Effective Central Interaction for Nuclear Hartree-Fock Calculations*

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(Received 14 March 1967)

Previous successful finite-nucleus Hartree-Fock calculations, such as those of the Baranger group, have required of the effective two-nucleon interaction simply that it correctly saturate nuclear matter in first-order perturbation theory, with the second-order correction being small. In an attempt to relate the effective central interaction to the real nucleon-nucleon force, we follow the spirit of the separation method and insist that the two be identical in the long-range region. As for the short-range region, we argue that the considerable uncertainty in this part of the real force does not justify the application of a reaction-matrix theory which itself is not completely unequivocal in this region. Instead, the short-range part of our interaction is purely phenomenological, determined by still requiring that the total interaction satisfy the nuclear-matter condition, a condition which in any case appears from the work of Bhaduri and Tomusiak to be necessary for correct results in finite nuclei. The role in finite-nucleus Hartree-Fock calculations of different effective interactions is then investigated by calculating the binding energy and rms radius of O^{16} and Ca^{40} in the well-tested approximation of their wave functions being pure oscillator in form. It is found that interactions that give identical saturation of nuclear matter are not necessarily completely equivalent in finite nuclei. However, requiring that the effective interaction conform in the long-range region to the real force leads to finite-nucleus results that are just as acceptable as those given by phenomenological interactions fitted to the same nuclear-matter saturation point but bearing no apparent relation to the real force.

I. INTRODUCTION

NUCLEAR Hartree-Fock calculations have become a practical proposition with the advent of the method of reiterative diagonalization of the energy matrix in an oscillator basis.[1] One such successful calculation is that of the Baranger group,[2] who took a highly simplified two-body interaction adjusted to give satisfactory results in a Hartree-Fock calculation of nuclear matter. That is to say, the interaction was chosen to give the correct nuclear-matter saturation properties in the first order of perturbation theory and small second-order correction terms. Without any further adjustment, this same interaction was then found to give reasonable results in Hartree-Fock calculations on several finite nuclei.

Actually, the nuclear-matter condition is a natural criterion to impose on the effective interaction for Hartree-Fock calculations, since it follows by definition that any effective interaction which pretends to be *the* correct one, insofar as it is meaningful to speak of such, must not only be equally applicable to all finite nuclei without *ad hoc* adjustment of parameters but must remain valid as one passes to the limiting case of nuclear matter. Indeed, it has been shown explicitly[3] that an effective interaction that does not saturate nuclear matter correctly gives rise to serious difficulties in nuclei with $A>40$.

Passing from the question of necessity to that of sufficiency, it is noted that the Hartree-Fock calculation on nuclear matter will not define a *unique* effective interaction, so that it is conceivable that there exist other interactions giving similar results for nuclear matter but differing in finite nuclei. To resolve such questions and to increase one's general confidence in the physical significance of the Hartree-Fock method, it is clearly desirable that one be able to derive the smooth effective interaction in a unique way from the real nucleon-nucleon interaction, which is singular enough to induce short-range correlations in the real wave function.[4]

One prescription for extracting an effective interaction from the real one is provided by the canonical transformation of Villars,[5] which has in fact already served as the basis for a Hartree-Fock calculation.[6] Another approach is via reaction-matrix theory.[7] It is, in fact,

* Work supported by the National Research Council of Canada.
[1] M. Baranger, in *Cargèse Lectures in Theoretical Physics*, edited by M. Lévy (W. A. Benjamin, Inc., New York, 1963), Chap. 5, p. 29.
[2] K. T. R. Davies, S. J. Krieger, and M. Baranger, Nucl. Phys. 84, 545 (1966). Referred to hereafter as DKB.
[3] R. K. Bhaduri and E. L. Tomusiak, Nucl. Phys. 88, 353 (1966).
[4] We are adopting here the point of view that the real nucleon-nucleon force does indeed induce correlations and hence cannot be identical to the effective Hartree-Fock interaction. However, it has recently been shown [F. Tabakin and K. T. R. Davies, Phys. Rev. 150, 793 (1966)] that an interaction with a Gaussian velocity dependence can be fitted to the nucleon-nucleon data and at the same time give the correct saturation properties for nuclear matter in Hartree-Fock approximation, the second-order terms being small. The question as to whether or not such a smooth interaction may be regarded as the "real" one can only be decided by assessing the experimental evidence for the existence of correlations: The historical argument that the nucleon-nucleon data alone imply a correlation-producing singularity is no longer valid. Should the evidence for correlations be accepted as conclusive then the potential of Tabakin and Davies would have to be regarded simply as an effective and not as the real interaction. But in this event the imposition of the phase parameter criterion would be difficult to justify: While the smooth effective interaction should certainly be related to the real singular one it is by no means obvious that they should yield the same free nucleon-nucleon scattering, except perhaps at low energies.
[5] F. Villars, in *Proceedings of the Enrico Fermi International School of Physics, Course XXIII, 1961* (Academic Press Inc., New York, 1963).
[6] C. M. Shakin, J. Svenne, and Y. R. Waghmare, Phys. Letters 21, 209 (1966).
[7] T. T. S. Kuo and G. E. Brown, Nucl. Phys. 85, 40 (1966).

a highly simplified form of this, namely, the separation method of Moszkowski and Scott[8] in first order, that serves as the basis of the present work.

To describe this method very roughly, the suppression of the short-range correlations, which is implicitly made in the Hartree-Fock method, can be compensated by removing the highly singular short-range part of the potential, which gives rise to the correlations. Then, since the long-range part of the interaction remaining after this separation is smooth, its contribution may be calculated by ordinary first-order perturbation theory. Thus, according to the argument, this long-range part of the real interaction should constitute a reliable effective interaction for Hartree-Fock calculations.

Unfortunately, this prescription for extracting an effective interaction from the real interaction is incomplete, as Bhaduri and Tomusiak[3] have emphasized. In the first place, the theory of Moszkowski and Scott requires that the distance of separation vary slightly with energy, with the result that the corresponding long-range residue is non-Hermitian when taken by itself. At the same time, the strongly singular short-range part of the force gives rise to second-order corrections to the Moszkowski-Scott theory which, although small, do play an important role in determining the correct saturation of nuclear matter,[9] since there is a considerable cancellation between the mean interaction energy and the kinetic energy.

But just because these short-range effects are so small in comparison with the rest of the interaction energy, it should be possible to simulate them by replacing the singular short-range part of the real force with a new short-range force that is smooth enough for its effect to be calculated in ordinary perturbation theory. At the same time this new short-range force may perform the second function of simulating the energy dependence of the separation distance. It is the combination of this smooth short-range force and the long-range part of the real interaction remaining after separation that we propose as an effective interaction for Hartree-Fock calculations.

In contrast to our treatment of the long-range part of the interaction, we do not offer any prescription for relating the short-range part to the free nucleon-nucleon interaction. *A priori*, it is quite arbitrary and we determine it in fact by returning once more to the nuclear-matter criterion, i.e., we adjust the short-range term such that the *total* interaction saturates nuclear matter correctly in Hartree-Fock approximation, while keeping the second-order term of ordinary perturbation theory sufficiently small.

Thus we abandon the attempt to derive the *entire* effective interaction from the real one: Only the long-range part is so determined. On the other hand it is to be noted that reaction-matrix theory is not free from uncertainty in the short-range region. Furthermore, the short-range part of the real nucleon-nucleon interaction itself is by no means uniquely determined. Hence, perhaps it may be said of the present program that it relates the effective interaction to the real one insofar as it is possible to do this without ambiguity.

We describe the construction of our effective interaction in Sec. II. Two consequences follow from our being forced back on to the nuclear-matter criterion for completing the definition of the effective interaction. Firstly, we shall only be able to establish an effective central interaction, since the vector (i.e., two-body spin-orbit) and tensor components make no contribution to nuclear matter in Hartree-Fock approximation. Secondly, since the form of the short-range term that we add is completely arbitrary, we shall still not arrive at a *unique* effective interaction, although the ambiguity is much less than before imposing the long-range condition on the interaction.

The consequences for finite nuclei of the imposition of this long-range condition on the effective interaction and of the residual uncertainty are discussed in Sec. III, where we consider the closed-shell cases of O^{16} and Ca^{40}. Since all we are concerned with is a *comparison* of the different effective interactions, we deemed it unnecessary to make complete Hartree-Fock calculations on these nuclei. Rather, we have limited ourselves to pure oscillator wave functions, treating the oscillator strength as a variational parameter.[10]

II. EFFECTIVE INTERACTION

For the real nucleon-nucleon interaction in singlet-even states we take a potential containing a hard core of radius 0.4 F, a one-pion exchange potential (OPEP) tail ($g_\pi^2=14.8$), and an intermediate-range Yukawa term adjusted to fit with precision[11] the ^1S phase shifts of the Livermore analysis[12] and the low-energy parameters

$$\begin{aligned} {}^1V_{\text{real}}+ &= \infty, && r<0.4\text{ F} \\ &= -466\frac{e^{-1.95r}}{r}\text{ MeV}+\text{OPEP}, && r>0.4\text{ F}. \end{aligned} \quad (1)$$

Since we do not take account of the tensor force in this work, the simplest assumption for the triplet-even states is to regard the potential (1) as spin-independent (this is certainly better than having no triplet-even interaction at all).

The separation distance was found to be $d=1.00$ F

[8] S. A. Moszkowski and B. L. Scott, Ann. Phys. **11**, 65 (1960).
[9] These corrections also restore the missing Hermiticity to the theory.
[10] A preliminary version of this work was presented at the Annual Meeting of the American Physical Society, New York, 1967. See J. M. Pearson and Gérard Saunier, Bull. Am. Phys. Soc. **12**, 47 (1967).
[11] J. M. Pearson, Can. J. Phys. **45**, 1289 (1967).
[12] R. A. Arndt and M. H. MacGregor, Phys. Rev. **141**, 873 (1966).

TABLE I. Interactions 1 to 5 represent the phenomenological components of effective interactions saturating nuclear matter according to Eq. (3). The form of this phenomenological term, which is combined with the basic realistic interaction described in Sec. II, is given by Eq. (2). The parameters A (F^{-2}), α (F^{-1}), B, and β (F^{-1}) are listed in columns 2 to 9. (The even states are spin-independent, and there is no velocity-dependent term for the odd states.) The last column gives the second-order contribution (in MeV) to nuclear matter at saturation. The last interaction, labeled "Test," also saturates nuclear matter according to Eq. (3), and has the form given by Eq. (2), but acts alone, not being combined with the basic realistic interaction.

Interaction	Even states A	α	B	β	Singlet-odd states A	α	Triplet-odd states A	α	$\Delta e^{(2)}$
1	−0.866	0.63	0.56	0.60	0.077	0.71	0.0086	0.71	−2.0
2	−1.45	0.77	0.56	0.60	−0.016	0.71	−0.0018	0.71	−1.5
3	−3.04	1	0.76	0.67	0.722	0.77	0.080	0.77	−1.5
4	−2.85	1	0.57	0.59	−0.365	0.77	−0.041	0.77	−1.5
5	−8.24	1.41	0.91	0.69	0.52	0.77	0.06	0.77	−1.9
Test	−2.3	0.77	0.49	0.55	0	···	0	···	−1.1

for the ^1S state at low energies. This value was adopted for all energies and in all even states, the short-range part of the interaction defined in this way, containing the hard core and the deepest part of the attractive pocket that lies beyond, then being discarded.

The simplest assumption for the odd states is the Serber one, but to increase the reality of our interaction we took the OPEP for these states, this giving a very rough fit to the scattering data. Being smooth, a separation of this potential is unnecessary (indeed, it is impossible, because the potential is monotonic).

The basic interaction consists, then, of the long-range residue of potential (1) for the even states and pure OPEP for the odd states. To this we add in all states[13] a short-range phenomenological potential having quadratic velocity dependence,

$$V_{\text{phen}}=\frac{\hbar^2}{M}Ae^{-\alpha^2r^2}+\frac{B}{M}(p^2e^{-\beta^2r^2}+e^{-\beta^2r^2}p^2), \qquad (2)$$

where M is the nucleon mass and the parameters A, B, α, and β are adjusted such that the *entire* interaction saturates nuclear matter at

$$e=-16.3 \text{ MeV}, \quad k_F=1.35 \text{ F}^{-1} \qquad (3)$$

in first-order perturbation theory.

The results of perturbation theory for nuclear matter are consigned to the Appendix. Construction of the first-order saturation curve for a given interaction involves simply the evaluation of the radial integral in Eq. (A9). The second-order correction requires the calculation of the quadruple integral in Eq. (A11). Using Gauss quadrature, this could be obtained to about 5% precision within approximately one minute on the CDC 3400 computer.

The range of possibilities for effective interactions satisfying the nuclear-matter criterion (3) is considerably limited by the conflict between the requirement that the second-order term be small and the condition that V_{phen} be of short range: Our calculation would lose its point if the phenomenological term were to dominate the realistic long-range component. Fortunately, these two conditions are not irreconcilable, and the potentials presented in Table I are representative of what is possible. In particular, it seems that one cannot reduce the ranges of the two terms of V_{phen} below the values of case No. 5 without raising the second-order correction beyond 2.0 MeV, which we regard as the maximum acceptable.[14]

III. RESULTS FOR FINITE NUCLEI

The binding energy and rms radius of the closed-shell nuclei O^{16} and Ca^{40} are now calculated for each of the interactions listed in Table I. Instead of performing a complete Hartree-Fock calculation, where the forms of the radial parts $R_{nl}(r)$ of the single-particle wave functions are varied in the search for the energy minimum, we restrict ourselves to the oscillator form, varying only the oscillator strength parameter $a=(M\omega/\hbar)^{1/2}$. This is equivalent to using the Baranger method[1] with only one oscillator term in the expansion, so that no reiteration will be required.

With the assumption of an oscillator form the radial wave functions become[15]

$$R_{nl}(r)=\left[\frac{2^{l-n+2}(2l+2n+1)!!n!a^{(2l+3)/2}}{\pi^{1/2}}\right]^{1/2} \times r^l\exp(-ar^2/2)\sum_{k=0}^{n}\frac{(-2ar^2)^k}{(n-k)!k!(2l+2k+1)!!}, \qquad (4)$$

where n and l are the radial and orbital angular-momentum quantum numbers, respectively. The nor-

[13] The short-range phenomenological terms in the odd states will not have the same interpretation in terms of Moszkowski-Scott theory as do the corresponding terms in the S state. It is only in this latter state that there exists strong evidence for a correlation-producing singularity. We have included such terms in the odd states simply in order to widen the range of phenomenological possibilities while maintaining long-range conformity.

[14] Whatever the range of the static part of V_{phen}, the velocity-dependent term seems to be essential to guarantee saturation with small second-order corrections.

[15] See, for example, R. D. Lawson and M. Goeppert-Mayer, Phys. Rev. **117**, 174 (1960).

malization is defined according to

$$\int_0^\infty R_{nl}^2(r) r^2\, dr = 1. \tag{5}$$

The general expression

$$E = \langle \Psi | \sum_{i=1}^{A} t_i + \sum_{i>j}^{A} v_{ij} | \Psi \rangle \tag{6}$$

for the total nuclear energy, t_i being the kinetic-energy operator and v_{ij} the two-body interaction, reduces for closed-shell "oscillator" nuclei (with equal numbers of neutrons and protons) to

$$E = 2\hbar\omega \sum_{n_i,l_i} (2n_i + l_i + \tfrac{3}{2})(2l_i+1) + \sum_{n_i,l_i,n_j,l_j} \sum_{nl N\Lambda L} (2L+1)\{\langle nl,N\Lambda,L | n_i l_i, n_j l_j, L\rangle\}^2 \times \langle nl \| V_l \| nl \rangle. \tag{7}$$

Here the summation of the quantum numbers n_i, l_i, n_j and l_j goes over all closed shells. In the second term, which represents the potential energy, we have made the usual Talmi transformation[16] of the two-particle states,

$$|n_i l_i, n_j l_j, L\rangle = \sum_{nlN\Lambda} \langle nl,N\Lambda,L | n_i l_i, n_j l_j, L\rangle | nl, N\Lambda, L\rangle, \tag{8}$$

where n and l are the quantum numbers of the relative oscillator motion of the two nucleons, N and Λ are the corresponding quantities for the center-of-mass motion, and L is the total orbital-angular momentum of the pair. The coefficients of the transformation are tabulated.[17] The potentials V_l appearing in the radial matrix elements $\langle nl \| V_l \| nl \rangle$ are just the $V_\pm$ of Eq. (A3), according to whether l is even or odd, respectively.[18]

The rms radius of an oscillator nucleus is given approximately by[19]

$$R \simeq (0.93/a) A^{1/6}. \tag{9}$$

To compare the oscillator approximation with a full-scale Hartree-Fock calculation, we applied it first to the interaction used by DKB. It will be seen from Table II that there is excellent agreement. This is probably fortuitous to some extent, and we find it difficult to believe that such close agreement would be found for all nuclei and all acceptable effective interactions. Nevertheless, there can be no doubt that the oscillator approximation does permit a valid comparison of different effective interactions and should indicate at least qualitatively the effect in a complete Hartree-Fock calculation of a modification of the effective interaction.

TABLE II. Comparison of the complete Hartree-Fock calculation of DKB with the "oscillator" approximation for the same interaction. Calculated are the energy e per nucleon (MeV) and the rms radii R (F) for O^{16} and Ca^{40}.

Calculation	O^{16} e	O^{16} R	Ca^{40} e	Ca^{40} R
Hartree-Fock	−5.1	2.91	−7.1	3.47
Oscillator	−4.8	2.9	−7.1	3.4

TABLE III. Results for finite nuclei in the "oscillator" approximation. Shown are the equilibrium values for the energy e per nucleon (MeV), the oscillator parameter $a = (M\omega/\hbar)^{1/2}$ (F)$^{-1}$, and the equivalent rms radius R (F). Experimental data are presented in the last line.

Interaction number	O^{16} e	O^{16} a	O^{16} R	Ca^{40} e	Ca^{40} a	Ca^{40} R
1	−6.2	0.54	2.7	−8.5	0.51	3.4
2	−7.0	0.57	2.6	−9.2	0.52	3.3
3	−7.1	0.57	2.6	−9.6	0.52	3.3
4	−7.1	0.60	2.5	−9.6	0.53	3.3
5	−7.2	0.60	2.5	−9.6	0.53	3.3
Test	−6.8	0.59	2.5	−9.3	0.54	3.3
Experiment[a]	−7.98		2.64	−8.55		3.52

[a] The binding-energy data come from A. H. Wapstra, Physica **21**, 367 (1955); **21**, 385 (1955). The experimental radii are given by R. Hofstadter, Ann. Rev. Nucl. Sci. **7**, 231 (1957).

We show in Table III the results of oscillator calculations on O^{16} and Ca^{40} with each of the interactions of Table I. It is seen that while all these interactions give identical saturation of nuclear matter, they are not completely equivalent in finite nuclei. However, the differences are quite small and tend to vanish altogether as the range of the phenomenological term is shortened, an observation which may be taken to indicate that it is the long-range part of the interaction that is of dominant importance in the Hartree-Fock method.

While the agreement of all these results with experiment is seen to be good, it might be unwise to attach too much importance to this in view of our incomplete treatment of the tensor interaction. Of more significance is a comparison of the results with an effective interaction that has been fitted to exactly the same saturation point of nuclear matter but without having long-range conformity to the real interaction. For this we take a spin-independent Serber potential having the same form as the phenomenological components of our other interactions, as given by Eq. (2).

We refer to this potential as "Test" in Tables I and III. From the latter it is seen that the imposition of the long-range condition on the interaction introduces no significant differences for finite nuclei.[20] Indeed, comparison with the results for the DKB interaction (Table II), suggests that the form of the interaction is

[16] I. Talmi, Helv. Phys. Acta **25**, 185 (1952); M. Moshinsky, Nucl. Phys. **13**, 104 (1959).

[17] T. A. Brody and M. Moshinsky, *Table of Transformation Brackets* (Monografias del Instituto de Fisica, Mexico, 1960).

[18] For the treatment of the handling of the velocity-dependent component in the matrix element see, for example, B. H. J. McKellar, Phys. Rev. **134**, B1190 (1964).

[19] S. A. Moszkowski, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 469.

[20] *Note added in proof.* Subsequent calculation [J. M. Pearson and G. Saunier (unpublished)] shows that this agreement is to some extent fortuitous.

of less importance than the precise position of the saturation point of nuclear matter to which the interaction is fitted, since in place of Eq. (3) DKB took $e=-15.5$ MeV, $k_F=1.42$ F.

IV. CONCLUDING REMARKS

We have shown that it is possible to make an effective central two-nucleon interaction conform to the real two-nucleon interaction in its long-range part while continuing to saturate nuclear matter correctly in first-order approximation with a small second-order correction. Imposing this long-range conformity on the effective interaction is probably as far as one can reasonably go in relating it to the real nucleon-nucleon interaction: the fact that the short-range part of this is by no means well determined seems to make it somewhat pointless to apply to it a complicated reaction-matrix theory which is itself not completely unambiguous. Indeed, it is conceivable that an effective interaction derived in a completely *a priori* way from a supposedly real interaction would fail to saturate nuclear matter correctly, an apparently necessary condition for successful Hartree-Fock calculations in finite nuclei.

The two criteria of long-range conformity and the fit to nuclear matter do not, however, determine a unique effective interaction, but we believe that the five interactions, the short-range phenomenological parts of which are displayed in Table I, span the full range of possibilities for a quadratic velocity dependence, both for the even and odd states. Using then the well-tested oscillator approximation, we found that all these interactions gave comparable and reasonable results for the binding energies and sizes of O^{16} and Ca^{40}. It is therefore conceivable that further refinement of reaction-matrix theory and of our knowledge of the short-range part of the nucleon-nucleon interaction would not lead to any significant changes in Hartree-Fock calculations, at least as far as the central part of the interaction is concerned.

While it may be said that the ability to use realistic interactions gives added physical significance to the Hartree-Fock method, it is still a great convenience to be able to use simple Gaussian-shaped interactions, especially in calculations on deformed nuclei. Our having shown that these give essentially the same results, at least in spherical nuclei, as our own more realistic interactions gives one increased confidence in their use, although, of course, their validity should be checked in deformed nuclei as well.

ACKNOWLEDGMENTS

Dr. Jean LeTourneux is thanked for discussions and a critical reading of the manuscript of this paper. We wish also to thank Mlle. Lucille Roy of the Centre de Calcul at the Université de Montréal for her cooperation and generous scheduling of computer time. Saunier acknowledges the award of a Canada Council scholarship and Pearson that of a post-doctorate fellowship of the National Research Council of Canada. The latter author also wishes to express his appreciation of the hospitality extended by Professor Maurice Jean of the Laboratoire Joliot-Curie during the period in which this work was initiated.

APPENDIX: PERTURBATION THEORY FOR NUCLEAR MATTER

We first explain some points of notation. The central two-nucleon interaction V_{ST} of the different spin-isospin states consists of static and dynamic terms, thus:

$$V_{ST}=V_{ST}{}^{s}(r)+\frac{1}{\hbar^2}[f(p^2)V_{ST}{}^{d}(r)+V_{ST}{}^{d}(r)f(p^2)]. \quad \text{(A1)}$$

Then the total direct and exchange interactions,

$$V_{\rm dir}=V_{+}+V_{-}, \quad V_{\rm exc}=V_{+}-V_{-}, \quad \text{(A2)}$$

where

$$V_{+}=3(V_{01}+V_{10}), \quad V_{-}=V_{00}+9V_{11}, \quad \text{(A3)}$$

can be decomposed likewise, e.g.,

$$V_{\rm dir}=V_{\rm dir}{}^{s}(r)+\frac{1}{\hbar^2}[f(p^2)V_{\rm dir}{}^{d}(r)+V_{\rm dir}{}^{d}(r)f(p^2)], \quad \text{(A4)}$$

and similarly for $V_{\rm exc}$, V_{+} and V_{-}.

The energy per nucleon in nuclear matter of density corresponding to Fermi momentum k_F may be written as the sum of the unperturbed kinetic energy and the mean interaction energy,

$$e=\frac{3}{10}\frac{\hbar^2}{M}k_F{}^2+\bar{v}(k_F). \quad \text{(A5)}$$

In first-order perturbation theory, $\bar{v}(k_F)$ is given simply as the sum over all states below the Fermi surface of diagonal plane-wave matrix elements of the total interaction,

$$\bar{v}(k_F)=\frac{1}{2n}\sum_{i,j<k_F}(V)_{ij,ij}, \quad \text{(A6)}$$

where n is the number of nucleons per unit volume. Then

$$\bar{v}(k_F)=\frac{2}{\pi}\int_0^{k_F}\left(k^2-\frac{3}{2}\frac{k^3}{k_F}+\frac{1}{2}\frac{k^5}{k_F{}^3}\right)I(k)dk, \quad \text{(A7)}$$

where

$$I(k)=\int_0^{\infty}\left[V_{\rm dir}{}^{s}+\frac{\sin 2kr}{2kr}V_{\rm exc}{}^{s}+2f(k^2)\left(V_{\rm dir}{}^{d}+\frac{\sin 2kr}{2kr}V_{\rm exc}{}^{d}\right)\right]r^2dr. \quad \text{(A8)}$$

For the case of quadratic momentum dependence, $f(p^2)=p^2$, the integration over k can be performed analytically and we obtain

$$\bar{v}(k_F)=\frac{1}{\pi}\int_0^\infty dr\left\{\frac{k_F{}^3r^2}{12}V_{\rm dir}{}^s+\frac{k_F{}^5r^2}{20}V_{\rm dir}{}^d+\left[\frac{3}{8}\left(\frac{1}{k_Fr^2}+\frac{1}{k_F{}^3r^4}\right)-\frac{3}{4}\frac{\sin 2k_Fr}{k_F{}^2r^3}+\frac{3}{8}\left(\frac{1}{k_Fr^2}-\frac{1}{k_F{}^3r^4}\right)\cos 2k_Fr\right]V_{\rm exc}{}^s\right.$$
$$\left.+\left[-\frac{9}{4k_Fr^4}\left(1+\frac{5}{2k_F{}^2r^2}\right)+\frac{15}{4r^3}\left(\frac{3}{k_F{}^2r^2}-1\right)\sin 2k_Fr+\left(\frac{45}{8}\frac{1}{k_F{}^3r^6}-\frac{9}{k_Fr^4}+\frac{3}{4}\frac{k_F}{r^2}\right)\cos 2k_Fr\right]V_{\rm exc}{}^d\right\}, \quad \text{(A9)}$$

a result which is valid for arbitrary radial forms of the potentials.

For the second-order correction to this result we have

$$\Delta e^{(2)}=\frac{1}{4n}\sum_{i,j<k_F}\sum_{i',j'>k_F}\frac{|(V)_{ij,i'j'}|^2}{\epsilon_i+\epsilon_j-\epsilon_{i'}-\epsilon_{j'}}, \quad \text{(A10)}$$

in which the plane-wave elements of the total interaction are summed over all particle states i, j below the Fermi surface and over all particle states i', j' above the Fermi surface. The quantities ϵ_i, etc., represent the unperturbed single-particle energies: $\epsilon_i=\hbar^2k_i^2/2M$.

The angular integrations over momentum space may be performed analytically. Limiting ourselves to central forces, which we suppose to give negligible contribution for $l>1$, we have

$$\Delta e^{(2)}=\frac{9M}{\pi^2k_F{}^3}\int_0^{2k_F}K^2dK\left[\int_0^{(k_F{}^2-K^2/4)^{1/2}}dk\int_{(k_F{}^2-K^2/4)^{1/2}}^\infty dk'xy(G_{10}+G_{01}+G_{00}+9G_{11})\right.$$
$$\left.+\frac{1}{2}\int_{k_F-K/2}^{(k_F{}^2-K^2/4)^{1/2}}dk\int_{(k_F{}^2-K^2/4)^{1/2}}^{k_F+K/2}dk'xy(1+x^2y^2-x^2-y^2)(G_{10}+9G_{11})\right]. \quad \text{(A11)}$$

Here we have defined

$$x=\frac{k_F{}^2-k^2-\frac{1}{4}K^2}{kK}, \quad k>k_F-\tfrac{1}{2}K$$
$$=\quad 1\quad , \quad k<k_F-\tfrac{1}{2}K, \quad \text{(A12a)}$$

and

$$y=\quad 1\quad , \quad k'>k_F+\tfrac{1}{2}K$$
$$=\frac{k'^2+\frac{1}{4}K^2-k_F{}^2}{k'K}, \quad k'<k_F+\tfrac{1}{2}K. \quad \text{(A12b)}$$

Also

$$G_{ST}=\frac{k^2k'^2}{k^2-k'^2}\left[\int_0^\infty j_l(kr)j_l(k'r)\{V_{ST}{}^s+[f(k^2)+f(k'^2)]V_{ST}{}^d\}r^2dr\right]^2, \quad \text{(A13)}$$

in which l is 0 or 1 according to whether S and T correspond to even or odd states, respectively.