Mechanical Energy Density of Second Sound in Liquid Helium II†

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The mechanical energy density in a second-sound beam was determined experimentally between 1.2°K and the lambda point by measuring the force that the beam exerts on the vane of a torsion balance. At low power input, the experimental energy density conforms to the expression $E=(\rho CTu_2^2)^{-1}\langle\dot{Q}^2\rangle_{\mathrm{av}}$ derived from low-velocity nondissipative two-fluid theory. At higher power input the experimental values are too low. The discrepancy does not seem to arise from large amplitude effects in the second-sound wave. The temperature and power dependence of the observed attenuation is approximately the same as in Vinen's experiment on the attenuation of second sound propagated across a steady heat current in a tube. Vinen shows that the attenuation is caused by turbulence in the heat current, although the exact nature of the turbulence and the role played by tube diameter and surface imperfections of the tube walls is still controversial. It appears that in the present experiment the attenuation also results from scattering of second sound by turbulence in a heat current. This heat current arises from the dc component of the heat input to the second-sound source, and it flows freely through the bulk liquid, in the same direction as the second-sound beam. Since the observed attenuation is essentially the same as in Vinen's experiment, it is reasonable to conclude that the turbulence is approximately isotropic and that tube diameter and wall effects are not decisive factors for nucleation and growth of turbulence in a heat current in helium II.

INTRODUCTION

THE unique mechanism of heat transfer in liquid helium II allows the measurement of heat currents by mechanical effects. In terms of the two-fluid theory, heat is transferred in liquid helium II by the flow of a viscous normal fluid coupled to the counterflow of an inviscid superfluid in such a manner that there is no net transfer of matter. The normal fluid, carrying all the entropy, flows away from the source of heat, the superfluid towards it. Thus heat currents in liquid helium II imply a finite flux of momentum and exert finite forces on surfaces in their path.

In the low-velocity, nondissipative theory, for free counterflow, the flux of momentum density in a heat current is given by[1]

$$\rho_s v_s^2+\rho_n v_n^2=(\rho CTu_2^2)^{-1}\dot{Q}^2, \qquad (1)$$

where $\dot{Q}$ is the instantaneous heat current density. Since for a traveling plane wave the time average of the left side of the equation also represents the mechanical energy density E, one obtains

$$E=(\rho CTu_2^2)^{-1}\langle\dot{Q}^2\rangle_{\mathrm{av}}. \qquad (2)$$

Equations (1) and (2) relating the mechanical to the thermal quantities apply to steady heat currents as well as to alternating heat currents, i.e., second sound.

The first measurements of mechanical effects of heat currents were performed by Kapitza[2] some years before the discovery of second sound. The mechanical reaction on a source of heat was measured by Hall.[3] His results were in substantial agreement with Eq. (1) but an arbitrary scaling factor was needed for numerical comparison. Detection and measurement of second sound by purely mechanical means was achieved by Pellam and his co-workers who adapted the Pitot tube[4] and the Rayleigh disk[5] to the purpose. The experimental results fully confirmed the validity of Eq. (1) for the observed range of values.

Other acoustical measuring devices which, just as the Rayleigh disk, respond to second-order, i.e., quadratic effects, are also adaptable to the measurement of second sound. One such device is the radiation balance frequently employed for the measurement of the intensity of high-frequency ultrasound beams. It is based on the phenomenon of "radiation pressure" leading to a force exerted by a high-frequency beam on an intercepting surface. The force is related to the energy density in the beam as will be shown in the next section.

The radiation balance should provide, in some respects, a more direct method of measurement than the thermal Rayleigh disk, because it does not rely on resonance and the determination of a resonance re-

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[1] See for example: K. R. Atkins, *Liquid Helium* (Cambridge University Press, New York, 1959); C. T. Lane, *Superfluid Physics* (McGraw-Hill Book Company, Inc., New York, 1962); and C. F. Squire, *Low Temperature Physics* (McGraw-Hill Book Company, Inc., New York, 1953).

[2] P. L. Kapitza, J. Phys. (USSR) **4**, 181 (1941).

[3] H. E. Hall, Proc. Phys. Soc. (London) **A67**, 485 (1954).

[4] J. R. Pellam, Phys. Rev. **78**, 818 (1950).

[5] J. R. Pellam and P. Morse, Phys. Rev. **78**, 474 (1950); J. R. Pellam and W. Hanson, *ibid.* **85**, 216 (1952).

inforcement factor. Because of its basic simplicity, the radiation balance lends itself particularly well to a controlled variation of the parameters; temperature, input power, and frequency, over a wide range of values and to a study of their influence on the energy density of second sound.

Some time ago one of us adapted the radiation balance to second sound and was able to detect and measure second-sound radiation pressure.[6] The particular balance used was a torsion balance. However, the force measurement as determined by deflection of the balance was unsatisfactory. Moreover, reflections from the walls of the Dewar, and the presence of a steady heat current superimposed on the second sound affected the reliability of the results.

These defects have been largely overcome. Force measurements are now determined by a null method, an absorber prevents reflection from the Dewar walls, and a new procedure corrects for the force of the steady heat current.[7]

RADIATION PRESSURE OF ULTRASONIC AND SECOND-SOUND WAVES

The mechanical energy density in an acoustic wave can be identified with a radiation pressure. The term "pressure" is misleading because the stress in an acoustic wave in not a pure compression, equal in all directions, but is given by a tensor, the momentum flux density tensor Π_{ik}[8]:

$$\Pi_{ik}=p\delta_{ik}+\rho v_i v_k, \tag{3}$$

where p is the hydrostatic pressure, ρ the fluid density, and $\mathbf{v}$ the fluid velocity at a given time and a given point fixed in space.

When seeking to evaluate the force due to radiation pressure on a material obstacle exposed to a sound field, the specific boundary conditions in the particular experiment must be taken into account.

The case of interest here is a parallel beam of high-frequency sound of cross section A passing through undisturbed fluid and incident obliquely on an intercepting plane solid target where it is partially or fully reflected. No radiation is transmitted by the target. In the fluid region adjacent to the target the incident and the reflected beam overlap and a standing-wave region is formed.

The average force $\langle\mathbf{F}\rangle_{\text{av}}$ exerted by the fluid on the target is obtained by calculating the momentum flux through a closed surface S surrounding the target and passing everywhere through the fluid:

$$\langle F_i\rangle_{\text{av}}=\oint\overline{\Pi}_{ik}n_k dS, \tag{4}$$

where $\hat{n}$ is a unit vector along the outward normal to the fluid.

The surface S is chosen so that it bypasses the standing-wave region and crosses perpendicularly through the incident and the reflected traveling plane wave. For a traveling plane wave it can be shown[8] that, to second order the time average of the momentum flux density tensor is given by

$$\overline{\Pi}_{ik}=p_0\delta_{ik}+\rho_0\langle v_i v_k\rangle_{\text{av}}, \tag{5}$$

where p_0 and ρ_0 are the equilibrium pressure and density in the fluid.

In the calculation of the average force on the target the term $p_0\delta_{ik}$ integrates to zero. The second term is equal to the energy density E in the direction of propagation of the wave and is zero perpendicular to that direction. Thus, integration over the closed surface S leads to the result that the time average of the force on the target is the sum of two forces, one along the incident and one along the reflected beam, each directed towards the target and equal in magnitude to the product of the cross section and the energy density in the respective beam.

Similarly for a traveling plane wave of second sound the momentum flux density tensor is given by[8]

$$\Pi_{ik}=p\delta_{ik}+\rho_n v_{ni}v_{nk}+\rho_s v_{si}v_{sk}. \tag{6}$$

Assuming that for second sound also the time average of p is, to second order, equal to the equilibrium pressure p_0, one obtains, in the same way as before, that the time average $\langle\mathbf{F}\rangle_{\text{av}}$ of the net force on a plane intercepting target is the resultant of a force along the incident, and a force along the reflected beam, and that the magnitude of either of these forces is given by

$$\langle F\rangle_{\text{av}}=AE, \tag{7}$$

where E refers to the energy density in the respective beam of second sound.

APPARATUS

The general arrangement inside the cryostat is shown in Fig. 1. The small torsion balance is suspended from a fine glass fiber. The balance can be rotated to adjust its azimuthal orientation and this orientation can be determined by a light beam reflected from the mirror to an external scale. A vertical rod, at a distance 1.25 cm from the axis of the balance, holds the vane in a vertical plane. The angle between the vane and the balance arm is set at 45°. The circular ring below the mirror is mounted with its axis horizontal and parallel to the balance arm. Its function is to provide a countertorque to the second-sound beam torque as will be described below.

The second-sound source is a gold film about 100 Å thick evaporated on a glass plate with an active area about 8 mm on a side. The gold film heater is mounted on a support, not shown in Fig. 1, so that it can be

[6] E. M. J. Herrey, in *Conference de Physique des Basses Temperatures, Paris, 1955* (Centre Nationale de la Recherche Scientifique, Paris, 1956); J. Acoust. Soc. Am. **27**, 891 (1955).
[7] C. D. Metz, thesis, Columbia University, 1965 (unpublished).
[8] L. D. Landau and E. M. Lifshits, *Fluid Mechanics* (Pergamon Press, Ltd., London, 1959).

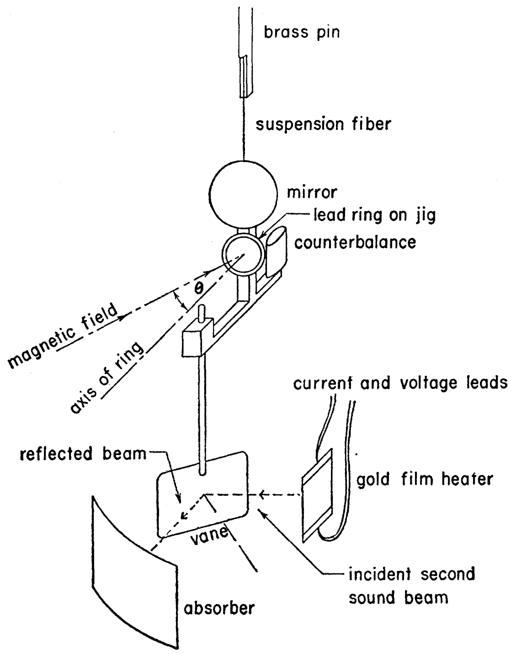

FIG. 1. Schematic drawing of the experimental apparatus. The gold film heater is in the position used to measure the energy density of a second-sound beam incident on the vane.

rotated from outside the cryostat about a vertical line 1.25 cm away from the suspension fiber. The balance is adjusted so that in its rest position the center of the vane lies on this vertical line. Thus it is ensured that the center of the beam emerging from the gold film is directed toward the center of the vane whatever the setting of the heater. When the plane of the heater is set parallel to the balance arm as in Fig. 1, only the component of the force normal to the heater surface produces a torque on the radiation balance, while with the plane of the heater perpendicular to the balance arm only the component parallel to the heater surface is effective. Thus force components parallel and perpendicular to the heater surface are separately measurable. By taking measurements with heater positions 180° apart and averaging, any errors due to horizontal misalignment can be reduced to a minimum. Disturbances from reflections at the Dewar walls are eliminated by placing a styrofoam absorber in the path of the reflected beam as shown in Fig. 1. The holes in the styrofoam are of the same order of magnitude as the wavelength of the second sound used.

The torque exerted on the torsion balance is measured by a null method, the countertorque being provided by the action of a variable magnetic field H which induces a persistent current in the superconducting lead ring described above.[9] If the field is at an angle θ with the normal to the ring, the net current i_p around the ring is given by

$$i_p = aH\cos\theta, \tag{8}$$

where a is a constant depending on the geometry of the ring. Thus the ring acquires a magnetic moment $\mathbf{m}$ perpendicular to its plane and proportional to i_p and experiences a torque $\mathbf{m}\times\mathbf{H}$, proportional to $H^2\sin\theta\cos\theta$, tending to rotate the ring into a plane parallel to the field.

In this apparatus a horizontal magnetic field is generated by a set of Helmholtz coils outside the cryostat. The coils are mounted so that they can be rotated about a vertical line in the plane of the lead ring and passing through its center. The torque Γ on the lead ring expressed in terms of the current i through the Helmholtz coils is given by

$$\Gamma = ci^2\sin\theta\cos\theta, \tag{9}$$

where c is a constant depending on geometry.

The constant c can be determined experimentally by measuring the period of small oscillations of the balance when the field is parallel to the plane of the lead ring in its null position. This period τ is given by

$$\tau^2 = 4\pi^2 I/(k+ci^2), \tag{10}$$

where k is the torsion constant of the suspension fiber and I the moment of inertia of the balance.

For actual measurements of second-sound forces the angle θ is adjusted to be about 45°. To be sure that no magnetic lines are locked in by the lead ring while it is cooled below its transition temperature, the horizontal component of the local magnetic field is cancelled by an equal and opposite field produced by a second set of Helmholtz coils, outside the first set. The appropriate coil azimuth and current were determined with the cryostat removed, by means of a flip coil device.

MEASUREMENTS

The method by which the second sound is generated introduces a complication. An ideal generator would produce a surface temperature fluctuating above and below the ambient temperature of the liquid. But if the temperature oscillation is produced by supplying alternating power of angular frequency ω and effective value per unit area W to the heater, one obtains for the instantaneous power input per unit area

$$2W\cos^2\omega t = W(1+\cos 2\omega t). \tag{11}$$

This expressions shows the presence not only of an alternating heat input but also of a steady heat input. Therefore in order to obtain the force $\langle F\rangle_{\text{av}}$ resulting from the second-sound beam alone the force $\langle F_{\text{ac}}\rangle_{\text{av}}$ measured with ac power must be corrected by subtracting the force $\langle F_{\text{dc}}\rangle_{\text{av}}$ measured with dc power of the same magnitude:

$$\langle F\rangle_{\text{av}} = \langle F_{\text{ac}}\rangle_{\text{av}} - \langle F_{\text{dc}}\rangle_{\text{av}}. \tag{12}$$

For a parallel beam of second sound and no power

[9] D. Shoenberg, *Superconductivity* (Cambridge University Press, New York, 1938); J. de Launay, Naval Research Laboratory Report P-3441, 1949 (unpublished).

losses, the associated alternating heat current density $\dot{Q}$ is equal to the alternating part of the power input per unit area, $W\cos 2\omega t$. The mean-square value is given by

$$\langle \dot{Q}^2 \rangle_{\mathrm{av}} = \tfrac{1}{2} W^2. \qquad (13)$$

In all calculations the cross section A of the second-sound beam was taken to be constant and equal to the heater area. Of course this is strictly true only in the limit of very high frequencies, but the assumption is sufficiently good even if there is some beam spreading as long as the vane of the radiation balance is large enough to intercept the diverging beam. For the second-sound frequencies used, from 10 to 40 kc/sec, no systematic increase of force with frequency was observed, hence any error due to beam spreading was negligible compared with other sources of error.

The steady part of the current probably spreads out before reaching the vane. In any case, the measured values of $\langle F_{\mathrm{dc}} \rangle_{\mathrm{av}}$ were only a fraction of $\langle F_{\mathrm{ac}} \rangle_{\mathrm{av}}$ at the power inputs used in the experiments.

To measure $\langle F_{\mathrm{ac}} \rangle_{\mathrm{av}}$ and $\langle F_{\mathrm{dc}} \rangle_{\mathrm{av}}$ for the incident second-sound beam the heater is set with its plane parallel to the direction of the balance arm in its rest position as shown in Fig. 1. Then the angle of incidence of the second-sound beam on the vane is 45°, with the incident beam perpendicular and the reflected beam parallel to the balance arm. Thus the torque on the balance is not affected by the reflected beam. A few measurements taken with the heater rotated through 90° showed that there was a reflected beam of approximately the same energy density as the incident beam.

EXPERIMENTAL RESULTS AND DISCUSSION

Experimental values for the radiation force produced by the incident second-sound beam as a function of temperature were obtained at frequencies of 20, 30, and 40 kc/sec and power inputs of 50, 100, 200, and 300 mW/cm². For comparison of the results with Eq. (2) we use a "normalized mechanical energy density," the energy density for unit heat current. The experimental value of this normalized mechanical energy density D_e is calculated from the data by means of Eqs. (7), (12), and (13), while the theoretical value D_t is defined according to Eq. (2), i.e.,

$$D_t = (\rho C T u_2^2)^{-1}. \qquad (14)$$

Figure 2 is a plot of normalized mechanical energy density versus temperature. The points are experimental values and the solid line is calculated from Eq. (14). For this calculation the fluid density measurements by Kerr,[10] the calculation of C by ter Harmsel,[11] and the measurement of u_2 by Peshkov[12] were used. The latter are corrected to the 1958 temperature scale because Peshkov took 2.176°K instead of 2.172°K as the temperature of the lambda point. The combined error in D_t amounts to approximately 2%. The probable error of the experimental points is estimated to be about 5%.

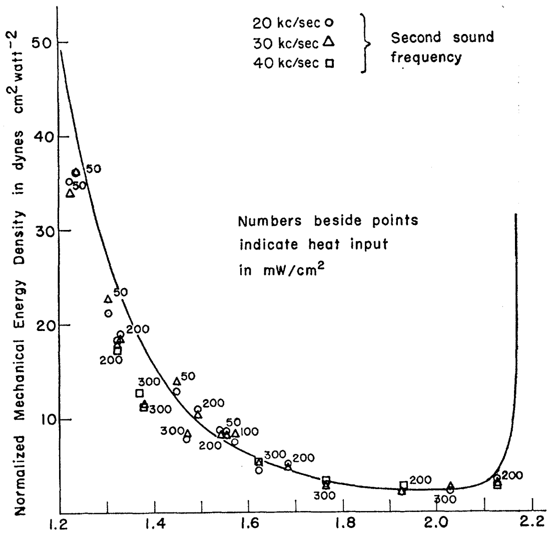

FIG. 2. Normalized mechanical energy density versus temperature. The points are calculated from measured forces assuming no attenuation. The solid line is given by $(\rho C T u_2^2)^{-1}$.

Inspection of Fig. 2 reveals that for small power inputs and at temperatures above 1.4°K the measured points agree with the theoretical curve within the estimated errors thus demonstrating that—at least within these limits—radiation-pressure measurements can be used to obtain the energy density of second sound and that its magnitude is in satisfactory agreement with the formula derived from low-velocity nondissipative two-fluid theory.

Nevertheless, discrepancies occur, especially at the lowest temperatures. Figure 2 shows that the greater the power input, the lower the experimental points are below the theoretical curve. The simplest explanation is of course that at the higher input powers losses occur so that the heat current density at the vane is actually less than that calculated from Eq. (13). Reduction of the amplitude of the temperature oscillation because of inertia of the source is unlikely, thermal relaxation times of thin metal films being of the order of nanoseconds.[13] Moreover, any losses in the heater would be of constant magnitude throughout a temperature interval as small as 1°K, while the observed effect is strongly temperature-dependent. On the other hand, it is quite possible that attenuation of second sound occurs

[10] E. C. Kerr, J. Chem. Phys. **26**, 511 (1957).

[11] The values of C_s were calculated by H. ter Harmsel (private communication) from the original data of W. H. Keesom, and K. Clusius; W. H. Keesom and A. P. Keesom; H. C. Kramers, J. D. Wasscher, and C. J. Gorter; O. V. Lounasmaa and E. Kojo; W. M. Fairbank, J. J. Buckingham, and C. F. Kellers; and R. W. Hill and O. V. Lounasmaa. This calculation has revised all the measurements to the 1958 He^4 temperature scale.

[12] V. P. Peshkov, Zh. Eksperim. i Teor. Fiz. **38**, 799 (1960) [English transl.: Soviet Phys.—JETP **11**, 580 (1960)].

[13] R. J. von Gutfeld, A. H. Nethercot, Jr., and J. A. Armstrong, Phys. Rev. **142**, 436 (1966).

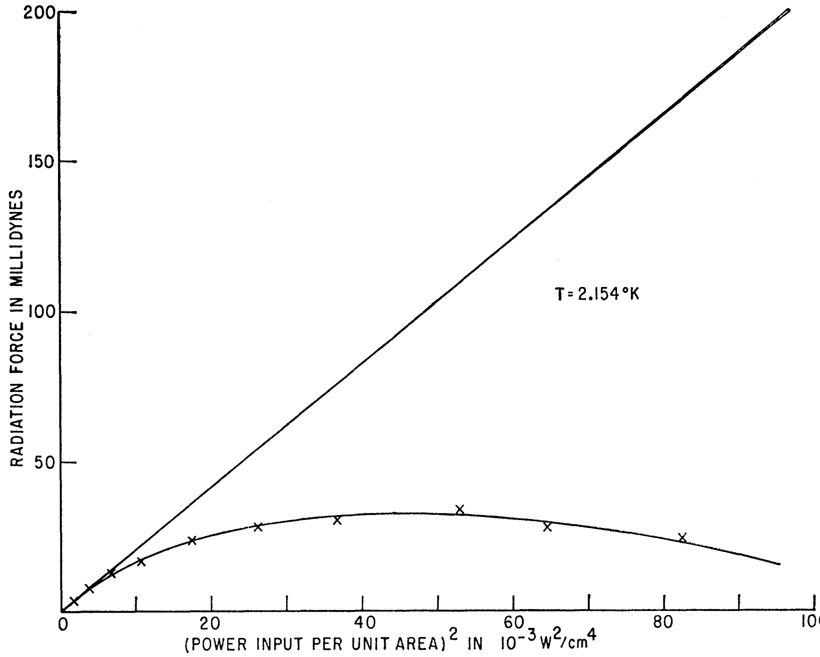

FIG. 3. Radiation force versus the square of the power input per unit area at $T=2.154$°K. The straight line is theoretical assuming no losses.

between the source and the vane. Such attenuation, in order to be noticeable as a reduction of the radiation force, would have to be due to scattering of the second sound. Absorption by itself has no effect, since in absorption momentum is conserved. Only if part of the momentum current then misses the vane will the force on the vane be diminished.

Assuming that scattering is the cause of the low value of the force, the ratio of the intensities at the vane and at the source is equal to the ratio D_e/D_t. The associated attenuation coefficient α can then be obtained from the relation

$$D_e = D_t e^{-2\alpha x}, \qquad (15)$$

where x is the distance from the source to the vane.

In order to explore the relationship between attenuation and power input and also to extend the temperature range further towards the lambda point an experiment was made at 10 kc/sec keeping the temperature constant at 2.154°K and varying the power input in small steps. A plot of the observed radiation force versus the square of the input power per unit area is shown in Fig. 3. The straight line is calculated using Eqs. (2), (7), and (13), i.e., assuming no attenuation. At low power input the experimental curve coincides with the line but as the power is increased there is increasing attenuation, until beyond 200 mW/cm^2 increasing the power input at the heater actually reduces the radiation force at the vane. This could not happen if the attenuation coefficient were increasing with the amplitude of the second-sound wave, a situation often encountered with large amplitude ordinary sound waves in liquids. The reason must be sought in a change of the scattering properties of the medium through which the second-sound beam passes. The most plausible explanation is that the flow which is generated in the liquid by the dc component of the heat input becomes increasingly turbulent.

Dziwornooh and Mendelssohn[14] have shown that the turbulence arising from interrupted steady heating approaches the turbulence resulting from steady heating of the same average value for pulses shorter than 3 sec. In our case the heat current varies at a high frequency between zero and a maximum and the same result must obtain. The strength of the turbulence will be that due to the steady part of the input power per unit area in Eq. (11), i.e., to W.

Attenuation of second sound by a heat current of sufficient intensity has been observed previously by Vinen,[15] who studied the propagation of second sound across a heat current in a rectangular tube serving as a resonator. The attenuation coefficient was deduced from the width of the resonance curve. It was found that the attenuation is linear and that the attenuation coefficient obeys the relation

$$\alpha = C(\dot{Q} - \dot{Q}_0)^2, \qquad (16)$$

where $\dot{Q}$ refers to the heat current and Q_0 is a small constant needed to fit the data into the square relationship. Calculating α for the present data from Eq. (15) a relation of the same form is obtained.

[14] P. A. Dziwornooh and K. Mendelssohn, in Proceedings of the International Conference on Low Temperature Physics, Moscow, 1966 (to be published).

[15] W. F. Vinen, Proc. Roy. Soc. (London) **A240**, 114 (1957).

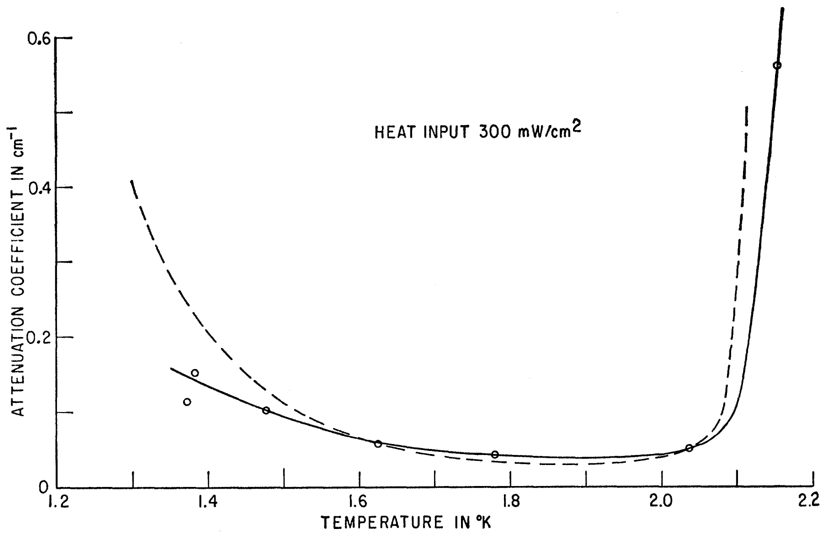

FIG. 4. Attenuation coefficient versus temperature. The solid curve is fitted to experimental points for attenuation of second sound in a freely flowing heat current which is emitted together with the second sound at heat input 300 mW/cm². The broken curve is a reconstruction of Vinen's data on attenuation of second sound across a heat current of 300 mW/cm² in a tube.

In Fig. 4, the attenuation coefficients calculated according to Eq. (15) for the data taken at 300 mW/cm² are plotted versus temperature. The points at any temperature are mean values taken from all measurements at that temperature. The point at 2.154°K is extrapolated from Fig. 3. The broken curve is a reconstruction of Vinen's data for the same heat current 300 mW/cm² ignoring the small constant $\dot{Q}_0$ in Eq. (16).

Comparing the broken curve with the experimental curve, the agreement is surprisingly close considering the great differences in experimental setup and measuring method. Though the exact agreement at some points must be considered fortuitous, there can be no doubt that the order of magnitude and the general temperature dependence is the same. Vinen explains his experimental observations by assuming turbulence in the superfluid involving quantized vortex lines. The present experiment showing that the attenuation of second sound arises from the scattering of the second-sound wave, is consistent with this view but other forms of turbulence of either one or of the two fluids could lead to the same result. In any case, since in Vinen's experiment the second sound was propagated perpendicular to the heat current while here it is in the same direction, the state of turbulence must be approximately isotropic.

The present experiment also throws some light on the mechanism which is responsible for the nucleation and growth of turbulence. In all the earlier experiments the heat current flows through a tube, while here it passes freely through bulk helium. The observed turbulence may still be influenced by the heater but cannot depend to any great extent on tube diameter, roughness of walls, conditions at the entrance and exit, protuberances, and the like. Recent experiments by Herrey[16] with the same apparatus have revealed that at low dc power input critical effects occur supporting the assumption that at the power inputs used in this experiment turbulence is present.

CONCLUSION

The experimental values of the energy density in a second-sound beam obtained from the measurement of the radiation force exerted on the vane of a radiation balance confirm the theoretical expression derived from the basic equations of low-velocity, nondissipative two-fluid theory. The observed discrepancies at higher power input and below 1.4°K are not caused by changes in the mechanism of second-sound wave motion but by turbulence in the steady heat current which is generated together with the second sound. The turbulence grows with increasing power input and causes scattering of the second sound thereby reducing the energy density measured at the vane of the radiation balance. For the same power input the attenuation is least at about 1.9°K increasing toward lower and toward higher temperatures. Since the steady heat current in this experiment is not channelled the observations show that tube wall effects are not decisive for the generation and growth of turbulence in liquid-helium-II heat flow.

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[16] E. M. J. Herrey, in Proceedings of the International Conference on Low Temperature Physics, Moscow, 1966 (to be published).