E)$ given in Ref. 2. For a discussion of the uncertainties in the experimental values, see Ref. 2.

E_{Lab} (MeV)	δ_0^1 (theory) (deg)	δ_0^1 (experimental) (deg)
2.0	118.0	126.4
2.5	112.5	116.5
3.0	108.5	109.2
3.5	105.0	104.4
4.0	102.0	106.0
4.5	99.5	68.0
5.0	98.0	84.6
6.0	94.0	69.9
7.0	91.5	78.4
8.0	89.0	84.8
9.0	87.0	83.6
10.0	85.0	80.5

TABLE II. Comparison of d-wave phase shift derived from Eq. (13) using one-pole calculation of $p_2(E)$, with experimental $\delta_2^3(E)$ given in Ref. 2.

E_{Lab} (MeV)	δ_2^3 (theory) (deg)	δ_2^3 (experimental) (deg)
2.0	175.0	171.7
2.5	173.5	171.5
3.0	172.0	167.6
3.5	170.0	169.6
4.0	168.0	172.4
4.5	165.5	180.0
5.0	163.0	164.5
6.0	155.5	159.7
7.0	145.5	156.5
8.0	131.5	154.6
9.0	116.5	156.8
10.0	98.0	154.0

two-body unitarity, and as we have seen, approximately satisfies three-body unitarity. Putting $\xi=0$ in (13) and (14), we recover the ordinary effective-range expansion. If $|\xi|^2$ is taken to be a free parameter, the generalized effective-range expansion [Eq. (13) together with (14)] is a three-parameter fit to the scattering amplitude.

We now discuss the application of our model to the analysis of the properties of the deuteron-α system. Restricting ourselves to a spinless formulation (note that spin can easily be included via the matrix ND^{-1} method[9]), we consider any physical J^+ channel to have definite $L=J-1$. The lowest-lying states in Li^6 are the $T=0$, $J^\pi=1^+$ level at -3.697 MeV below the breakup threshold (ground state) and the $T=0$, $J=3^+$ resonant state at -1.513 MeV (with $\Gamma=21$ keV). [The $J=0^+$, $T=1$ state at -0.137 MeV is clearly irrelevant to d-He^4 elastic scattering.] The parameter ξ can be evaluated assuming a Hulthén form for the deuteron wave function [see Eq. (7)].[10] Since the dimensions of $|\xi|^2$ are those of volume, it is most natural to consider $|\xi|^2/R^3$, with R the deuteron radius; this number turns out to be $\simeq 0.02$. To complete the specification of our model scattering amplitude, we solved the ND^{-1} equations in the one-pole[11] approximation, with sufficient subtractions to obtain convergence. These subtractions were chosen in such a way that they did not increase the number of free parameters beyond the two from the pole approximation.

Equation (11) gives (s waves) $\alpha_0\simeq 0.014$, seemingly indicating that the contribution from the three-body threshold is negligible. Actually, this is not the case: Because of the logarithmic singularity, the three-body channel contributes a substantial attractive force below threshold. In order to assess the qualitative importance of the three-body closed channel, we fitted the binding energy and scattering length in the 1^+ channel with the two parameters of the pole approximation. We compared our 1^+ phase shift [derived from Eq. (13) with $l=0$] with the $J^\pi=1^+$ (lower) eigenphase shift given by McIntyre and Haeberli.[2] The results are contained in Table I. We then took $|\xi|=0$, and using the previously determined pole parameters, recalculated the 1^+ amplitude below the elastic threshold. We found that the bound state shifted from -3.697 to -3.12 MeV. This represents a change in binding energy of $\sim 40\%$ (as measured from the elastic threshold) which corresponds to a change of $\sim 15\%$ in the strength of an effective Hulthén potential with range corresponding to the position of the left-hand pole. In other words, the closed three-body channel contributes about 15% to the attraction of the effective d-α potential.

The 3^+ resonance in the $L=2$ partial wave is much closer to the inelastic threshold and should therefore be much more sensitive to its presence or absence than was the 1^+ state. We found, when we fit the position and width of the 3^+ resonance with our pole approximation, that this was indeed the case. Putting $|\xi|=0$ and keeping the fitted values of the pole parameters led to the complete disappearance of this resonance (we searched up to 2.5 MeV). We also compared our 3^+, $L=2$ phase shift with the experimentally determined phase shift of Ref. 2. This comparison is presented in Table II.

Previous work[12] on Coulomb perturbation of weakly bound states and investigations currently being pursued by one of us (JVN) indicate that correct inclusion of the Coulomb repulsion between He^4 and the proton does not substantially modify our results or our conclusions.

Inherent in our treatment is the possibility of calculating the breakup amplitude using an Omnes-type equation. One can also more fully parametrize the partial-wave elastic amplitude to include both eigenphase shifts and the mixing parameter. Work on these extensions is currently in progress.

We wish to thank Professor R. D. Amado for much advice on the organization of this manuscript, and Professor W. Haeberli for communicating to us a table of his phase shifts.

[9] J. D. Bjorken, Phys. Rev. Letters 4, 473 (1960).
[10] Y. Yamaguchi, Phys. Rev. 95, 1628 (1954).
[11] See, e.g., S. C. Frautschi, *Regge Poles and S-Matrix Theory* (W. A. Benjamin and Company, Inc., New York, 1963), p. 5 ff.
[12] J. V. Noble, Phys. Rev. 148, 1528 (1966).