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# Classical Noise. V. Noise in Self-Sustained Oscillators*

MELVIN LAX
*Bell Telephone Laboratories, Murray Hill, New Jersey*

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A spectrally pure self-sustained oscillator, described by a positive- and negative-impedance series circuit $Z=Z_n+Z_p$ such that $Z\ e^{i\omega t}=e^{i\omega t}Z(\omega,\ D)$, yields a single frequency output depending nonlinearly on a parameter $D$ related to an instantaneous or recent-time-averaged power. The oscillator operates at a point $D=D_0$ at which gains and losses cancel, $R_p+R_n\equiv R(\omega_0,\ D_0)=0$, and at a frequency $\omega_0$ determined by $X_p+X_n\equiv X(\omega_0,\ D_0)=0$. Since $R=0$, the oscillator linewidth *vanishes* in the absence of noise. We endow the resistances $R_p$ and $R_n$ with Langevin noise sources. Amplitude fluctuations produce a broad *additive* background. The oscillator is unstable against phase fluctuations, which broaden the signal into finite width. A quasilinear treatment, well above threshold, demonstrates that the phase executes a Brownian motion. If $\partial R/\partial\omega\neq 0$ or $\partial X/\partial D\neq 0$ at the operating point, phase and amplitude fluctuations are coupled. Nevertheless, we succeed in calculating the linewidth and proving that it is independent of the rate at which power (or $D$) relaxes. A comparison is made with the "linear" treatment of oscillators as amplified noise. A *reduced* random process is set up, valid for time intervals obeying $\omega_0\Delta t\gg 1$. Although the phase of the oscillator involves a *nonlinear*, *nonstationary* action on the Gaussian input noise, it is shown that well above threshold, for the *reduced* process, the phase is again properly described as a Gaussian variable subject to the expected Brownian-motion diffusion. For all well-designed oscillators, even near threshold, we establish that the reduced random process is that of a rotating-wave van der Pol oscillator. A comparison is made between quasilinear solutions of the rotating-wave van der Pol oscillator and exact solutions of the Fokker-Planck equation computed in the next paper, VI, in this series. For intensity fluctuations it is demonstrated that quasilinear methods are quantitatively valid away from the threshold region and qualitatively valid near threshold, provided that the quasilinear approximation is made in the correct variable.

## 1. INTRODUCTION

IN this paper we shall give a Langevin-noise-source treatment of classical self-sustained oscillators.[1]

A detailed consideration of the classical case is warranted at this time, because we have established a connection[2] between noise in quantum oscillators (masers or lasers) and *corresponding* classical oscillators. Thus the solution of the classical problem provides the key to the solution of the quantum-mechanical one.

Self-sustained oscillators differ from ordinary nonlinear systems in that the nonlinearities cannot be regarded as small since they control the operating level of the oscillator. Moreover, all autonomous oscillators (that is to say, oscillators described by differential equations whose coefficients are not explicitly dependent on the time) possess a form of instability which in simple cases is phase instability. Because of this instability, the usual quasilinear methods which assume that fluctuations from some operating point are small cannot be applied directly.

We shall, however, try to take special advantage of certain properties of what might be called an ideal oscillator. Such an ideal oscillator is spectrally quite pure, that is to say its output is not merely a nearly periodic function of the time but is in fact a nearly sinusoidal function of the time. An ideal oscillator will have only a small amount of harmonic content, and in our present discussion we shall make approximations which neglect this harmonic content. Such approximations are of minor importance and can easily be corrected after the present calculations are completed.

One way to take advantage of the spectral purity in dealing with the motion of a self-sustained oscillator in the absence of noise is to take the nonlinear terms and

* This work was previously presented at the 1964 Durham Conference on High Intensity Photon Beams (unpublished), and at the New York meeting of the American Physical Society, Bull. Am. Phys. Soc. **11**, 111 (1966).

[1] This paper has been written in a nearly self-contained fashion. A more detailed discussion of the tools needed for this treatment are provided in the author's previous papers: I. Rev. Mod. Phys. **32**, 25 (1960); II. J. Phys. Chem. Solids **14**, 248 (1960); III. Rev. Mod. Phys. **38**, 359 (1966); IV. *ibid.* **38**, 541 (1966); V. Bull. Am. Phys. Soc. **11**, 111 (1966) (and this paper); VI. with R. D. Hempstead, Bull. Am. Phys. Soc. **11**, 111 (1966), and thesis by R. D. Hempstead, Department of Electrical Engineering, Massachusetts Institute of Technology, September 1965 and Phys. Rev. (to be published). See also M. Lax, "Fluctuation and Coherence Phenomena in Classical and Quantum Physics," in the 1966 Brandeis Summer Institute of Theoretical Physics Lectures (Gordon and Breach Science Publishers, to be published).

[2] See paper QIX by M. Lax and W. H. Louisell. The QIX refers to the ninth paper in the series by M. Lax on quantum noise: QI. Phys. Rev. **109**, 1921 (1958); QII. Phys. Rev. **129**, 2342 (1963); QIII. J. Phys. Chem. Solids **25**, 487 (1964); QIV. Phys. Rev. **145**, 110 (1966); QV. In *Physics of Quantum Electronics*, edited by P. L. Kelley, B. Lax, and P. E. Tannenwald (McGraw-Hill Book Company, Inc., New York, 1966), p. 735; QVI. "Moment Treatment of Maser Noise" (with D. R. Fredkin, to be published); QVII. J. Quantum Electron., **QE-3**, 37 (1967); QVIII. H. Cheng and M. Lax in *Quantum Theory of the Solid State*, edited by Per-Olav Löwdin (Academic Press Inc., New York, 1966); QIX. with W. H. Louisell, J. Quantum Electron., **QE-3**, 47 (1967); (Note: papers QVII and QIX were presented as part of one long talk at the Phoenix International Conference on Quantum Electronics, April 1966); Phys. Rev. **157**, 213 (1967). For a summary of the preceding papers see M. Lax, "Quantum Theory of Noise in Masers and Lasers," in *1966 Tokyo Summer Lectures in Theoretical Physics, Part I*, (Syskabo, Tokyo and W. A. Benjamin, Inc., New York, 1967).

perform an average over one cycle.[3] We shall find it more expedient to introduce complex amplitudes. It will then turn out that retaining appropriate powers of the complex amplitudes automatically performs a selection for us of the appropriate terms of the correct frequency and discards the harmonic terms that would otherwise appear.

One of our complex amplitudes will have the simple variation $\exp(i\omega_0 t)$, and the other amplitude will vary in the complex-conjugate manner. We shall make the rotating-wave approximation which neglects the coupling between these two complex amplitudes. This approximation is exceedingly good as long as the $Q$ of the oscillator is a large number. Indeed this approximation has essentially the same spirit as that of neglecting the higher harmonics. All of these approximations are excellent for an oscillator of high $Q$.

### A. Nature of an Oscillator Without Noise

The essential characteristic of a self-sustained oscillator is that at zero amplitude the system has a negative resistance which causes the system to be unstable. The amplitude then grows until sufficient nonlinear positive resistances come into play so as to cancel the negative resistance. The oscillator then settles down to an operating point at which the positive and negative resistances cancel precisely. Moreover we shall see in the subsequent analysis that the system settles at a point at which the reactance vanishes as well, and this will determine the operating frequency of the oscillator.

The cancellation of positive and negative resistances essentially means that in the absence of noise the linewidth of an oscillator goes to zero even though dissipative elements are present in the system.

### B. Self-Sustained Oscillator With Noise

In the usual quasilinear treatment of noise in nonlinear systems, the noise produces an effect additive to the signal. If this were the case, the output of our oscillator would possess a $\delta$-function spectrum plus a background. This is not satisfactory for our purposes. We anticipate that the noise will spread this $\delta$-function spectrum into a finite width. Thus the noise mixes with the signal in a complex fashion in a self-sustained oscillator that is quite different from ordinary nonlinear systems.

The reason for the ability (at least in an approximate sense) to talk about signal plus noise in ordinary nonlinear systems is that such systems are stable; thus the quasilinear treatment of[1] I which assumes that the effects of the noise are small is an adequate treatment. Therefore, in order to broaden our $\delta$-function spectrum into a finite linewidth, we must find a breakdown of the quasilinear approximation. This breakdown indeed follows from the previously mentioned phase instability of autonomous oscillators. To understand this instability, we merely note that if our differential equations do not depend explicitly upon the time and we have a solution that starts at a time $t_0$, we can construct from it other solutions that start at slightly different times. There is no cost in energy in passing from one such solution to another, so that a fluctuation of phase which takes us from one solution to another will not be suppressed. Indeed such phase fluctuations will be shown to be quite analogous to the Brownian motion of a free particle. Thus the phase deviations can become quite large. Hence the phase fluctuation cannot be treated in the quasilinear approximation, although as we shall see, all coordinates in the problem which are independent of the phase will be stable and hence can be treated by quasilinear methods. Thus the spectral linewidth which involves the phase fluctuations cannot be treated by quasilinear methods, but we shall see that amplitude fluctuations, in which the phase cancels, can be treated by the usual quasilinear methods. Thus we shall find that the effect of amplitude fluctuations is to add a background to the signal, but only the phase fluctuations broaden the signal from a $\delta$-function spectrum into one of finite width.

For purposes of understanding this central broadened line, it is then quite adequate to study the phase fluctuations only, neglecting all amplitude fluctuations. In doing so we arrive at an equation of universal form:

$$d\phi/dt = G(t)\cos(\phi+\omega_0 t). \qquad (1.1)$$

In this equation $\omega_0$ represents the frequency of the oscillator, $\phi+\omega_0 t$ its (total) phase, and $G(t)$ is a random variable linearly proportional to the input noise source. The latter is assumed to be Gaussian and approximately white.

If the noise source is precisely white, Eq. (1.1) describes a *nonstationary* Markoffian process (see Sec. IV) whose diffusion constant is time-dependent and proportional to $\cos^2(\omega_0 t+\phi)$. Over a time interval $\Delta t \gg (\omega_0)^{-1}$, we show in Sec. 5 by using the methods of IV, Sec. 5, that the process (1.1) can be replaced accurately by a simpler "reduced process," a *stationary* Markoff process with diffusion constant obtained by replacing the above $\cos^2(\omega_0 t+\phi)$ by its average value $\frac{1}{2}$. Thus over *long time intervals* ($t \gg \omega_0^{-1}$), there is no distinction between the process (1.1), the simpler process $d\phi/dt = G(t)\cos\omega_0 t$, and the reduced process $d\phi/dt = 2^{-1/2}G(t)$. For the reduced process, $d\phi/dt$ and $\phi$ are clearly Gaussian. Indeed, this reduced process is the usual Brownian-motion process.

[3] See, for example, B. van der Pol, *Selected Scientific Papers* (North-Holland Publishing Company, Amsterdam, 1960), Vol. 1, pp. 346, 361; N. M. Krylov and N. N. Bogolyubov, *Introduction to Nonlinear Mechanics* (Princeton University Press, 1943); N. N. Bogolyubov and Yu. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillators* (Daniel Davey and Company, Inc., New York, 1965). Yu. A. Mitropolsky, *Problems of the Asymptotic Theory of Nonstationary Vibrations* (Daniel Davey and Company, Inc., New York, 1965); N. Minorsky, *Nonlinear Oscillations* (D. Van Nostrand Company Inc., Princeton, New Jersey, 1962).

The spectral density can then be computed by taking the Fourier transform of the autocorrelation function

$$\langle a(t)^* a(0)\rangle = \langle |a(t)| e^{i\phi(t)} |a(0)| e^{-i\phi(0)}\rangle \exp i\omega_0 t$$
$$\approx \langle |a|^2\rangle \langle \exp i[\phi(t)-\phi(0)]\rangle \exp i\omega_0 t. \tag{1.2}$$

In passing from the first to the second step in Eq. (1.2) we have neglected amplitude fluctuations. It follows from the fact that the Fourier transform of a Gaussian variable is also Gaussian that we can write

$$\langle \exp i[\phi(t)-\phi(0)]\rangle = \exp\{-\tfrac{1}{2}\langle[\phi(t)-\phi(0)]^2\rangle\}. \tag{1.3}$$

For $t \gg (\omega_0)^{-1}$ we may use the reduced process. Then $\phi$ exhibits a simple Brownian motion and the mean square of the displacement of $\phi$ increases linearly with the time:

$$\langle[\phi(t)-\phi(0)]^2\rangle = W|t|. \tag{1.4}$$

Thus our autocorrelation can be written in the form

$$\langle a^*(t)a(0)\rangle = \langle |a|^2\rangle \exp(-\Lambda_p|t|)\exp(i\omega_0 t);\ \Lambda_p = \tfrac{1}{2}W, \tag{1.5}$$

so that the noise spectrum[4] takes the well-known[5] Lorentzian form

$$\tfrac{1}{2}G(\omega,a) \equiv \langle a^*_\omega a_\omega\rangle = \int_{-\infty}^{\infty} e^{-i\omega t}dt\langle a^*(t)a(0)\rangle$$
$$\approx 2\langle |a|^2\rangle \Lambda_p/[\Lambda_p^2+(\omega-\omega_0)^2]. \tag{1.6}$$

The factor of $\frac{1}{2}$ in front of the noise power $G$ arises because the usual engineering definition of noise power assigns all of the energy to the positive-frequency range, so that, for example,

$$\int_0^\infty G(\omega,a)\frac{d\omega}{2\pi} = \int_{-\infty}^{\infty}\langle a^*_\omega a_\omega\rangle\frac{d\omega}{2\pi} = \langle |a(0)|^2\rangle. \tag{1.7}$$

The total power is determined by the mean square fluctuation at one time. The noise power $G$ has been integrated from zero to infinity whereas the quantity $\langle a^*_\omega a_\omega\rangle$ has been integrated from $-\infty$ to $\infty$. We shall show in Sec. 4 that the linewidth $\Lambda_p$ associated with phase fluctuations is inversely proportional to the power at which the oscillator operates. If we define

$$\rho(t) = |a(t)|^2, \tag{1.8}$$

then the power will be proportional to $\rho$. In Sec. 9, which describes the rotating wave van der Pol oscillator, we shall find it possible to introduce new units of time and power such that the dimensionless variables corresponding to $\rho$ and $\Lambda$, denoted by primes, obey the very simple equation

$$\Lambda_p' = 1/\langle\rho'\rangle \quad \text{above threshold}$$
$$= 2/\langle\rho'\rangle \quad \text{below threshold.} \tag{1.9}$$

Thus we shall see that the approximations made above threshold in Sec. 4 and that made below threshold in Sec. 7 appear to disagree at threshold. However, we shall show in our next paper[1] (Classical Noise VI) that both of these answers are indeed correct. It is simply that there is a narrow region near threshold (which is treated in VI by an exact solution of the appropriate Fokker–Planck equation) over which the linewidth varies from one form to the other (see Fig. 1).

The very simple results we have just quoted apply to an ideal *stabilized* oscillator, namely one whose frequency is not a function of its operating point. When the frequency is a function of the operating point, amplitude and phase fluctuations are coupled together. The calculation of the linewidth in the presence of such coupling is, so far as we know, a previously unsolved problem (at least in the nonlinear region above threshold). We shall show that when such amplitude

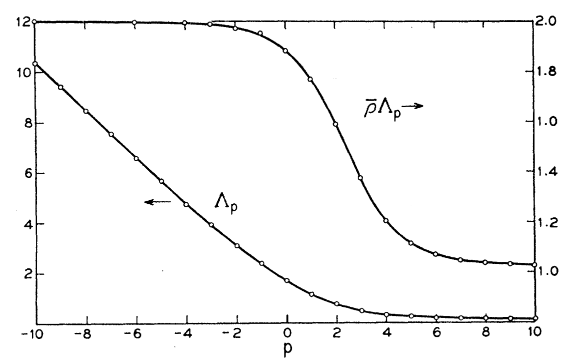


FIG. 1. The half-width $\Lambda_p$ of the spectrum (1.6) of a van der Pol oscillator plotted versus dimensionless pump rate $p$. In VI, the spectrum is shown to be nearly Lorentzian. The half-width $\Lambda_p$ is then essentially equal to the lowest eigenvalue of the Fokker-Planck operator. In the dimensionless units of Sec. 9, the Fokker-Planck equation (8.10) takes the form

$$\partial P(r,\varphi,t)/\partial t = H_r P + (1/r^2)\partial^2 P/\partial\varphi^2$$

in radial and phase variables, where

$$H_r P \equiv \partial/\partial r[(r^3 - pr - r^{-1})P] + (\partial^2/\partial r^2)P.$$

The eigenfunctions of the Fokker-Planck operator $H_r + (1/r^2)\partial^2/\partial\varphi^2$ have the form $R_n(r)\exp(i\lambda\varphi)$ and eigenvalues $\Lambda_{\lambda,n}$ determined by

$$(H_r - \lambda^2/r^2)R_n(r) = -\Lambda_{\lambda,n}R_n(r).$$

The eigenfunctions that contribute to the spectrum (1.6) have $\lambda=1$, and the lowest of them $\Lambda_{1,0}$ is called $\Lambda_p$ ($p$ for "phase") and is plotted above against pump rate. The "exact" numerical computations of $\Lambda_{1,n}$ for $n=0, 1, \cdots, 9$ are described in VI.

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[4] For definitions appropriate to complex variables, see IV, Ref. 13 and QIV, Ref. 24.

[5] See, for example, W. A. Edson, Proc. IRE **48**, 1454 (1960); J. A. Mullen, *ibid.* **48**, 1467 (1960); M. E. Golay, *ibid.* **48**, 1473 (1960). R. Esposito, J. Electron. Control **12**, 251 (1962). An excellent collection of Russian papers is presented by P. I. Kuznetsuv, R. L. Stratonovich, and V. I. Tikhonov, *Nonlinear Transformations of Stochastic Processes* (Pergamon Press Ltd., Oxford, England, 1965).

and phase coupling is present, the linewidth is increased over that shown in Eq. (1.9) by a factor that depends on this coupling. Effects of this coupling are discussed in Sec. 6 and the resulting linewidth is presented there and in the Summary.

While our formula for the linewidth associated with phase fluctuations takes a universal form regardless of the detailed nature of the amplitude fluctuations, a discussion of the spectrum of amplitude noise requires the specification of a more detailed model of the nature of the nonlinearities. If the nonlinear dependence occurs only through the parameter $\rho$, then the amplitude spectrum is determined by the autocorrelation

$$\langle\Delta\rho(t)\Delta\rho(0)\rangle\approx\langle(\Delta\rho)^2\rangle\exp(-\Lambda_a|t|), \qquad (1.10)$$

which again leads to an approximate spectrum of Lorentzian shape. In the quasilinear approximation, the halfwidth of the amplitude spectrum takes the following form in appropriate dimensionless units:

$$\Lambda_a'=2\langle\rho'\rangle+4/\langle\rho'\rangle. \qquad (1.11)$$

In the same units,

$$\langle(\Delta\rho')^2\rangle=2\langle\rho'\rangle^2/(\langle\rho'\rangle^2+2). \qquad (1.12)$$

These equations are meant to be valid above threshold, below threshold, and even in the threshold region. However, we would anticipate that the quasilinear approximation is not as accurate in the threshold region as elsewhere. Comparisons between the phase noise of Eq. (1.9) and the amplitude noise of Eq. (1.11) with exact solutions for these quantities for a rotating wave van der Pol oscillator, as obtained in VI, are shown in Figs. 1 and 2.

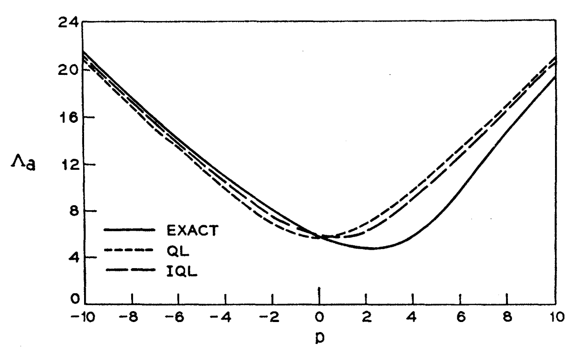


FIG. 2. The half-width $\Lambda_a$ for amplitude fluctuations [see (1.10)] is plotted versus dimensionless pump rate $p$. In VI, we evaluate the spectrum for amplitude fluctuations directly and find it to be nearly Lorentzian. The eigenvalues that contribute to pure amplitude fluctuations are $\Lambda_{0,n}$ (see caption, Fig. 1). The lowest eigenvalue $\Lambda_{0,0}=0$ corresponds to the steady state, and $\Lambda_a\equiv\Lambda_{0,1}$, the lowest nonvanishing eigenvalue is plotted above. This result agrees closely with the actual half-width of the directly computed amplitude spectrum. See VI for details, as well as for $\Lambda_{0,n}$ for $n=1, 2, \cdots, 10$. The quasilinear approximation (QL) is given by $(\Lambda_a)_{\rm QL}=2(p^2+8)^{1/2}$ according to (9.16). The intelligent quasilinear (IQL) approximation is obtained by replacing $(\Lambda_a)_{\rm QL}=2\rho_0+4/\rho_0$ by $(\Lambda_a)_{\rm IQL}=2\langle\rho\rangle+4/\langle\rho\rangle$, where $\langle\rho\rangle$ is the exact mean value of $\rho$ shown in Fig. 4.

It will often happen that the nonlinear response of an oscillator depends on some parameter $D$ that does not depend on the instantaneous value of $\rho$, but rather on some time integral over the previous history of $\rho$. The fact that $D$ lags behind $\rho$ can introduce spiking into the behavior of an oscillator as discussed[2] in QVII. Our general results for the phase linewidth have already included the possibility of such a lagging response. For the amplitude noise, however, separate consideration must be given to the case of the lagging response and this is done in Sec. 10 of the present paper.

The previous literature on noise in classical oscillators[5] has generally dealt with more specialized models, and/or has been less systematic and rigorous than the present approach. More careful discussions have just begun to appear in connection with noise in lasers.[6]

## 2. COMPLEX AMPLITUDE FORMULATION FOR NONLINEAR PROBLEMS

### A. Complex Amplitude Transformation

In this section, we shall consider a nonlinear system, not necessarily a self-sustained oscillator, and shall show that the introduction of appropriate complex variables produces an automatic simplification in the discussion of nonlinear problems. We will start with a resonant circuit perturbed by a nonlinear impedance:

$$LdI/dt+Q/C=f(Q,I). \qquad (2.1)$$

In order to have a definite example in mind, we shall consider the special case

$$f(Q,I)=-\mu LI^2Q. \qquad (2.2)$$

The procedure we shall adopt will be valid for a general function $f$ and we shall indicate both special and general results as we go along. The form of Eq. (2.2) suggests that in series with a capacitance $C$ there is another capacitance whose inverse capacity is proportional to the square of the current. This suggests that the two inverse capacitances must be added and that the resonant frequency will be modified in accord with

$$\omega_0^2=\omega_c^2+\mu\langle I^2\rangle?; \qquad \omega_c^2=(LC)^{-1}. \qquad (2.3)$$

We shall demonstrate shortly that while this conclusion is qualitatively correct, it is quantitatively incorrect. To demonstrate this point we introduce a pair of complex variables

$$a=I-i\omega_0Q, \qquad a^*=I+i\omega_0Q \qquad (2.4)$$

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[6] C. Schmid and H. Risken, Z. Phys. **189**, 365 (1966); H. Haken and W. Weidlich, *ibid.* **189**, 1 (1966); H. Sauerman, *ibid.* **188**, 480 (1965); **189**, 312 (1966); W. Weidlich and F. Haake, *ibid.* **186**, 203 (1965); H. Risken, *ibid.* **191**, 302 (1966); H. Risken, C. Schmid, and W. Weidlich, Phys. Letters **20**, 489 (1966).

together with the inverse transformation

$$I=\tfrac{1}{2}(a+a^*), \qquad Q=(2i\omega_0)^{-1}(a^*-a). \quad (2.5)$$

Note that $\omega_0$ is as yet unknown and that the parameter $\mu$ in the nonlinear term is not assumed small.

### B. Pure-Spectrum Approximation

In the spirit of what was said in the introduction, we shall seek a "pure-spectrum" solution of the form

$$a=Ae^{-i\omega_0 t}, \qquad a^*=A^*e^{i\omega_0 t}, \quad (2.6)$$

the variable $a$ containing only a single negative frequency, and the variable $a^*$ containing a single positive frequency. The nonlinear term $f$ is then expanded as a polynomial in $a$ and $a^*$. In accord with our pure-spectrum approximation, we shall retain only the terms on the right-hand side that contain the frequencies $+\omega_0$ and $-\omega_0$:

$$f=(i\mu L/8\omega_0)[a^*\mid a\mid^2-a\mid a\mid^2+(a^*)^3-a^3]$$
$$\approx(i\mu L/8\omega_0)(a^*\mid a\mid^2-a\mid a\mid^2). \quad (2.7)$$

In the case of an arbitrary nonlinear form, $f$, we can make the general expansion

$$f(Q,I)=\tfrac{1}{2}\sum_{n=1}^{\infty}[F_n(\mid a\mid^2)(a^*)^n+F_n(\mid a\mid^2)^*a^n]. \quad (2.8)$$

The pure-spectrum approximation that we propose consists in retaining only the terms

$$f(Q,I)\approx\tfrac{1}{2}[F_1(\mid a\mid^2)a^*+F_1(\mid a\mid^2)^*a]. \quad (2.9)$$

We shall refer to an oscillator as an ideal oscillator if these are the only terms that are in fact present among all of the nonlinear terms. It is to be noted that the coefficient $F_1$ does not display any frequency dependence. A somewhat more general form of an ideal oscillator can be defined by means of

$$f(Q,I)=Z(-id/dt,\mid a\mid^2)\tfrac{1}{2}(a^*+a), \quad (2.10)$$

where

$$Z(-id/dt,\mid a\mid^2)e^{i\omega t}=e^{i\omega t}Z(i\omega,\mid a\mid^2),$$

in which the impedance $Z$ produces a single-frequency output when acted on by any single-frequency input yet the amplitude of that output depends on the instantaneous value of $\mid a\mid^2$. Thus the choice (2.9) is a special case of (2.10) with the parameters chosen in accord with

$$Z(\omega_0,\mid a\mid^2)=F_1(\mid a\mid^2);$$
$$Z(-\omega_0,\mid a\mid^2)=Z(\omega_0,\mid a\mid^2)^*, \quad (2.11)$$

and with the impedance independent of frequency.

### C. Equation for $a$ and Steady-State Solution

Making use of Eqs. (2.1) and (2.4), we find that our complex variable $a$ obeys

$$\frac{da}{dt}=\frac{dI}{dt}-i\omega_0 I=-\omega_c^2Q-i\omega_0 I+\frac{f(Q,I)}{L}, \quad (2.12)$$

where our truncated nonlinear impedance can be written in the form

$$F_1\equiv R+iX; \qquad f(Q,I)=\tfrac{1}{2}R(a+a^*)+\tfrac{1}{2}iX(a^*-a), \quad (2.13)$$

where $R$ and $X$ are the real and imaginary parts of the impedance $Z$ or the coefficient $F_1$. The equation for $a$ can now be rewritten in the form

$$da/dt=-\tfrac{1}{2}i[(\omega_c^2/\omega_0)+\omega_0+(X/L)]a$$
$$+\tfrac{1}{2}i[(\omega_c^2/\omega_0)-\omega_0+(X/L)]a^*$$
$$+\tfrac{1}{2}(R/L)(a+a^*). \quad (2.14)$$

A solution of the form (2.6) will satisfy (2.14) only if the terms in $\exp(i\omega_0 t)$ and $\exp(-i\omega_0 t)$ vanish separately. The imaginary and real coefficients of $\exp(i\omega_0 t)$ yield

$$\omega_0^2=\omega_c^2+\omega_0 X/L, \qquad R=0. \quad (2.15)$$

By what appears to be a miraculous coincidence, the complex coefficient of $\exp(-i\omega_0 t)$ then also vanishes automatically. In our special example, the form (2.7) for $f$ yields a contribution to the reactance $X$ but no contribution to the resistance $R$; thus the form (2.6) is in fact an exact solution of Eq. (2.14) with *frequency* determined by the first part of (2.15). In general, however, there is a contribution to the resistance $R$ and the *level* of oscillation is determined by

$$R\equiv R(\omega_0,\mid a\mid^2)=0 \quad (2.16)$$

is obeyed. This condition corresponds to the cancellation of resistance that must occur at the steady operating point of a self-sustained oscillator. Indeed this condition can be used to determine $\mid a\mid^2$ at the operating point, whereas the previous condition (2.15) determines the frequency at this operating point. If we consider the special example (2.7), we find that our reactance $X$ is defined by

$$i[\mu L/(8\omega_0)]\mid a\mid^2=\tfrac{1}{2}F_1=\tfrac{1}{2}iX, \quad (2.17)$$

which combined with Eq. (2.15), leads to the correct expression for the operating frequency $\omega_0$;

$$\omega_0^2=\omega_c^2+\tfrac{1}{4}\mu\mid a\mid^2=\omega_c^2+\tfrac{1}{2}\mu\langle I^2\rangle. \quad (2.18)$$

A comparison between (2.18) and (2.3) shows that the correction in the squared frequency is half as large as was previously conjectured.

### D. Truncated Impedance Expression

To display the relation between our simple method of truncating the impedance and the method commonly used in textbooks on nonlinear mechanics,[3] we introduce the transformation

$$a=|a|e^{-i\omega_0 t};\qquad a^*=|a|e^{i\omega_0 t} \tag{2.19}$$

into the expression for $f$ and write that expression as a Fourier series in the form

$$f(\omega_0^{-1}|a|\sin\omega_0 t,\,|a|\cos\omega_0 t)=\tfrac{1}{2}\sum F_n|a|^n e^{in\omega_0 t}+\text{c.c.} \tag{2.20}$$

Our truncation consists in retaining the $n=1$ term (and its complex conjugate) in that Fourier series. This first term, by the usual theorem on Fourier series, is given by

$$R+iX\equiv F_1=\frac{2}{|a|}2\pi^{-1}\times\int_0^{2\pi}d\theta\, e^{-i\theta}f(\omega_0^{-1}|a|\sin\theta,\,|a|\cos\theta). \tag{2.21}$$

The similarity between Eq. (2.21) and the "stroboscopic" methods or the methods of averaging over one cycle is now obvious.[3] It is to be noted, however, that our procedure is slightly more general than the one commonly used: our nonlinear term is not assumed small and the frequency $\omega_0$ is calculated self-consistently. Thus the frequency which appears underneath the integral sign of Eq. (2.21) is not known until the calculation is finished. Let us furthermore note that if the nonlinear function is a polynomial, the need for performing the integral in (2.21) is eliminated and the procedure of simply picking out the appropriate terms as was done in Eq. (2.7) can be used.

## 3. MODEL OF A SELF-SUSTAINED OSCILLATOR

We have seen in the preceding section that the usual treatment of nonlinear oscillations proceeds by truncating the impedance in such a way as to discard terms containing undesired frequencies. We shall therefore introduce our model of an ideal self-sustained oscillator as one whose impedance does not produce any of these undesired frequencies to begin with. Our model is displayed in Fig. 3 and in the following equation:

$$Z(-id/dt,\,D)I(t)=e(t). \tag{3.1}$$

The impedance is assumed to be a series combination of a positive and negative impedance as shown in Fig. 3:

$$Z(\omega,D)=Z_p(\omega,D)+Z_n(\omega,D)=R(\omega,D)+iX(\omega,D). \tag{3.2}$$

In Sec. 2, our controlling parameter $D$ was simply

$$D=|a(t)|^2. \tag{3.3}$$

However, as discussed in the Introduction, it is often the case that the nonlinear impedance responds not to the instantaneous value of $|a|^2$ but rather to some time average over its previous history such as that given, for example, in

$$dD/dt=\mathcal{R}-\Gamma(-id/dt)D-r(-id/dt)|a(t)|^2+F(t). \tag{3.4}$$

(We can think of $\mathcal{R}$ as the pump rate in a maser.) The operating point of our oscillator will be determined by

$$R(\omega_0,D_0)=0,\qquad X(\omega_0,D_0)=0. \tag{3.5}$$

The first of these equations should be thought of as determining the operating parameter $D_0$, and the second of these equations, the reactance equation, can be thought of as determining the operating frequency $\omega_0$. The steady operating value for the $|a|$ can be determined from the corresponding value for $D$ by setting the equation for $dD/dt=0$, in other words

$$0=\mathcal{R}-\Gamma(0)D_0-r(0)|a_0|^2. \tag{3.6}$$

### A. Noise Sources

As shown in Fig. 3, our positive and negative impedances each have associated with them a noise source so that the total noise voltage is the sum of the separate voltages, which are to be regarded as independent. In particular this makes the noise power additive:

$$e(t)=e_n(t)+e_p(t);\qquad (e^2)_\omega=(e_p^2)_\omega+(e_n^2)_\omega, \tag{3.7}$$

where, for example,

$$(e_p^2)_\omega\equiv\int_{-\infty}^{\infty}e^{-i\omega t}dt\langle e_p(0)e_p(t)\rangle=2kT_pR_p'(\omega,\bar{D})C(\omega,T_p) \tag{3.8a}$$

and

$$(e_n^2)_\omega=2kT_nR_n'(\omega,\bar{D})C(\omega,T_n)>0. \tag{3.8b}$$

If our resistances were in thermal equilibrium, the noise sources would be Johnson noise and the primes on $R_p$ and $R_n$ could be omitted. We retain this prime to remind us that there can be a change when these resistances are operated in a nonequilibrium situation. We have chosen to evaluate these resistances at the

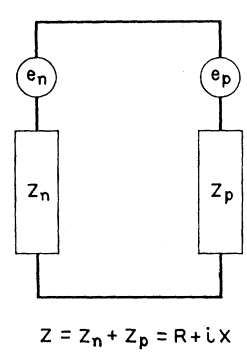


FIG. 3. Model of a self-sustained oscillator based on a series combination of positive impedances each with its own noise source.

mean value of the parameter $D$:

$$\bar{D}=\langle D\rangle\approx D_0. \qquad (3.9)$$

The correction factors $C$ are chosen in such a way that they reduce to unity at high temperature but at low temperatures yield a correction which make the results valid in the quantum-mechanical case. This correction factor has the form[7]

$$C(\omega,T)=\tfrac{1}{2}(\hbar\omega/kT)\coth(\tfrac{1}{2}\hbar\omega/kT). \qquad (3.10)$$

### B. Complex Amplitude Equations

As a preliminary to introducing our complex amplitudes, let us define an inductance by means of

$$L\equiv\tfrac{1}{2}(\partial X/\partial\omega)_{\omega=\omega_0,D=D_0}. \qquad (3.11)$$

Our total impedance can then be decomposed into a resonant part and a remainder:

$$Z(\omega,D)\equiv L(\omega-\omega_0^2/\omega)+Z_1(\omega,D). \qquad (3.12)$$

Our model equation can now be rewritten in the form

$$LdI/dt+L\omega_0^2Q+Z_1I=e(t). \qquad (3.13)$$

Our complex amplitude now obeys

$$\frac{da}{dt}=\frac{dI}{dt}-i\omega_0\frac{dQ}{dt}=\frac{dI}{dt}-i\omega_0 I. \qquad (3.14)$$

When Eq. (3.13) is introduced, we obtain the form

$$da/dt=-i\omega_0a-(Z_1/2L)(a+a^*)+e(t)/L. \qquad (3.15a)$$

In view of (3.5), $Z_1(\omega_0,D_0)=0$. The other complex amplitude $a^*$ obeys

$$da^*/dt=i\omega_0a^*-(Z_1/2L)(a+a^*)+e(t)/L. \qquad (3.15b)$$

### C. Slowly Varying Amplitudes

In the absence of the noise voltage $e(t)$, our solution had the form of a simple exponential. Therefore, in the presence of these noise sources we can assume solutions of the form

$$a^*=A^*(t)e^{i\omega_0t},\qquad a=A(t)e^{-i\omega_0t}, \qquad (3.16)$$

where the variables $A$ and $A^*$ will be slowly varying functions of the time. After neglecting the small counter-rotating terms, these slowly varying functions obey

$$\frac{dA^*}{dt}=-Z_1\left(\omega_0-i\frac{d}{dt}\right)\frac{A^*}{2L}+e^{-i\omega_0t}\frac{e(t)}{L}, \qquad (3.17)$$

$$\frac{dA}{dt}=-Z_1\left(-\omega_0-i\frac{d}{dt}\right)\frac{A}{2L}+e^{i\omega_0t}\frac{e(t)}{L}, \qquad (3.18)$$

and depend on the impedance $Z_1$ in the neighborhood of the frequency $\omega_0$.

### D. Neighborhood of the Operating Point

In the neighborhood of our operating point $\omega_0$, $D_0$, we can therefore expand our impedance in the form

$$\begin{aligned}Z_1(\omega,D)&\equiv Z(\omega,D)-R(\omega_0,D_0)-\tfrac{1}{2}(\partial X/\partial\omega_0)i(\omega-\omega_0^2/\omega),\\&=(\partial R(\omega,D)/\partial\omega)_{\omega_0,D_0}(\omega-\omega_0)+(\partial Z/\partial D)_{\omega_0,D_0}\Delta D\\&\quad+(\partial^2Z/\partial\omega\partial D)_{\omega_0,D_0}(\omega-\omega_0)\Delta D+\cdots,\end{aligned} \qquad (3.19)$$

where

$$\Delta D=D-D_0. \qquad (3.20)$$

When this expansion (3.19) is introduced into (3.17) we obtain

$$\left[1-i\frac{\partial R/\partial\omega}{2L}-\frac{\partial^2Z/\partial\omega\partial D}{2L}\Delta D\right]\frac{dA^*}{dt}=-\frac{\partial Z}{\partial D}\frac{\Delta DA^*}{2L}+\frac{e_+(t)}{L}, \qquad (3.21)$$

where $e_+(t)=e^{-i\omega_0t}e(t)$. Dividing by the expression within brackets on the left-hand side, and neglecting small quantities of second order, we obtain

$$dA^*/dt=\lambda\Delta DA^*+e_+(t)/L', \qquad (3.22)$$

where $L'$ is a complex inductance defined by

$$L'=L\left[1-i\frac{\partial R}{\partial\omega}\Big/2L\right]=-\frac{i}{2}\frac{\partial Z}{\partial\omega}\bigg|_{\omega_0,D_0}, \qquad (3.23)$$

and the parameter $\lambda$ is defined by

$$\lambda=-\frac{\partial Z}{\partial D}\Big/(2L')=-i\frac{\partial Z}{\partial D}\Big/\frac{\partial Z}{\partial\omega}=+i\frac{\partial\omega_0}{\partial D_0}=|\lambda|e^{i\beta}. \qquad (3.24)$$

### E. Phase and Amplitude Equations

In the region above threshold for self-sustained oscillations, the phase and amplitude of an oscillator will be nearly independent variables. It is therefore expedient to introduce the transformation

$$A^*=-i|a_0|\exp(u+iv). \qquad (3.25)$$

The variable $v$ is the phase of our oscillator, and the variable $u$ is related to the amplitude of our oscillator. Together they obey the complex equation

$$d(u+iv)/dt=\lambda\Delta D+i\exp[-i(v+\omega_0t)]\times\exp(-u)e(t)/|L'a_0|. \qquad (3.26)$$

[7] For a discussion of this factor, see I, Sec. 7.

The real and imaginary parts of Eq. (3.26) lead to

$$dv/dt=|\lambda|\sin\beta\Delta D+e^{-u}\cos(v+\omega_0 t)e(t)/|L'a_0|, \quad (3.27)$$

$$du/dt=|\lambda|\cos\beta\Delta D+e^{-u}\sin(v+\omega_0 t)e(t)/|L'a_0|. \quad (3.28)$$

We note that when $\beta$ is not equal to 0, the phase and amplitude are coupled together. In spite of this, it is possible to introduce a linear combination of $u$ and $v$ which, in the absence of noise, is a precise constant of the motion. This combination obeys

$$d(v\cos\beta-u\sin\beta)/dt=e^{-u}\cos(v+\omega_0 t+\beta)e(t)/|L'a_0|. \quad (3.29)$$

## 4. PHASE FLUCTUATIONS IN A FREQUENCY-STABILIZED OSCILLATOR

Examining Eq. (3.24) we see that when $\beta=0$ a change in the operating point $D_0$ produces no change in the real part of the frequency $\omega_0$. Such an oscillator can be called a frequency-stabilized oscillator since its operating frequency does not depend on its operating level. We summarize this statement in

$$\partial\omega_0/\partial D_0=\text{pure imaginary}, \quad \beta=0. \quad (4.1)$$

For simplicity we start with this frequency-stabilized case. Moreover, as remarked in the introduction, amplitude fluctuations produce only an additive background. So for the discussion of this case we shall neglect the amplitude fluctuations. Then Eq. (3.27) reduces to the simple form

$$dv/dt=G(t)\cos(v+\omega_0 t), \quad (4.2)$$

where

$$G(t)=e(t)/|L'a_0|. \quad (4.3)$$

We have thus arrived at our universal equation, (4.2), for the phase diffusion in a self-sustained oscillator.

We shall postpone to the next section the rigorous discussion of this nonlinear random equation. In this section, we shall make the customary approximation[5] of setting $v=0$ on the right-hand side. In this case, we obtain

$$v(t)-v(0)\approx\int_0^t G(s)\cos\omega_0 s\,ds=\text{Gaussian}. \quad (4.4)$$

Neglecting amplitude fluctuations, our desired autocorrelation then takes the form

$$\langle a^*(t)a(0)\rangle\approx|a_0|^2 e^{i\omega_0 t}\langle\exp-i[v(t)-v(0)]\rangle = |a_0|^2 e^{i\omega_0 t}\exp\{-\tfrac{1}{2}\langle[v(t)-v(0)]^2\rangle\}. \quad (4.5)$$

The last step follows from the Gaussian nature of the variable $v(t)$.

### A. Phase Diffusion

We now wish to show that by choosing a time $t$ such that $\omega_0 t$ is large compared to 1, we will obtain a simple diffusion for the phase in the form

$$\langle[v(t)-v(0)]^2\rangle\approx\tfrac{1}{2}(G^2)_{\omega_0}|t|=W|t|. \quad (4.6)$$

Let us start with

$$\langle[v(t)-v(0)]^2\rangle=\int_0^t ds\int_0^t ds'\cos\omega_0 s\cos\omega_0 s'\langle G(s)G(s')\rangle. \quad (4.7)$$

By the Wiener-Khinchin theorem, the autocorrelation of the noise variable $G(s)$ is expressed in the form

$$\langle G(s)G(s')\rangle=(2\pi)^{-1}\int_{-\infty}^{\infty}\exp[i\omega(s-s')](G^2)_\omega d\omega. \quad (4.8)$$

In the special case, when the spectrum of $G$ is white, this autocorrelation has the form of a delta function in the time difference. But the proof which follows does not make use of the assumption of a white spectrum. Introducing the trigonometric identity,

$$\cos\omega_0 s\cos\omega_0 s'=\tfrac{1}{2}\cos\omega_0(s-s')+\tfrac{1}{2}\cos\omega_0(s+s'),$$

the time integration can be performed and we obtain

$$\tfrac{1}{2}\int_0^t\int_0^t ds ds'\exp[i\omega(s-s')]\cos\omega_0(s-s') = \frac{\sin^2[\frac{1}{2}(\omega+\omega_0)t]}{(\omega+\omega_0)^2}+\frac{\sin^2[\frac{1}{2}(\omega-\omega_0)t]}{(\omega-\omega_0)^2} \to\tfrac{1}{2}\pi t[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]. \quad (4.9)$$

For $\omega_0 t\gg 1$, the right-hand side of (4.9) reduces to the familiar $\delta$-function form. Thus our results will only depend on the spectrum of $G$ at the frequency $\omega_0$. This is why it was unnecessary to assume the white spectrum.

In (4.9) we have retained only the term involving the cosine of the time difference. This is the important term; the other term, involving the cosine of the time sum, leads to

$$\tfrac{1}{2}\int_0^t ds\int_0^t ds'\exp[i\omega(s-s')]\cos\omega_0(s+s') = \cos\omega_0 t\,\frac{(\cos\omega_0 t-\cos\omega t)}{\omega^2-\omega_0^2} \approx+\frac{\pi}{2\omega_0}\cos\omega_0 t\sin\omega_0 t[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]. \quad (4.10)$$

If these results are combined, we obtain

$$\langle[v(t)-v(0)]^2\rangle=\tfrac{1}{2}(G^2)_{\omega_0}[t-(\sin 2\omega_0 t)/(2\omega_0)]. \quad (4.11)$$

In the region $\omega_0 t \gg 1$, the second term in (4.11) is negligible compared to the first term, and we obtain Eq. (4.6) as we set out to do. Our autocorrelation then takes the form

$$\langle a^*(t)a(0)\rangle \approx |a_0|^2 \exp[-\tfrac{1}{2}W|t|] \exp(i\omega_0 t), \quad (4.12)$$

and its Fourier transform leads to the spectrum

$$\langle a_\omega{}^* a_\omega\rangle = \int_{-\infty}^{\infty} e^{-i\omega t}\langle a^*(t)a(0)\rangle dt$$
$$= |a_0|^2 (\tfrac{1}{2}W)/[(\omega-\omega_0)^2+(\tfrac{1}{2}W)^2]. \quad (4.13)$$

### B. Full Width at Half-Power

The spectrum (4.13) is a Lorentzian whose full width at half-power is given by

$$W = \tfrac{1}{2}(G^2)_{\omega_0} = \tfrac{1}{2}(e^2)_{\omega_0}/|L'a_0|^2. \quad (4.14)$$

Introducing the noise sources (3.8), this width takes the explicit form

$$W = (kT_pR_p'C_p + kT_nR_n'C_n)/|L'a_0|^2, \quad (4.15)$$

$$C_p = C(\omega_0, T_p), \qquad C_n = C(\omega_0, T_n). \quad (4.16)$$

The power dissipated in the positive resistance can be defined by means of

$$P = R_p\langle I^2\rangle = \tfrac{1}{2}R_p|a_0|^2. \quad (4.17)$$

If we define a linewidth $\Delta\omega$ by means of

$$(\Delta\omega)^2 = R_pR_p'/|L'|^2, \quad (4.18)$$

our linewidth takes the form

$$W = (\Delta\omega)^2[kT_pC_p + k|T_n|C_n|R_n'/R_p'|](2P)^{-1}. \quad (4.19)$$

### C. Classical and Quantum Limits

In the high-temperature or low-frequency limit, we obtain the classical linewidth

$$W_{\text{classical}} \approx (\Delta\omega)^2 kT_p[1+(T_nR_n'/T_pR_p')](2P)^{-1}, \quad (4.20)$$

whereas in the opposite limit we obtain the linewidth given by

$$W_{\text{quantum limit}} \approx (\Delta\omega)^2 \hbar\omega_0[1+|R_n'/R_p'|](4P)^{-1}. \quad (4.21)$$

If the noise sources are correctly given by the Nyquist or Johnson noises

$$R_n' = R_n(\omega_0), \qquad R_p' = R_p(\omega_0), \qquad R_n(\omega_0) = -R_p(\omega_0), \quad (4.22)$$

then our results reduce to the thermal-equilibrium expression

$$W_{\text{Th}} = (\Delta\omega)^2(kT_pC_p + k|T_n|C_n)/(2P), \quad (4.23)$$

where the linewidth $\Delta\omega$ is defined by

$$\Delta\omega = R_p/|L'|. \quad (4.24)$$

## 5. JUSTIFICATION OF THE DIFFUSION TREATMENT

Our heuristic discussion of diffusion in the preceding section had two approximate elements: one involved the neglect of the frequency variation of the noise-source spectrum, and the second involved the neglect of the nonlinear dependence on the phase. The "white-noise" approximations of Eqs. (4.9) and (4.10) are rewritten in the form

$$\int_{-\infty}^{\infty} \frac{\sin^2[(\omega-\omega_0)\Delta t/2]}{(\omega-\omega_0)^2} d\omega (G^2)_\omega \approx (\tfrac{1}{2}\pi)(G^2)_{\omega_0}\Delta t, \quad (5.1)$$

$$\int_{-\infty}^{\infty} \frac{\cos\omega_0\Delta t - \cos\omega\Delta t}{\omega^2-\omega_0^2} d\omega (G^2)_\omega \approx \pi (G^2)_{\omega_0} \frac{\sin\omega_0\Delta t}{\omega_0}. \quad (5.2)$$

The approximation of the first factor in each of these integrands by a $\delta$ function in $\omega$ can be seen to be justified for a time interval $\Delta t$ such that

$$\Delta t\, \delta\omega \gg 1, \quad (5.3)$$

where $\delta\omega$ is the frequency width over which the noise spectrum $(G^2)_\omega$ has an appreciable variation. In the following analysis, we shall see that $\Delta t$ must be chosen large compared to $1/\omega_0$, but it need only be chosen small compared to the reciprocal of the linewidth. Thus the criterion (5.3) is equivalent to the statement that the input noise spectrum must not vary much over the linewidth. Since the linewidth will turn out to be exceedingly narrow, this is in fact a very mild restriction. Thus for all practical purposes, the phase broadening of an oscillator line with a nonwhite-noise spectrum is essentially the same as the phase broadening associated with a white spectrum. In what follows, we shall assume that the spectrum is white in order to convert our process to a Markoffian process.

Let us now rewrite our nonlinear random process, (4.2), which is a universal description for phase fluctuations in an autonomous oscillator, in the form

$$d\phi/dt = G(t)\cos(\omega_0 t+\phi) = F(t)e^{-i\phi} + F(t)^* e^{i\phi} \quad (5.4)$$

and

$$F(t) = \tfrac{1}{2}G\, e^{-i\omega_0 t} \qquad F(t)^* = \tfrac{1}{2}G\, e^{i\omega_0 t}. \quad (5.5)$$

We shall now follow the methods described in IV, Sec. 5 for the Markoffian representation of processes with short correlation times. We choose a time interval $\Delta t$ long compared to $1/\omega_0$. The random forces for the "reduced" random process are then describable in terms of a diffusion constant $D_{FF^*}$,

$$\langle F(t)F(t')^*\rangle \approx 2D_{FF^*}\delta(t-t'), \qquad (5.6)$$

where this diffusion constant can be evaluated by means of

$$2D_{FF^*}=(\Delta t)^{-1}\int_t^{t+\Delta t} ds\int_t^{t+\Delta t} ds' \langle F(s)F(s')^*\rangle$$

$$=(2\pi\Delta t)^{-1}\int d\omega (G^2)_\omega \frac{\sin^2[(\omega-\omega_0)\Delta t/2]}{(\omega-\omega_0)^2}$$

$$\approx \tfrac{1}{4}(G^2)_{\omega_0}. \qquad (5.7)$$

It is easy to choose a $\Delta t$ large enough so that (5.3) is obeyed and also small compared to the reciprocal of the resulting linewidth. Over such an interval, $\Delta t$, the noise spectrum appears to be flat and the process has a Markoffian behavior.

A similar calculation leads to

$$\langle F(t)F(t')\rangle = 2D_{FF}\delta(t-t'), \qquad (5.8)$$

$$2D_{FF}=(\Delta t)^{-1}\int_t^{t+\Delta t} ds\int_t^{t+\Delta t} ds' \langle F(s)F(s')\rangle$$

$$=\frac{e^{-2i\omega_0 t}e^{-i\omega_0\Delta t}}{4\pi\Delta t}\int (G^2)_\omega d\omega \frac{\cos\omega_0\Delta t-\cos\omega\Delta t}{\omega^2-\omega_0^2}$$

$$=e^{-2i\omega_0 t}e^{-i\omega_0\Delta t}\frac{\sin\omega_0\Delta t}{4\omega_0\Delta t}(G^2)_{\omega_0}\approx 0. \qquad (5.9)$$

Thus we see that because of the rapid oscillations associated with the frequency $\omega_0$, the noise source $F$ does not correlate with itself but only with its complex conjugate.

In accord with III, (5.13), the distribution function in the phase obeys the generalized Fokker-Planck equation

$$\frac{\partial P(\phi,t)}{\partial t}=\sum(-1)^n\frac{\partial}{\partial\phi^n}[D_n(\phi)P(\phi,t)], \qquad (5.10)$$

where the diffusion coefficients are defined by

$$n!D_n(\phi)=\langle[\phi(t+\Delta t)-\phi(t)]^n\rangle/\Delta t. \qquad (5.11)$$

The change in $\phi$ over the time interval $\Delta t$ can be computed using Eq. (5.4) in the form

$$\Delta\phi=\phi(t+\Delta t)-\phi(t)=\int_t^{t+\Delta t}[F(s)e^{-i\phi(s)}+F(s)^*e^{i\phi(s)}]ds. \qquad (5.12)$$

Unfortunately the right-hand side of (5.12) still contains the unknown function $\phi$. We handle this by setting up an iteration procedure using

$$\phi(s)=\phi(t)+\int_t^s\frac{d\phi(s')}{ds'}ds', \qquad (5.13)$$

which leads to a sum of terms

$$\Delta\phi=\Delta\phi_1+\Delta\phi_2+\cdots. \qquad (5.14)$$

The first of these terms is given by

$$\Delta\phi_1=e^{-i\phi(t)}\int_t^{t+\Delta t}F(s)ds+e^{i\phi(t)}\int_t^{t+\Delta t}F(s)^*ds. \qquad (5.15)$$

The second-order correction is given by

$$\Delta\phi_2=-i\int_t^{t+\Delta t}ds\int_t^s ds'F(s)F(s')^*+\text{c.c.}$$

$$-ie^{-2i\phi(t)}\int_t^{t+\Delta t}ds\int_t^s ds'F(s)F(s')+\text{c.c.} \qquad (5.16)$$

Our functions $F$ and $F^*$ in (5.15) and (5.16) are assumed to be Gaussian random variables. If $G_1, G_2, \cdots$ are any set of Gaussian random variables, they obey

$$\langle G_1G_2\cdots G_n\rangle=0 \qquad (n \text{ odd}) \qquad (5.17)$$

$$\langle G_1G_2\cdots G_{2n}\rangle=\langle G_1G_2\rangle\langle G_3G_4\rangle\cdots\langle G_{2n-1}G_{2n}\rangle + \text{other pair decompositions}. \qquad (5.18)$$

Thus the $2n$-fold time integral

$$\int_t^{t+\Delta t}dt_1\cdots\int_t^{t+\Delta t}dt_{2n}\langle G_1\cdots G_{2n}\rangle$$

$$=N\left[\int_t^{t+\Delta t}dt_1\int_t^{t+\Delta t}dt_2\langle G(t_1)G(t_2)\rangle\right]^n\sim(\Delta t)^n \qquad (5.19)$$

can be decomposed into a number $N$ of products of pairs each of which is given by the double integral shown. Moreover, if $\Delta t$ is any time interval large compared to the correlation time of $G(t_1)$ and $G(t_2)$, then this double integral is linear in $\Delta t$, and the $n$-fold product of such double integrals varies as $(\Delta t)^n$ as shown in Eq. (5.19). Equation (5.11) tells us, however, to find only the part of the change in $(\Delta\phi)^n$ that is linear in $\Delta t$ as $\Delta t$ becomes small. We see from (5.19) that such a linear term $\Delta t$ only occurs when there are no more than two Gaussian factors. Thus we can conclude

$$\langle D_n\rangle=0 \qquad n\geq 3. \qquad (5.20)$$

If we wish to retain only terms that are of second order in $F$ and $F^*$, the second-order diffusion coefficient can be computed using only the first-order approximation to $\Delta\phi$ as in

$$2\langle D_2\rangle=\langle(\Delta\phi_1)^2\rangle$$

$$=(\Delta t)^{-1}\int_t^{t+\Delta t}ds\int_t^{t+\Delta t}ds'\langle F(s)F(s')^*\rangle+\text{c.c.}$$

$$=2D_{FF^*}+\text{c.c.}=\tfrac{1}{2}(G^2)_{\omega_0}, \qquad (5.21)$$

where the terms depending on $\phi$ disappear in view of (5.9). In calculating $D_1$, however, the contribution of $\Delta\phi_2$ is essential since we see from Eq. (5.16) that it contains terms of second order. The $\Delta\phi_1$ contribution is only linear in $F$ and vanishes automatically. The second term in (5.16) vanishes because of Eq. (5.9). The first term in Eq. (5.16) cancels against its complex conjugate; thus we can conclude that

$$2\langle D_1\rangle=\langle\Delta\phi_1\rangle+\langle\Delta\phi_2\rangle+\cdots=0. \qquad (5.22)$$

The generalized Fokker–Planck Eq. (5.10) now simplifies to the usual Fokker–Planck equation; indeed, to the simple diffusion equation

$$\partial P/\partial t=\tfrac{1}{4}(G^2)_{\omega_0}\partial^2P/\partial\phi^2\equiv D\,\partial^2P/\partial\phi^2 \qquad (5.23)$$

as the description of our "reduced random process." Thus, as claimed in the introduction, the phase executes a simple Brownian motion. The conditional probability distribution, which is the Green's-function solution of Eq. (5.23), is well known in the form

$$P(\phi(t),t\,|\,\phi(0),0)=(4\pi Dt)^{-1/2}\times\exp\{-[\phi(t)-\phi(0)]^2/4Dt\}. \qquad (5.24)$$

Moreover, we see from (5.24) that the phase difference $\phi(t)-\phi(0)$ is a Gaussian random variable, as was previously assumed in the evaluation of the linewidth.

## 6. PHASE-WIDTH ENHANCEMENT DUE TO AMPLITUDE FLUCTUATIONS

When amplitude and phase are coupled together, as in Eqs. (3.27) and (3.28), the situation is slightly more complicated than that discussed in the preceding sections. Equation (3.29) suggests, however, that a simplification can be performed if one introduces a new variable $\phi$:

$$\phi=v\cos\beta-u\sin\beta;\qquad v=(\phi/\cos\beta)+u\tan\beta, \qquad (6.1)$$

so chosen that in the absence of noise it does not couple to any amplitude fluctuations. This variable then obeys the nonlinear random equation

$$d\phi/dt=G(t)e^{-u}\cos[\omega_0t+\beta+u\tan\beta+(\phi/\cos\beta)]. \quad (6.2)$$

Since the amplitude fluctuations are bounded, it is permissible to treat them in a quasilinear fashion by setting the variable $u=0$, which leads to the simpler equation

$$d\phi/dt\approx G(t)\cos[\omega_0t+\beta+(\phi/\cos\beta)]. \qquad (6.3)$$

This equation differs only slightly from that discussed in the preceding section. We saw there that the phase on the right-hand side of the equation cancels out of the various moments needed in calculating the diffusion constants because the random variable $F$ is coupled only to its complex conjugate. This in turn assures that factors of $\exp(i\phi)$ are canceled by factors of $\exp(-i\phi)$. Thus the correct answer can be obtained by ignoring the phase on the right-hand side of (6.3), and with a simple shift of time origin our random problem is reduced to the trivial form

$$d\phi/dt\approx G(t)\cos(\omega_0t). \qquad (6.4)$$

Using the second part of Eq. (6.1), we see that the diffusion in our original variable $v$ differs by a constant factor from the diffusion in our new variable $\phi$:

$$\langle[v(t)-v(0)]^2\rangle=(\cos\beta)^{-2}\langle[\phi(t)-\phi(0)]^2\rangle+\text{finite terms}. \qquad (6.5)$$

The mean-square displacement of $\phi$ increases linearly with time, so that the finite terms on the right-hand side of (6.5) are unimportant in determining the linewidth. The net result is that, in the presence of the coupling between phase and amplitude fluctuations, our linewidth is simply increased by a constant factor

$$W_{\text{enhanced}}=W_{\text{original}}/\cos^2\beta. \qquad (6.6)$$

## 7. MEAN-VALUE APPROXIMATION METHOD

The mean-value method which has been employed a number of times in dealing with noise in masers[8] consists in neglecting the dynamic aspects of the variable[9] $D$ and replacing it simply by a mean value. This leads to

$$L\,dI/dt+L\omega_0^2Q+Z_1(-id/dt,\bar{D})I=e(t). \qquad (7.1)$$

Since Eq. (7.1) is now a linear equation, it is permissible to use the usual Fourier-transform methods. We follow the notation explained in Ref. 13 of IV and find that the Fourier component is given by

$$\{L[i\omega+(\omega_0^2/i\omega)]+Z_1(\omega,\bar{D})\}I_\omega=e_\omega. \qquad (7.2)$$

We can then obtain the spectral density of the current as

$$\langle|I_\omega|^2\rangle=\frac{\langle|e_\omega|^2\rangle}{|iL(\omega-\omega_0^2/\omega)+(\partial R/\partial\omega)(\omega-\omega_0)+(\partial Z/\partial D)\Delta D|^2}, \qquad (7.3)$$

where

$$\Delta D=\bar{D}-D_0. \qquad (7.4)$$

[8] A. L. Schawlow and C. H. Townes, Phys. Rev. **112**, 1490, (1958); J. A. Fleck, Jr., J. Appl. Phys. **34**, 2997 (1963); R. V. Pound, Ann. Phys. (N. Y.) **1**, 24 (1957); M. P. W. Strandberg, Phys. Rev. **106**, 617 (1957); J. Weber, Rev. Mod. Phys. **31**, 681 (1959); W. H. Wells, Ann. Phys. (N. Y.) **12**, 1 (1961); G. Kemeny, Phys. Rev. **133**, A69 (1964); H. Risken, Z. Phys. **180**, 150 (1964); W. G. Wagner and G. Birnbaum, J. Appl. Phys. **32**, 1185 (1961); See also Sec. 6 of QV.

[9] In masers, $D$ represents the population difference between the upper and lower state.

In the positive-frequency region, we can approximate the denominator in the form

$$iL\frac{(\omega+\omega_0)(\omega-\omega_0)}{\omega}+\frac{\partial R}{\partial\omega}(\omega-\omega_0)\approx 2iL'(\omega-\omega_0) \quad (7.5)$$

for $\omega>0$, using (3.23). Thus the spectral density of the current reduces to the form

$$\langle|I_\omega|^2\rangle=\frac{(e^2)_\omega/4\,|\,L'\,|^2}{|\,\omega-\omega_0+(2iL)^{-1}(\partial Z/\partial D)\Delta D\,|}. \quad (7.6)$$

Making use of Eqs. (3.23) and (3.24), this spectral density can be written in the suggestive form

$$(I^2)_\omega=\frac{(e^2)_\omega/4\,|\,L'\,|^2}{|\,\omega-\omega_0-(\partial\omega_0/\partial D_0)\Delta D\,|^2}$$

$$\approx\frac{(e^2)_{\omega_0}}{4\,|\,L'\,|^2}[(\omega-\omega_0-\Delta\omega)^2+(\tfrac{1}{2}W_L)^2]^{-1}. \quad (7.7)$$

We see that the real part of the derivative $\partial\omega_0/\partial D_0$ gives rise to a frequency shift $\Delta\omega$, and the imaginary part of this derivative gives rise to a linewidth which we have denoted $W_L$, the $L$ to remind us that a linear approximation has been made.

The mean square of the current is given by the total spectral density in the form

$$\langle[I(t)]^2\rangle=(2\pi)^{-1}\int_{-\infty}^{\infty}(I^2)_\omega d\omega=\pi^{-1}\int_0^{\infty}(I^2)_\omega d\omega$$

$$\approx\frac{(e^2)_{\omega_0}}{4\,|\,L'\,|^2}\frac{2}{W_L}. \quad (7.8)$$

By solving this equation for a linewidth, we obtain an expression for the linewidth in terms of the total power or the value of $a$ at the operating point

$$W_L=(e^2)_{\omega_0}/2\,|\,L'\,|^2\langle I^2\rangle=(e^2)_{\omega_0}/|\,L'a_0\,|^2=2\,(\cos^2\beta)\,W. \quad (7.9)$$

Comparing the real and imaginary parts of Eq. (3.24), we find that the linewidth and frequency shift are related by

$$\Delta\omega=\tfrac{1}{2}W_L\tan\beta. \quad (7.10)$$

## 8. THE TUNED ADIABATIC OSCILLATOR

Our results for the phase noise of our oscillator do not depend on the details of the approach to equilibrium for the amplitude. Indeed (having treated the amplitude quasilinearly), the distribution of phases can be written down once and for all for a nearly arbitrary model. A discussion of amplitude noise requires however that we adopt a specific model for the return to equilibrium of the control parameter $D$. For simplicity, let us start with an oscillator that possesses no reactance other than that associated with a tuned circuit and provides no coupli g between phase and amplitude fluctuations. In particular, we assume

$$X_1=0, \qquad \partial R/\partial\omega=0. \quad (8.1)$$

Our equation of motion then takes the form

$$LdI/dt+L\omega_0^2Q+R(\rho)I=e(t), \quad (8.2)$$

where

$$\rho=|\,a\,|^2=|\,A\,|^2; \qquad I-i\omega_0Q=a=Ae^{-i\omega_0t}. \quad (8.3)$$

In Eq. (8.2) we have assumed that the control parameter $D$ is simply the variable $\rho$ itself. Thus the control parameter $D$ is sensitive to, or *adiabatically* follows, the *instantaneous* value of the amplitude of the oscillator; hence the name "adiabatic oscillator." In terms of our complex variable $A$, Eq. (8.2) reduces to the form

$$dA/dt=-(2L)^{-1}R(\rho)A+L^{-1}e_-(t); \qquad e_-\equiv e^{i\omega_0t}e(t). \quad (8.4)$$

### A. Cartesian Representation of Adiabatic Oscillator

For purposes of visualization it is often convenient to make the transformation to real variables

$$A=x-iy, \qquad \rho=x^2+y^2=r^2. \quad (8.5)$$

Equation (8.4) is then equivalent to the pair of equations

$$dx/dt=-(2L)^{-1}R(\rho)x+F_x,$$

$$dy/dt=-(2L)^{-1}R(\rho)y+F_y, \quad (8.6)$$

where

$$F_x(t)=L^{-1}e(t)\cos\omega_0t, \qquad F_y(t)=L^{-1}e(t)\sin\omega_0t. \quad (8.7)$$

For time intervals $\Delta t$ large compared to $1/\omega_0$, we obtain a *reduced* random process whose diffusion constants computed by the methods of Sec. 5 take the simple values

$$\langle F_x(t)F_x(u)\rangle=2D_{xx}\delta(t-u); \qquad D_{xx}=\tfrac{1}{4}(e^2)_{\omega_0}/L^2; \quad (8.8)$$

$$D_{yy}=D_{xx}; \qquad D_{xy}=0. \quad (8.9)$$

The mean values of these forces are zero. Using Eq. (5.13) from III, we can immediately write the Fokker-Planck equation for the reduced process:

$$\frac{\partial P(x,y,t)}{\partial t}=\frac{\partial}{\partial x}\left[\frac{R(\rho)x}{2L}P\right]+\frac{\partial}{\partial y}\left[\frac{R(\rho)y}{2L}P\right]+D_{xx}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]P. \quad (8.10)$$

### B. Equations for Radial (Amplitude) Fluctuations

By multiplying Eq. (8.4) by $A^*$ and adding the complex conjugate we obtain an equation for the radial variable $\rho$:

$$d\rho/dt=-L^{-1}R(\rho)\rho+L^{-1}(e_+A+e_-A^*). \quad (8.11)$$

The nonlinear random terms have a nonvanishing mean value which can be obtained by the methods of IV, Sec. 3:

$$\begin{aligned}\langle e_+A\rangle &= (\Delta t)^{-1}\int_t^{t+\Delta t}\left\langle e_+(s)\,ds\left[A(t)+\int_t^s\frac{dA(s')}{ds'}\,ds'\right]\right\rangle\\ &=(\Delta t)^{-1}L^{-1}\int_t^{t+\Delta t}ds\int_t^s ds'\langle e_+(s)e_-(s')\rangle\\ &=L^{-1}\tfrac{1}{2}(e^2)_{\omega_0}. \quad (8.12)\end{aligned}$$

Thus our Langevin equation for $\rho$ can be written in the form

$$d\rho/dt=[-L^{-1}R(\rho)\rho+4D_{xx}]+F_\rho, \quad (8.13)$$

where the Langevin force $F_\rho$ has mean zero and second moments given by

$$\begin{aligned}\langle F_\rho(t)F_\rho(u)\rangle &=L^{-2}|A|^2[\langle e_+(t)e_-(u)\rangle+\langle e_-(t)e_+(u)\rangle]\\ &=L^{-2}\rho 2(e^2)_{\omega_0}\delta(t-u)=2D(\rho)\delta(t-u), \quad (8.14)\end{aligned}$$

where

$$D(\rho)=\rho(e^2)_{\omega_0}/L^2=4\rho D_{xx}. \quad (8.15)$$

The drift term, the first term in (8.13), and the diffusion coefficient given in (8.15) could equally well have been obtained from the corresponding drift and diffusion coefficients in the $x$ and $y$ representation by means of the transformation equations (3.27) and (3.28) of IV:

$$\begin{aligned}A(\rho)&=\frac{\partial\rho}{\partial x}A_x+\frac{\partial\rho}{\partial y}A_y+\frac{\partial^2\rho}{\partial x^2}D_{xx}+\frac{\partial^2\rho}{\partial y^2}D_{yy}\\ &=2xA_x+2yA_y+4D_{xx}=-L^{-1}R(\rho)\rho+4D_{xx}, \quad (8.16)\end{aligned}$$

$$D(\rho)=(\partial\rho/\partial x)^2D_{xx}+(\partial\rho/\partial y)^2D_{yy}=4\rho D_{xx}. \quad (8.17)$$

With the drift vector $A(\rho)$ given by (8.16) and the diffusion constant $D(\rho)$ given by (8.17), the probability distribution for amplitude fluctuations obeys the Fokker–Planck equation:

$$\frac{\partial P(\rho,t)}{\partial t}=-\frac{\partial}{\partial\rho}[A(\rho)P]+\frac{\partial^2}{\partial\rho^2}[D(\rho)P]. \quad (8.18)$$

### C. Operating Point and Phase Fluctuations

As discussed in I, the operating point is determined by the point at which the first diffusion coefficient (the drift vector) vanishes. This leads to the condition

$$A(\rho_0)=-L^{-1}R(\rho_0)\rho_0+4D_{xx}=0. \quad (8.19)$$

This condition is not entirely invariant to the choice of variables because the drift vector does not transform as an ordinary vector (see IV, Sec. 3). For example, if we had used the drift vector in Eq. (8.4), we would have arrived at the less satisfactory operating-point condition

$$R(\rho_{00})=0. \quad (8.20)$$

A simple approximation valid below threshold for the decay constant associated with phase fluctuations can be obtained from Eq. (8.4) by replacing the coefficient of $A$ on the right-hand side by a mean value. This leads to the decay constant

$$\Lambda_p=(2L)^{-1}\langle R(\rho)\rangle\approx(2L)^{-1}R(\rho_0)=2D_{xx}/\rho_0, \quad (8.21)$$

using (8.19). This procedure is equivalent to the mean-value approximation method previously discussed in Sec. 7. In contrast to (8.21), the corresponding decay constant correctly computed above threshold in (4.14) is equivalent to the result

$$(\tfrac{1}{2}W)=(e^2)_{\omega_0}/(4L^2|a_0|^2)=D_{xx}/\rho_0. \quad (8.22)$$

We see as usual that the mean-value method leads to a decay constant which is twice as large as that obtained by the method of dealing directly with the phase of the oscillator.

As indicated previously, the use of amplitude and phase is the correct precedure *above threshold* whereas the approximation of the nonlinear term by a mean value, which is equivalent to regarding as the real and imaginary parts of $A$ as independent variables, is the appropriate procedure *below threshold*. We have already displayed in Fig. 1 the transition between these two results.

It is not entirely trivial that the determination of the operating point by means of Eq. (8.19) is equivalent to the determination made in Sec. 7 by considering the output of the oscillator as linearly amplified noise.

### D. Decay Constant for Amplitude Fluctuations

The appropriate quasilinear procedure for obtaining the decay constant in a stable system (and our system is stable against amplitude fluctuations) is to obtain the decay constant from the linear expansion of the drift vector about the operating point. This leads to

$$\begin{aligned}\Lambda_a&=-\partial A(\rho)/\partial\rho\,|_{\rho=\rho_0}=L^{-1}R(\rho_0)+L^{-1}\rho_0\partial R/\partial\rho_0\\ &=4D_{xx}/\rho_0+L^{-1}\rho_0\partial R/\partial\rho_0. \quad (8.23)\end{aligned}$$

In the region below threshold, the parameter $\rho_0$ becomes small, and the first term in (8.23) dominates. We see by comparison with Eq. (8.21) that

$$\Lambda_a \approx 2\Lambda_p \text{ below threshold.} \tag{8.24}$$

*It is important to note that the noninvariance of the operating point under nonlinear transformation of variables introduces an element of judgment into the determining of the point about which a quasilinear analysis is to be made. If, for example, we had used $r=\rho^{1/2}$ instead of $\rho$, we would have found the incorrect result $\Lambda_a \approx \Lambda_p$.* The fact that $\Lambda_a \approx 2\Lambda_p$ is a simple consequence of the Gaussian behavior of our random variables below threshold:

$$\langle A^*(t)A(t)A^*(0)A(0)\rangle - \langle |A(t)|^2\rangle\langle |A(0)|^2\rangle = \langle A^*(t)A(0)\rangle\langle A(t)A^*(0)\rangle. \tag{8.25}$$

The absolute square shown on the right-hand side of (8.25) produces the doubling of the decay constant shown in Eq. (8.24).

The necessity for using $\rho$ rather than $r=\rho^{1/2}$ as a variable in a quasilinear technique is not well known, and is probably related to the fact that the condition $A(\rho_0)=0$ is essentially an energy-balance condition.

The decay constant for amplitude fluctuations shown in (8.23) depends upon the derivative of the resistance at the operating point. It may be convenient, for experimental purposes, to re-express this result in a form which is experimentally measurable. For this purpose, we consider modifying the load of the oscillator by adding an increase in the series resistance

$$R(\rho)\to R(\rho)+\Delta R_p. \tag{8.26}$$

If we resolve Eq. (8.19) for the shift in operating point due to this change in load, we obtain the result

$$\partial\rho_0/\partial R_p = -[\partial R(\rho_0)/\partial\rho_0]^{-1}. \tag{8.27}$$

Since the power dissipated in the positive resistance is given by

$$P=\tfrac{1}{2}R_p\rho_0, \tag{8.28}$$

we can, by taking the appropriate derivatives, obtain the relation

$$R_p\partial P/\partial R_p - P = \tfrac{1}{2}R_p^2\partial\rho_0/\partial R_p. \tag{8.29}$$

Thus our required derivative of the resistance at the operating point is expressible in the form

$$\partial R(\rho_0)/\partial\rho_0 = \tfrac{1}{2}R_p^2[P-R_p\,\partial P/\partial R_p]^{-1}, \tag{8.30}$$

and our decay constant for amplitude fluctuations can be rewritten in the form

$$\Lambda_a = 4\frac{D_{xx}}{\rho_0} + \frac{R_p}{L}\left[1-\frac{R_p}{P}\frac{\partial P}{\partial R_p}\right]^{-1}. \tag{8.31}$$

## 9. ROTATING-WAVE VAN DER POL OSCILLATOR (RWVP)

An important special case of the adiabatic oscillator discussed in the previous section is one in which the resistance is a linear function of $\rho$, or one in which it can be approximated as a linear function in the neighborhood of the operating point:

$$R=R(\rho_0)+\partial R(\rho_0)/\partial\rho_0(\rho-\rho_0)=R_0\rho-\Pi, \tag{9.1}$$

$$R_0\equiv\partial R/\partial\rho_0; \qquad \Pi=\rho_0\partial R/\partial\rho_0-R(\rho_0). \tag{9.2}$$

This approximation is valid for all well-designed oscillators[10] even near threshold.[11] It is then convenient to make a change in the scale of the amplitude and the time:

$$x=\xi x', \qquad y=\xi y', \qquad t=Tt', \tag{9.3}$$

which results in

$$dx'/dt' = -(TR_0\xi^2/2L)(x'^2+y'^2-p)x'+F_x', \tag{9.4}$$

where

$$p=\Pi/(\xi^2R_0) \tag{9.5}$$

and

$$F_x'=(T/\xi)F_x. \tag{9.6}$$

The new moments of the Langevin forces are given by

$$\begin{aligned}\langle F_x'(t')F_x'(u')\rangle &= (T/\xi)^2\langle F_x(t)F_x(u)\rangle \\ &= (T/\xi)^2 2D_{xx}\delta(t-u) \\ &= (T/\xi^2)(2D_{xx})\delta(t'-u').\end{aligned} \tag{9.7}$$

If we choose our transformation so that the new diffusion constant takes on the value unity, we obtain the condition

$$D_{xx}'=(T/\xi^2)D_{xx}=1. \tag{9.8}$$

In addition, we may require the coefficient of the non-

[10] This quasilinear approximation in $R(\rho)$ has a much wider range of applicability than one might at first imagine. The ratio of the (omitted) second-order term to the linear term is of order

$$(\rho-\rho_0)\partial^2R/\partial\rho_0^2[\partial R/\partial\rho_0]^{-1}=(\rho-\rho_0)/\rho_1\sim\langle(\Delta\rho)^2\rangle^{1/2}/\rho_1,$$

where $\rho_1$, which measures the change in $\rho$ required to produce an important change in $R(\rho)$, will usually be of order $\rho_0$. Thus, the error involved is small if the operating level $\rho_0$ is large compared to the noise level $\langle(\Delta\rho)^2\rangle^{1/2}$, a condition obeyed in all well designed oscillators even at threshold. (For further discussion of the latter point see Ref. 11.)

[11] Since $\langle\rho'\rangle$ at threshold ($p=0$) is near unity (see Fig. 4), $\langle\rho\rangle=\xi^2\langle\rho'\rangle$ is roughly $\xi^2$ at threshold. Assuming $R(\rho)$ has the form $Kf(\rho/\rho_1)$, the threshold value of $\rho$ is thus given by

$$\rho_t=(2D_{xx}L/R_0)^{1/2}=(\rho_1\rho_n)^{1/2},$$

where we obtain $D_{xx}$ from (8.8) and define

$$\rho_n=(e^2)_{\omega 0}/[KLf'(\rho_t/\rho_1)]$$

as a typical value of $\rho$ that would be produced by noise. (We assume $K$ is so chosen that $f'$ is near unity.) For a well-designed oscillator, a typical operating value $\rho_1$ obeys $\rho_1\gg\rho_n$, so that the threshold value also obeys $\rho_t\gg\rho_n$.

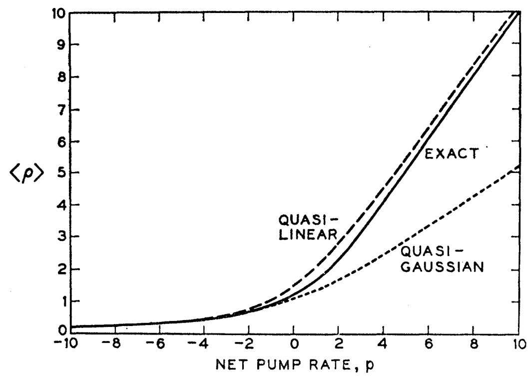


FIG. 4. The first moment $\langle\rho\rangle$ (proportional to the number of photons in a laser) is plotted versus dimensionless pump rate $p$. Using III (7B8) with $J=0$, the steady-state distribution is given by

$$W(\rho)=\exp[\tfrac{1}{2}(p\rho)-\tfrac{1}{4}\rho^2]\Big/\int_0^\infty d\rho\,\exp[\tfrac{1}{2}(p\rho)-\tfrac{1}{4}\rho^2],$$

and all moments can be computed from

$$\langle\rho^n\rangle=\int\rho^n W(\rho)\,d\rho$$

directly, or with the help of the recursion relation

$$\langle\rho^n\rangle=p\langle\rho^{n-1}\rangle+2(n-1)\langle\rho^{n-2}\rangle.$$

Since $\langle A(\rho)\rangle=\langle 4-2\rho(\rho-p)\rangle=0$, the quasilinear approximation $A(\rho_0)=0$ of (9.14) is equivalent to setting $\langle\rho^2\rangle\approx\langle\rho\rangle^2$. The quasi-Gaussian approximation, however, sets $\langle\rho^2\rangle\approx 2\langle\rho\rangle^2$. These approximations are respectively correct above and below thresholds. See Fig. 5.

linear term to take the value unity:

$$TR_0\xi^2/2L=1. \tag{9.9}$$

This results in the conditions

$$\xi^2=(2D_{xx}L/R_0)^{1/2};\qquad T=\xi^2/D_{xx}. \tag{9.10}$$

### A. Normalized Rotating-Wave van der Pol Oscillator (NRWVP)

When these scale changes have been made, we shall refer to our oscillator as a normalized rotating-wave van der Pol oscillator. All of the preceding equations for the RWVP oscillator remain valid providing we introduce the special values

$$L=\tfrac{1}{2},\qquad D_{xx}=1,\qquad R(\rho)=\rho-p,$$
$$A(\rho)=4-2\rho(\rho-p);\qquad D(\rho)=4\rho. \tag{9.11}$$

The fundamental stochastic equation (9.4) is replaced by

$$dA/dt=(p-|A|^2)A+h(t), \tag{9.12}$$

and the moments of the Langevin forces in this notation (with primes omitted) are given by

$$\langle h^*(t)h(u)\rangle=\langle h(t)h^*(u)\rangle=4\delta(t-u),$$
$$\langle h^*(t)h^*(u)\rangle=\langle h(t)h(u)\rangle=0. \tag{9.13}$$

In these units, the condition for the operating point leads to

$$A(\rho_0)=0\Longrightarrow\rho_0=\tfrac{1}{2}[p+(p^2+8)^{1/2}]. \tag{9.14}$$

See Fig. 4 for a comparison between $\rho_0$ and the exact $\langle\rho\rangle$ obtained from VI. The resulting decay constant associated with phase and amplitude fluctuations reduces to the form

$$\Lambda_p=2/\rho_0,\qquad\text{below threshold}$$
$$=1/\rho_0,\qquad\text{above threshold,} \tag{9.15}$$

and the decay constant for spectral linewidth associated with pure amplitude fluctuations reduces to the simple form

$$\Lambda_a\approx 4\rho_0-2p=2(p^2+8)^{1/2}=(2\rho_0+4/\rho_0). \tag{9.16}$$

With the present normalized form for the drift vector $A$ and the diffusion constant $D(\rho)$ the Fokker-Planck equation (8.18) reduces to the simple form

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial\rho}[(4-2\rho^2+2p\rho)P]+\frac{\partial^2}{\partial\rho^2}[4\rho P]. \tag{9.17}$$

## 10. NONADIABATIC DETUNED OSCILLATORS

### A. Phase and Amplitude Linewidths and Shifts Below Threshold

Our key equation (3.22) for the behavior of a nonadiabatic, possibly detuned, oscillator in the neighborhood of threshold is rewritten in the form

$$dA^*/dt=\lambda(D-D_0)A^*+e_+(t)/L'. \tag{10.1}$$

In the region below threshold we can estimate the linewidth associated with phase and amplitude fluctuations and the associated frequency shift by replacing the coefficient of $A^*$ by a mean value:

$$i\Delta\omega-\Lambda_p\approx\lambda(\langle D\rangle-D_0). \tag{10.2}$$

The real and imaginary parts of the above relationship lead to

$$\Lambda_p=(\mathrm{Re}\lambda)(D_0-\bar{D});\qquad\Delta\omega=-(\tan\beta)\Lambda_p\quad\text{below} \tag{10.3}$$

for the region below threshold.

To discuss amplitude fluctuations we use Eq. (10.1) and its complex conjugate to derive an equation for $\rho$ in the form

$$d\rho/dt=(2\mathrm{Re}\lambda)(D-D_0)\rho+4D_{xx}+F_\rho, \tag{10.4}$$

where

$$\langle F_\rho\rangle=0,\qquad D_{\rho\rho}=4\rho D_{xx};\qquad D_{xx}=(e^2)_{\omega_0}/(4|L'|^2). \tag{10.5}$$

The operating point is then determined by the condition

$$\langle A(\rho)\rangle = 2\mathrm{Re}\lambda\langle(D-D_0)\rho\rangle + 4D_{xx} = 0. \quad (10.6)$$

Equation (10.3) can now be rewritten in the form

$$\Lambda_p = (\mathrm{Re}\lambda)(D_0 - \bar{D}) \approx 2D_{xx}/\langle\rho\rangle \quad \text{below.} \quad (10.7)$$

The corresponding decay constant for pure amplitude fluctuations below threshold can be estimated by taking the mean value of the coefficient of $\rho$ in Eq. (10.4), which leads to the result

$$\Lambda_a \approx 2\mathrm{Re}\lambda(D_0 - \bar{D}) \approx 2\Lambda_p \quad \text{below.} \quad (10.8)$$

As in the adiabatic case, there is a factor-of-2 relationship between the amplitude and phase fluctuation decay constants below threshold because of the Gaussian nature of the fluctuations.

### B. Determination of Operating Point

To determine operating points for both $D$ and $\rho$, we not only need Eq. (10.6), but must also make use of the equation describing the equilibration of the operating parameter $D$. We shall take this equation to have the form

$$dD/dt = -f(D, \rho) + F_D(t). \quad (10.9)$$

Our condition for the operating point can be written in the form

$$f(D_1, \rho_0) = 0. \quad (10.10)$$

We have used the symbol $D_1$ to distinguish from the simpler approximation $D_0$ which was determined from Eq. (3.5). In the present analysis (3.5) is replaced by the slightly more accurate equation (10.6), which we rewrite in the form

$$2\mathrm{Re}\lambda(D_0 - D_1)\rho_0 = 4D_{xx}. \quad (10.11)$$

To make further explicit progress it is necessary to assume a form for the arbitrary function on the right-hand side of Eq. (10.9). This form can be obtained by expanding the function $f$ around the lowest approximation for the operating point $D_0$ and $\rho_0$ and retaining only the linear terms. We shall therefore take our equation for the operating parameter $D$ in the linear form

$$dD/dt = \mathcal{R} - \Gamma D - s\rho + F_D. \quad (10.12)$$

The condition (10.10) now takes the explicit form

$$D_1 = (\mathcal{R} - s\rho_0)/\Gamma. \quad (10.13)$$

Inserting this result into Eq. (10.11), we finally obtain a quadratic equation for $\rho_0$:

$$2\mathrm{Re}\lambda[(\mathcal{R} - s\rho_0)/\Gamma - D_0]\rho_0 + 4D_{xx} = 0. \quad (10.14)$$

Comparison with Eq. (8.16) indicates the relationship

$$L^{-1}R(\rho) \to 2\mathrm{Re}\lambda[(s/\Gamma)\rho - (\mathcal{R}/\Gamma - D_0)]. \quad (10.15)$$

Indeed if the parameter $\Gamma$ is large enough, an adiabatic approximation is valid, and our present results reduce to the earlier ones with the above transcription. This suggests, even in the nonadiabatic case, that we make a corresponding scaling transformation

$$\rho = \xi^2\rho'; \qquad \xi^4 = \Gamma D_{xx}/(s\mathrm{Re}\lambda). \quad (10.16)$$

The resulting operating point in the dimensionless notation again takes the simple form

$$\rho_0' = \tfrac{1}{2}[p + (p^2 + 8)^{1/2}], \quad (10.17)$$

where

$$p = (\mathrm{Re}\lambda/D_{xx})(\mathcal{R}/\Gamma - D_0). \quad (10.18)$$

An analysis of the amplitude fluctuations can now be made by making the quasilinear approximation which leads to the set of coupled equations

$$d(\Delta\rho)/dt = -(4D_{xx}/\rho_0)\Delta\rho + (2\mathrm{Re}\lambda)\rho_0\Delta D + F_\rho, \quad (10.19)$$

$$d(\Delta D)/dt = -s\Delta\rho - \Gamma\Delta D + F_D, \quad (10.20)$$

where

$$\Delta D = D - D_1, \qquad \Delta\rho = \rho - \rho_0. \quad (10.21)$$

Making use of the time-scale transformation

$$T = \xi^2/D_{xx} = [\Gamma/(sD_{xx}\mathrm{Re}\lambda)]^{1/2} \quad (10.22)$$

and a corresponding transformation of the decay constants

$$\Lambda_a' = \Lambda_a T, \qquad \Gamma' = \Gamma T, \quad (10.23)$$

the decay eigenvalues of the pair of equations (10.19) and (10.20) are obtained as a solution of a quadratic equation in the form

$$\Lambda_a' = \tfrac{1}{2}\{(4/\rho') + \Gamma' \pm [(4/\rho' - \Gamma')^2 - 8\Gamma'\rho']^{1/2}\}, \quad (10.24)$$

where $\rho'$ is written briefly for $\rho_0'$.

### C. Results in Adiabatic Limit with Detuning

The adiabatic limit occurs when the parameter $\Gamma'$ is sufficiently large. In this limit, one eigenvalue becomes very large and would not be observed in the noise, and the smaller eigenvalue reduces to the form

$$\Lambda_a' \to (4/\rho' + 2\rho'). \quad (10.25)$$

This result is in agreement with our previous result (9.16). In the present case, however, it is valid even when $\lambda$ is complex and $\beta$ is not equal to zero; in other words for the case in which amplitude and phase fluctuations are coupled. This coupling, of course, influences the location of the operating point, but the linewidth for amplitude fluctuations retains the same form (10.25) even in the presence of such coupling. The quasilinear estimate of the total amplitude fluctuation based on Eq. (5.20) of I can be given in the form

$$\langle(\Delta\rho)^2\rangle = (D_{\rho\rho}/\Lambda_a)_{\rho=\rho_0}. \quad (10.26)$$

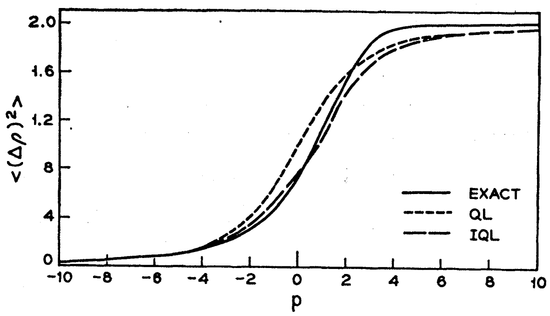


FIG. 5. The second moment of amplitude fluctuation, $\langle(\Delta\rho)^2\rangle = \langle\rho^2\rangle - \langle\rho\rangle^2$, is plotted versus dimensionless net pump rate $p$. For exact results see caption of Fig. 4. The Einstein relation yields the quasilinear approximation (QL):

$$\langle(\Delta\rho)^2\rangle \approx D/\Lambda \approx 4\rho_0[2\rho_0 + (4/\rho_0)]^{-1} = 1 + p(p^2+8)^{-1/2},$$

whereas the intelligent quasilinear approximation replaces $\rho_0$ by $\bar{\rho} = \langle\rho\rangle$:

$$\langle(\Delta\rho)^2\rangle \approx 4\bar{\rho}[2\bar{\rho} + (4/\bar{\rho})]^{-1}.$$

In terms of our dimensionless units, this result can be written in the simple form

$$\langle(\Delta\rho')^2\rangle \to 4\rho'/(4/\rho' + 2\rho'), \qquad (10.27)$$

which has the limiting behavior

$$\langle(\Delta\rho')^2\rangle/(\rho')^2 \to 1 \quad \text{as} \quad \rho' \to 0 \qquad (10.28)$$

appropriate to Gaussian fluctuations and a limiting behavior

$$\langle(\Delta\rho')^2\rangle \to 2 \quad \text{as} \quad \rho' \to \infty, \qquad (10.29)$$

which indicates the suppression of fluctuations as one moves well above threshold. See Figs. 5 and 6 for a comparison between the quasilinear second moment (10.27) and the corresponding exact result obtained from VI.

Spectra of noise generated by the pair of quasilinear equations (10.19) and (10.20) are obtainable by the standard quasilinear methods discussed in I. The noise, for example, is given explicitly in Eq. (1.13) of IV. The spectrum will involve both of the eigenvalues for decay constants of Eq. (10.24). The extent to which each of these decay constants appears in the spectrum is determined by the moments and correlations between the Langevin forces $F_\rho$ and $F_D$. Without a specific model, the moments of $F_D$ and its correlation with $F_\rho$ are not known. Therefore we shall not bother to write out any further specific results. Let us mention, however, that in the problem in which $\rho$ stands for the number of photons and $D$ for the population difference in a maser, these noise sources have been derived from first principles in QIV and QVII. A detailed evaluation of the spectrum involving both decay constants is given by McCumber[12] and also in paper QVII.

[12] D. E. McCumber, Phys. Rev. **141**, 306 (1966).

## 11. SUMMARY

In this paper we have introduced a model for a self-sustained oscillator based on Fig. 3;

$$Z[-i(d/dt), D]I(t) = e(t), \qquad (11.1)$$

where the operating parameter $D$ is itself controlled by a stochastic relation

$$dD/dt = \mathcal{R} - \Gamma D - s\rho + F_D, \quad (10.12), (11.2)$$

where

$$\rho = |a|^2, \qquad a = I - i\omega_0 Q. \qquad (11.3)$$

The lowest approximations to the operating frequency $\omega_0$ and parameter $D_0$ are given by

$$Z(\omega_0, D_0) = 0. \qquad (11.4)$$

Introducing the definitions

$$L' = -\tfrac{1}{2}i\partial Z/\partial\omega\,|_{\omega_0,D_0} \qquad (11.5)$$

$$-i(\partial Z/\partial D)/(\partial Z/\partial\omega)\,|_{\omega_0,D_0} \equiv \lambda \equiv |\lambda|\, e^{i\beta}, \quad (11.6)$$

$$D_{xx} = (e^2)_{\omega_0}/(4\,|L'|^2), \qquad (11.7)$$

where

$$(e^2)_\omega \equiv \int_{-\infty}^{\infty} e^{-i\omega t} dt \langle e(0)e(t)\rangle, \qquad (11.8)$$

we found that the autocorrelation important for the spectrum of oscillator noise was given by

$$\langle a^*(t)a(0)\rangle = \langle\rho\rangle \exp(-\Lambda_p|t|)\exp[-i(\omega_0+\Delta\omega)t], \quad (10.3), (11.9)$$

where $\langle\rho\rangle$, the mean signal, was reduced to an appropriate dimensionless form by means of

$$\langle\rho\rangle = \xi^2\langle\rho'\rangle, \qquad (11.10)$$

with the transformation defined by

$$\xi^4 = [\Gamma D_{xx}/(s\mathrm{Re}\lambda)]. \qquad (11.11)$$

In this dimensionless form, the operating point was

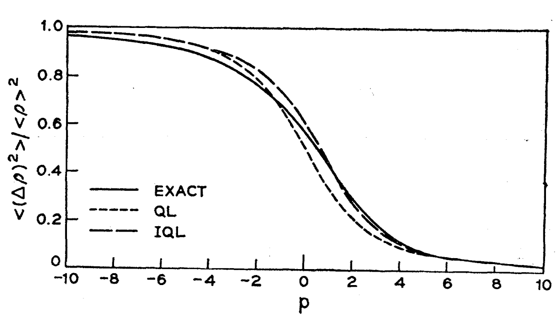


FIG. 6. The normalized second moment of amplitude fluctuations $\langle(\Delta\rho)^2\rangle\langle\rho\rangle^{-2}$ is plotted against net pump rate $p$, using $\langle(\Delta\rho)^2\rangle$ from Fig. 5.

found to be

$$\langle\rho'\rangle \approx \rho_0' = \tfrac{1}{2}[p+(p^2+8)^{1/2}], \quad (10.17), (11.12)$$

where the parameter $p$ represents the net pump rate in dimensionless units and is given by

$$p = (\mathcal{R}/\Gamma - D_0)\,\mathrm{Re}\lambda/D_{xx}. \quad (10.18), (11.13)$$

After introducing the new time unit

$$T = \xi^2/D_{xx} = [\Gamma/(sD_{xx}\mathrm{Re}\lambda)]^{1/2}, \quad (11.14)$$

we found that the spectrum for combined amplitude and phase fluctuations was given in Lorentzian form as the Fourier transform of Eq. (11.9) above, where the dimensional half-width $\Lambda_p$ is defined by

$$\Lambda_p T = (\langle\rho'\rangle\cos^2\beta)^{-1} \quad \text{above (6.6)}$$
$$= 2/\langle\rho'\rangle \quad \text{below } W_{\text{enhanced}}. \quad (7.9), (11.15)$$

With the notation

$$\Delta\rho = \rho - \langle\rho\rangle, \quad (11.16)$$

the autocorrelation for pure amplitude fluctuations for the case in which the parameter $D$ adiabatically follows $\rho$ was given by

$$\langle\Delta\rho(t)\Delta\rho(0)\rangle = \langle(\Delta\rho)^2\rangle \exp(-\Lambda_a|t|), \quad (11.17)$$

with the dimensional half-width $\Lambda_a$ for amplitude fluctuations given approximately, using the intelligent quasilinear approximation given in the caption of Fig. 2, by

$$\Lambda_a T \approx 2\langle\rho'\rangle + 4/\langle\rho'\rangle \quad (11.18)$$

and the total fluctuations in our dimensionless units given by

$$\langle(\Delta\rho')^2\rangle = 4\langle\rho'\rangle/[(4/\langle\rho'\rangle)+2\langle\rho'\rangle]. \quad (11.19)$$

In the nonadiabatic case we found that two different decay parameters are needed to describe the amplitude noise and these parameters are given in the intelligent quasilinear approximation by

$$\Lambda_a T = \tfrac{1}{2}\left\{\frac{4}{\langle\rho'\rangle} + \Gamma T \pm \left[\left(\Gamma T - \frac{4}{\langle\rho'\rangle}\right)^2 - 8\Gamma T\langle\rho'\rangle\right]^{1/2}\right\}. \quad (10.24), (11.20)$$

If one takes the limit $\Gamma T$ approaching infinity, the lower root of Eq. (11.20) reduces to the usual adiabatic result (11.18).

From Figs. 1 and 2, we see that the quasilinear methods discussed in this paper are quite accurate everywhere except in the small region $-10<p<10$ near threshold (i.e., $0.2<\langle\rho\rangle<10$), and that even in this region, the results are qualitatively correct.

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# Radiation from the $^4T_2$ State of $Cr^{3+}$ in Ruby and Emerald

P. Kisliuk and C. A. Moore
*Aerospace Corporation, El Segundo, California*
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The first appreciable decrease in the lifetime of the $^2E$ state of $Cr^{3+}$ in ruby and emerald as the temperature is raised is shown to be the result of populating the shorter-lived $^4T_2$ level in thermal equilibrium with the metastable $^2E$. The direct transition from the $^4T_2$ to the ground $^4A_2$ state is shown to be mostly radiative. In both crystals, the emission peak of the $^4T_2 \leftrightarrow {}^4A_2$ transition is shifted considerably toward longer wavelength compared to the absorption peak of that transition. The reduction in lifetime and quantum efficiency at very high temperatures is also considered.

## I. INTRODUCTION

THE energy levels of iron-group ions of the $d^3$ configuration in nearly octahedral crystal sites have been the subject of intensive theoretical and experimental investigation which has been summarized in a number of reviews.[1,2] The greatest effort has focused on the $Cr^{3+}$ ion in ruby, which is chosen as an example in both of the above references. The states therein described which enter into the following discussion are the ground $^4A_2$, the metastable $^2E$ lying at roughly 14 500 cm$^{-1}$, and the lowest excited quartet state—the $^4T_2$—which gives rise to the broad absorption band in ruby centered near 18 000 cm$^{-1}$. The only other state in this region is the $^2T_1$, lying about 550 cm$^{-1}$ above the $^2E$ in ruby.[3] However, since the $^4A_2 \rightarrow {}^2T_1$ transition is weakly absorbing and the $^2T_1$ state decays quickly to the $^2E$,[4] the $^2T_1$ plays but a small role in the economics of absorption and emission; in the remainder of this paper its population will usually be added to the $^2E$, these levels together forming the metastable doublet

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[2] J. Margerie, J. Phys. (Paris) **26**, 268 (1965).
[3] J. Margerie, Compt. Rend. **225**, 1598 (1962).
[4] J. L. Calviello, E. W. Fisher, and Z. H. Heller, J. Appl. Phys. **37**, 3156 (1966).