High-Energy Consistency Conditions and Superconvergence Sum Rules

M. Kugler
Rockefeller University, New York, New York
(Received 27 February 1967)

High-energy consistency conditions in the form of superconvergence sum rules are presented for three cases: (1) combinations of partial-wave amplitudes in potential scsttering; (2) a fixed-angle dispersion relation in relativistic theory; (3) partial-wave amplitudes in πN scattering. Resonance saturation of the last is attempted. In this approximation, one class of sum rules involving combinations of $f_l(w)$ agrees roughly with experiment; two which are more sensitive to approximation disagree.

I. INTRODUCTION

In a recent paper[1] we have introduced a set of high-energy consistency conditions for partial-wave scattering amplitudes. These consistency conditions were in the form of superconvergence[2,3] sum rules. The purpose of this paper is threefold: (i) generalize these sum rules to the case of πN scattering, (ii) produce analogous sum rules for potential scattering, and (iii) establish superconvergence sum rules for fixed-angle scattering.

Most of the recent successful sum rules are derivable from a low-energy theorem. This theorem is then used in conjunction with dispersion relations to produce a sum rule. In a similar spirit the superconvergence sum rules establish a high-energy theorem. Generally this is done by having a function vanish rapidly at high energy. A real analytic function satisfying the high-energy condition:

$$\lim_{s\to\infty} s^{1+\epsilon} f(s)=0 \tag{1}$$

will also satisfy

$$\int \operatorname{Im} f(s)ds=0 \tag{2}$$

where $\operatorname{Im} f(s)$ may include residues of poles. Equation (2) is satisfied, provided there is no essential singularity at infinity and all integrals converge.

Since total cross sections approach a constant at high energy, or at least vanish very slowly, one has to construct carefully amplitudes which satisfy Eq. (1). There have been several ways of constructing such amplitudes. The original superconvergence paper[2] considered a double helicity flip amplitude in $\rho\pi$ scattering. The double helcity exchange in the t channel was sufficient to reduce the Froissart bound[4] by two powers of s, thus giving a superconvergent function.[5]

A second approach to the construction of superconvergent amplitudes stems from symmetry considerations. The scattering amplitude can be decreased by considering the $I=2$ contribution[6] or the **27** of SU_3 in the t channel.[7] This approach suffers from a serious drawback, since superconvergence sum rules are extremely sensitive to symmetry breaking. Instead of satisfying Eq. (1), $f(s)$ constructed that way may contain terms proportional to $\epsilon s^{-\alpha}$ with $\alpha<1$ and ϵ of the order the symmetry breaking. In the presence of such terms, the integral in Eq. (2) would *diverge*, however small ϵ may be. For this reason superconvergence sum rules are extremely sensitive to symmetry breaking. Their validity is thus restricted to a "model world" only.[8]

In I we have considered partial wave amplitudes. These are bounded from unitarity by a constant. In order to achieve superconvergence the nonsuperconvergent terms had to be canceled off between several partial waves. This could be done exactly without resorting to approximations, since the l dependence of these terms was known.

As an illustration to our approach Sec. II discusses potential scattering. A high-energy theorem in this case is provided by the approach of the partial wave amplitude to the first Born approximation. Sum rules similar in form to those discussed in I are derived. The reader may skip this section if he has no interest in potential theory, as the rest of the paper is a self-contained unit.

Section III establishes the high-energy conditions we are using and connects these with Regge pole theory. It is shown that fixed-angle dispersion relations are superconvergent.

In Sec. IV we formulate sum rules for partial-wave amplitudes and construct several families of superconvergent functions.

Section V presents an attempt to saturate some of the low-l sum rules by a small number of Regge poles and

[1] M. Kugler, Phys. Rev. Letters **17**, 1166 (1966). Referred to as I.
[2] V. de Alfaro, S. Fubini, G. Rosetti, and G. Furlan, Phys. Letters **21**, 576 (1966).
[3] L. D. Solov'ev, Yadernaya Fiz. **3**, 188 (1966) [English transl.: Soviet J. Nucl. Phys. **3**, 133 (1966)].
[4] M. Froissart, Phys. Rev. **123**, 1053 (1961).
[5] For a general discussion of superconvergence due to helicity flip see T. L. Trueman, Phys. Rev. Letters **17**, 1198 (1966).
[6] P. Babu, F. J. Gilman, and M. Suzuki, Phys. Letters **24B**, 65 (1967).
[7] B. Sakita and K. C. Wali, Phys. Rev. Letters **18**, 29 (1967). G. Altarelli, F. Bucella, and R. Gatto, Phys. Letters **24B**, 57 (1967).
[8] Appearance of energy denominators which limit the range of integration make sum rules stemming from low-energy theorems much less sensitive to symmetry breaking. Since the range of integration is finite in practice, a reasonable criterion for applicability would be $\epsilon R<\Gamma$ where ϵ is the fraction of symmetry breaking, R the range of integration, and Γ a typical strong-interaction width. This makes electromagnetic breaking ($\epsilon=\alpha$) unimportant and SU_3 breaking ($\epsilon\cong0.1$) crucial.

resonances. Rough agreement with experiment is indicated for one class of sum rules. The other two, it is argued, are more sensitive to our crude approximation and disagree badly with experiment when such crude approximations are used.

An Appendix discusses the high-energy behavior of partial-wave amplitudes.

II. POTENTIAL SCATTERING

In this section we discuss high-energy consistency conditions in potential scattering. We restrict our discussion to a superposition of Yukawa potentials:

$$V(r)=\frac{1}{r}\int_{\mu_0}^{\infty}\sigma(\mu)e^{-\mu r}dr;\quad \mu_0>0 \tag{3}$$

$\sigma(\mu)$ will be restricted in Eq. (10). Our purpose is to discuss the high-energy behavior of the scattering amplitude and use it to obtain relations between various partial waves. In a sense, such consistency conditions are redundant. We could in principle solve the inverse scattering problem.[9] Knowing one partial wave we could obtain the potential. Once the potential is known, a solution of the Schrödinger equation would produce all partial-wave scattering amplitudes. This procedure is by no means an easy task. A simple consistency condition may be a substitute, though a weak one, avoiding the difficulties of solving the inverse scattering problem. Such a consistency could in principle be used to "detect" the presence of exchange forces by comparing odd and even values of angular momentum, or the presence of an l dependence of the potential.

The high-energy behavior of potential scattering has already been discussed in detail. The leading term in the high-energy behavior is given by the first Born approximation. The higher corrections to the full scattering amplitude were discussed by Hunziker.[10] For our purpose it is sufficient to consider the high-energy behavior of a partial-wave amplitude. Kohn[11] has shown that at high energy the Born series for a partial-wave scattering amplitudes converges. From his work we also learn the magnitude of correction terms needed. The high-energy behavior of a partial-wave amplitude $a_l(s)$ is given by

$$a_l(s)=a_{lB}(s)+O(s^{-3/2}\ln s^n), \tag{4}$$

where $s=k^2$ is the energy and $a_{lB}(s)$ is the first Born approximation. Since we are looking for superconvergence sum rules the correction terms in Eq. (4) are of no importance to us.

We now consider $a_{lB}(s)$, which is given by

$$a_{lB}(s)=\frac{1}{2s}\int_{\mu_0}^{\infty}d\mu\,\sigma(\mu)Q_l\left(1+\frac{\mu^2}{2s}\right). \tag{5}$$

At infinite s the function $a_{lB}(s)$ does not vanish rapidly enough for our purpose. There are two terms in the high-energy behavior of $a_{lB}(s)$ which decrease too slowly. One behaves like $s^{-1}\ln s$ and the second like s^{-1}. For our approach the important fact about these terms is that their dependence on l is known exactly. The leading behavior of $Q_l(z)$ near $z=1$ is given by

$$\lim_{z\to 1}Q_l(z)=-\tfrac{1}{2}\ln(\tfrac{1}{2}z-\tfrac{1}{2})-\gamma-\psi(l+1)+O(z-1), \tag{6}$$

where γ is the Euler constant and $\psi(l+1)$ the logarithmic derivative of the Γ function.[12] Using this knowledge we can find the l dependence of the leading terms of $a_l(s)$:

$$a_l(s)=-\frac{1}{4s}\int_{\mu_0}^{\infty}d\mu\,\sigma(\mu)\ln\left(\frac{\mu^2}{4s}\right)-\frac{1}{2s}[\gamma+\psi(l+1)]\times\int_{\mu_0}^{\infty}d\mu\,\sigma(\mu)+O(s^{-1-\epsilon}). \tag{7}$$

Once this dependence is known we can pick a linear combination of three partial waves which, provided all integrals converge, cancels the leading behavior of $a_l(s)$ and leaves a superconvergent function.

We have outlined this way of constructing a superconvergent function, in order to stress the similarity between potential scattering and the relativistic case discussed in Sec. IV. In potential scattering it is easier to use a relation between the associated Legendre functions.[12]

$$(2l+1)zQ_l(z)=(l+1)Q_{l+1}(z)+lQ_{l-1}(z). \tag{8}$$

Using this relation and Eqs. (5) and (6), we have

$$(2l+1)a_l(s)-(l+1)a_{l+1}(s)-la_{l-1}(s)=\frac{2l+1}{(2s)^2}\int_{\mu_0}^{\infty}d\mu\,\sigma(\mu)\mu^2Q_l\left(1+\frac{\mu^2}{2s}\right)+O(s^{-1-\epsilon}). \tag{9}$$

The right-hand side of Eq. (9) converges provided

$$\int_{\mu_0}^{\infty}d\mu\,\sigma(\mu^2)\mu^2\ln\mu<\infty. \tag{10}$$

We will restrict our discussion to potentials satisfying this condition.

For these potentials we define

$$\Delta_l(s)=a_l(s)-\frac{l+1}{2l+1}a_{l+1}(s)-\frac{l}{2l+1}a_{l-1}(s). \tag{11}$$

[9] See, for instance, V. de Alfaro and T. Regge, *Potential Scattering* (Interscience Publishers, Inc., New York, 1965).
[10] W. Hunziker, Helv. Phys. Acta **36**, 838 (1963).
[11] W. Kohn, Rev. Mod. Phys. **26**, 292 (1954).
[12] *Higher Transcendental Functions*, edited by A. Erdély (McGraw-Hill Book Company, New York, 1953).

By virtue of Eq. (9), $\Delta_l(s)$ is superconvergent, and satisfies the usual superconvergence sum rule

$$\int \mathrm{Im}\Delta_l(s)ds+\sum R_i=0\,, \qquad (12)$$

where R_i are the residues of $\Delta_l(s)$ at its poles. The integrals over the cuts in Eq. (12) will involve both the physical, right-hand cuts and the unphysical, left-hand potential cuts.

Considering the first Born approximation to the scattering amplitude we note that it contributes only to the left-hand cut in Eq. (12). Because of our construction the first Born approximation will contribute only superconvergent terms to $\Delta_l(s)$. It will therefore satisfy the sum rule in a trivial way. This is analogous to the vanishing of the "one Regge-pole exchange" contribution discussed in I and in Sec. V. Only the second and higher Born terms will contribute to Eq. (12), and only for these terms does our sum rule constitute a real constraint. Our discussions of the Born expansion does not imply that our sum rule depends on the convergence of the Born expansion at all energies. The sum rules are valid even when the Born expansion does not converge at low energies as in the presence of bound states.

In exact evaluation of our sum rules, the appearance of left-hand cut discontinuities is a major drawback. These are not easy to calculate even in potential theory.

It is of interest to compare the standard N/D approximation with our sum rules. In most of these approximations the left-hand cut discontinuity is chosen to be the first Born approximation. The high-energy behavior of the solution of such N/D equations is not necessarily identical to the first Born approximation, thus our sum rule will not necessarily be satisfied. This is not surprising since the approximations used are supposedly good for low-energy scattering even at the expense of mutilating the high-energy behavior.

In conclusion, we have constructed a sum rule for potential scattering. Our main aim was to illustrate a similar construction in the relativistic case. The sum rules described here do not apply to concrete physical problems, since nowhere do we expect potential scattering to be valid in the high-energy region. They are, however, of some importance as an additional tool in the so-called "theoretical laboratory" of potential scattering, in which a high-energy theorist hopes to gain experience and motivation for treating relativistic theories.

III. HIGH-ENERGY CONDITIONS AND FIXED-ANGLE SUM RULES

In this section we will formulate the high-energy conditions in which our sum rules are based. We will also explore briefly one of the more direct consequences of these conditions, namely, fixed-angle dispersion relations are superconvergent and give rise to superconvergence sum rules.

A. High-Energy Conditions

We will present our high-energy conditions, on the basis of Regge-pole theory. Only a very small part of Regge-pole theory will be used in the form of bounds on scattering amplitudes. The bounds we assume, are much less restrictive than Regge-pole theory and will indeed hold on the basis of other high-energy models, even if detailed Regge behavior is inconsistent with experiment.

In treating πN scattering we will treat the amplitudes $A(s,t)$ and $B(s,t)$ which obey the Mandelstam representation. We use s, t, and u the usual Mandelstam variables. First consider the forward region near $t=0$. In this region we assume

$$|A(s,t)|<Cs^{\alpha_0}\ln^2 s\,, \qquad (13)$$

$$|B(s,t)|<Cs^{\alpha_0-1}\ln^2 s\,. \qquad (14)$$

From the Froissart bound,[4] which has now been shown to follow rigorously from axiomatic field theory,[13] we know that $\alpha_0\leq 1$ for elastic scattering. Experimentally it seems that $\alpha_0=1$, but the possibility that α_0 is slightly less than one cannot be ruled out.[14] For charge-exchange scattering we have $\alpha_0<1$. The bound (14) is thus a consequence of all reasonable high-energy theories, and is not a severe assumption. Considering Regge-pole dominance would just omit the logarithmic factors in Eq. (14).

In the backward region near $t=-4k^2$ the exchange of baryon Regge trajectories will dominate the amplitude.[15] Using this behavior we are motivated to assume

$$|A(s,t)|<Cs^{\alpha_B-\frac{1}{2}}\,, \qquad (15a)$$

$$|B(s,t)|<Cs^{\alpha_B-\frac{1}{2}}\,. \qquad (15b)$$

From an experimental fit to the high-energy backward scattering[16] we learn that α_B, which is the value of $\alpha(u=0)$ for baryon Regge trajectories, is bounded by $\alpha_B<\frac{1}{2}$. This bound on α_B is far below the one we can prove from axiomatic theory. We will, however, make use of this bound.

Our last bound will concern the region of large momentum transfer. It is an experimental fact, that all large momentum-transfer amplitudes vanish rapidly with momentum transfer. Electromagnetic form factors seem to vanish like t^{-2} at high t.[17] Scattering seems to

[13] A. Martin, Nuovo Cimento 44, 1219 (1966).

[14] N. Cabibbo, J. J. J. Kokkedee, L. Horwitz, and Y. Ne'eman, Nuovo Cimento 45, 175 (1966).

[15] V. Singh, Phys. Rev. 129, 1889 (1963).

[16] V. Barger and D. Cline, Phys. Rev. Letters 16, 913 (1966); Phys. Rev. 156, 1522 (1967).

[17] For a discussion see S. D. Drell, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, California, 1967).

vanish exponentially with t or u.[18] Our next bound will give a mathematical formulation of this fact. We assume that for $t<t_0$ and $u<u_0$, t_0 and u_0 one fixed numbers, the following bound holds:

$$|A(s,t)|<Cs^{-\alpha_1}[t^{-\gamma}+u^{-\gamma}] \quad (16)$$

with $\alpha_1+\gamma>1+\epsilon$, and a similar bound for $B(s,t)$, where $\alpha_1+\gamma>\frac{3}{2}+\epsilon$. This bound follows from Regge-pole behavior. To prove this statement consider the contribution of a Regge pole in the t channel to $A(s,t)$:

$$A_R(s,t)\cong\beta(t)s^{\alpha(t)}. \quad (17)$$

Khuri[19] has shown that in order to maintain crossing symmetry we must have

$$|\beta(t)|<ct^{-1/2} \quad (18)$$

for large t. Without this bound, Regge poles in the t channel would influence the high-energy behavior in that channel. This runs counter to the basic assumption of Regge behavior at high energies. It is then sufficient to choose t_0 so that $\alpha(t_0)<-\frac{1}{2}$ in order to satisfy Eq. (16) for $A(s,t)$. Equation (16) for B follows in a similar fashion from the Regge behavior of B. Note that we have avoided the region near $\alpha=-1$ where essential singularities may exist. A similar discussion will hold for the contribution from Regge poles in the u channel and for the background integral. If there exist fixed Regge poles or Regge cuts, one would have to assume a sufficiently rapid decrease of $\beta(t)$ associated with these for Eq. (16) to hold.

It should be re-emphasized that Eq. (16) can hold even if Regge-pole theory is not valid. A rapid decrease of the scattering amplitude off the forward region is the common feature of many of the currently suggested models,[20] so these too may justify Eq. (16).

B. Fixed-Angle Sum Rules

We now turn to the first and most natural consequence of our high-energy bounds. This involves fixed-angle dispersion relations for $|\cos\theta|<1-\epsilon$. For fixed $\cos\theta$ in this region both t and u are proportional to s at high s. Denoting either $A(s,\cos\theta)$ or $B(s,\cos\theta)$ by $F(s,\cos\theta)$, we have as a consequence of (16) that

$$|F(s,\cos\theta)|<s^{-1-\epsilon}. \quad (19)$$

$F(s,\cos\theta)$ is therefore a superconvergent function. We can write a fixed-$\cos\theta$ dispersion relation for $F(s,\cos\theta)$:

$$F(s,\cos\theta)=\int_{\text{Left}}\frac{\text{Im}F(s'\cos\theta)}{s'-s}\frac{ds'}{\pi}+\int_{\text{Right}}\frac{\text{Im}F(s'\cos\theta)}{s'-s}\frac{ds'}{\pi}. \quad (20)$$

The appearance of a left-hand unphysical cut is the general feature of fixed-angle dispersion relation. The integral on the left should include all unphysical cuts. Using Eq. (20) we can write a superconvergence sum rule, provided all integrals converge and $F(s,\cos\theta)$ has no essential singularity at $s=\infty$.

$$\int_{\text{Left}}\text{Im}F(s,\cos\theta)ds+\int_{\text{Right}}\text{Im}F(s,\cos\theta)ds=0. \quad (21)$$

We will not exploit this sum rule further in the present paper. Some of the difficulties should be mentioned. This sum rule depends on the knowledge of the unphysical left-hand-cut discontinuity. Even the discontinuity in the physical region is not easily determinable. To find the imaginary part of a scattering amplitude in a nonforward direction a partial-wave analysis has to be performed. We prefer, thus, to formulate our sum rules directly in terms of partial-wave amplitudes, as we did in I. This is the subject of the Sec. IV.

IV. SUPERCONVERGENT PARTIAL-WAVE AMPLITUDES

In this section we construct superconvergent amplitudes on the basis of the high-energy behavior discussed in Sec. III. We treat the case of πN scattering. Our treatment follows that given in I for $\pi\pi$ scattering. It is, however, more complicated because of spin and unequal-mass kinematics.[21,22] The procedure we use is straightforward. We establish the l dependence of the nonsuperconvergent terms in each partial-wave amplitude. Once this is known we cancel these parts by choosing a linear combination of partial waves.

The standard partial-wave amplitudes $f_{l\pm}$ in πN scattering are defined by

$$f_{l\pm}(w)=\int_{-1}^{1}d\cos\theta[f_1P_l(\cos\theta)+f_2P_{l\pm1}(\cos\theta)]. \quad (22)$$

These amplitudes are functions of $w=\sqrt{s}$ and have a kinematic branch cut in the s plane. We can also define

$$A_l(s)=\int_0^{-4k^2}P_l\left(1+\frac{t}{2k^2}\right)A(s,t)\frac{dt}{2k^2}, \quad (23)$$

where

$$k^2=\frac{1}{4s}[s-(m+\mu)^2][s-(m-\mu)^2] \quad (24)$$

and

$$t=-2k^2(1-\cos\theta). \quad (25)$$

A similar definition holds for $B_l(s)$ expressed in terms of $B(s,t)$. $A_l(s)$ and $B_l(s)$ are functions of w^2 and have no kinematic singularities in the s plane. We can express

[18] J. Orear, Phys. Letters **13**, 190 (1964).
[19] N. N. Khuri, Phys. Rev. Letters **9**, 420 (1963); Phys. Rev. **132**, 914 (1963).
[20] R. Serber, Phys. Rev. Letters **10**, 357 (1963); T. T. Wu and C. N. Yang, Phys. Rev. **137**, B708 (1965).
[21] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).
[22] W. R. Frazer and J. R. Fulco, Phys. Rev. **119**, 1420 (1960).

$f_{l\pm}(w)$ in terms of $A_l(s)$ and $B_l(s)$;

$$f_{l\pm}(w)=\frac{1}{32\pi^2w^2}\{[(w+m)^2-\mu^2][A_l+(w-m)B_l]$$
$$+[(w-m)^2-\mu^2][-A_{l\pm1}+(w+m)B_{l\pm1}]\}. \quad (26)$$

The MacDowell symmetry which follows from Eq. (26) determines the left-hand physical cut by

$$f_{l+}(-w)=-f_{(l+1)-}(w). \quad (27)$$

We will also consider

$$g_l(s)=\frac{1}{4\pi}[B_{l-1}(s)-B_{l+1}(s)]$$
$$=\frac{2w}{(w+m)^2-\mu^2}[f_{(l-1)+}-f_{(l+1)-}]$$
$$+\frac{2w}{(w-m)^2-\mu^2}[f_{l-}-f_{l+}] \quad (28)$$

and

$$h_l(s)=\frac{1}{4\pi}[A_{l-1}-A_{l+1}-m(B_{l-1}-B_{l+1})]. \quad (29)$$

Express $h_l(s)$ by

$$h_l(s)=\frac{2w^2}{(w+m)^2-\mu^2}[f_{(l-1)+}-f_{(l+1)-}]$$
$$-\frac{2w^2}{(w-m)^2-\mu^2}[f_{l-}-f_{l+}]. \quad (30)$$

$g_l(s)$ is essentially the coefficient of $P_l'(\cos\theta)$ in the expansion of $B(s,t)$ in terms of the $f_l(w)$. $h_l(s)$ is related to a similar coefficient in the expansion of $A(s,t)$. $g_l(s)$ and $h_l(s)$ are by definition functions of s without kinematic singularities.

In an Appendix we use the high-energy condition of the preceding section to establish the l dependence of the high-energy behavior of $A_l(s)$ and $B_l(s)$. The relevant terms in this expansion are given by Eqs. (A13) and (A14)

$$A_l(s)=F_0(s)+\tfrac{1}{2}l(l+1)F_1(s)+(-1)^lF_0'(s)$$
$$+O(s^{\alpha_0-3})+O(s^{\alpha_B-\frac{5}{2}})+O(s^{-1-\epsilon}), \quad (31)$$

$$B_l(s)=G_0(s)+(-1)^lG_0'(s)+O(s^{\alpha_0-3})$$
$$+O(s^{\alpha_B-\frac{5}{2}})+O(s^{-\frac{3}{2}-\epsilon}). \quad (32)$$

We have omitted logarithmic factors in the estimate of the remainder terms. Again omitting logarithmic terms, we have

$$|F_0(s)|<Cs^{\alpha_0-1}, \quad |G_0(s)|<Cs^{\alpha_0-2},$$
$$|F_1(s)|<Cs^{\alpha_0-2}, \quad |G_0'(s)|<Cs^{\alpha_B-\frac{3}{2}}, \quad (33)$$
$$|F_0'(s)|<Cs^{\alpha_B-\frac{1}{2}}.$$

Equations (30) and (31) are analogous to Eq. (7) in potential scattering. A large number of superconvergent function can be constructed using the high-energy behavior resulting from Eqs. (31), (32), and (33). To construct such sum rules, the nonsuperconvergent terms have to be canceled between various functions.

For the purpose of exact evaluations of these sum rules, one is as good as the other. Such an evaluation of these sum rules, is certainly far beyond our present capability. When constructing our sum rules, we have, thus, to keep an approximation scheme in mind. It is quite reasonable that some of them will be much more sensitive to approximation than the others.

Since the main approximations will enter in evaluating unphysical cuts we will have to discuss these briefly. Most of the complications we discuss here are due to spin and unequal mass which were not present in I. Two regions will be discussed, the one near $s=(m-\mu)^2$ and the second near $s=0$. Contrary to the commonly held belief formulated in Refs. 21 and 22 the discontinuity near $s=(m-\mu)^2$ does not behave like $[s-(m-\mu)^2]^{l+\frac{1}{2}}$, but like $[s-(m-\mu)^2]^{1/2}$. This has been shown first by Frye and Warnock.[23] The physical reason for this behavior is that the s-wave threshold behavior in the u channel dominates the threshold behavior of this discontinuity. Near $s=0$ it has been shown recently by Freedman and Wang[24] that the scattering amplitude behaves in the spinless case like $s^{-\alpha(0)}$. $\alpha(0)$ in this context determines the high-energy behavior in the t and u channels. It is given by $t^{\alpha(0)}$ and $u^{\alpha(0)}$, respectively. $\alpha(0)$ is therefore the value of the Regge trajectory in the s *channel*. This is in contradiction to the behavior s^{-l} for which arguments were given in Refs. 21 and 22. In analogy to the proof of Freedman and Wang, $A_l(s)$ will be given by

$$A_l(s)\sim s^{-\alpha(0)+\frac{1}{2}} \quad (34)$$

with similar behavior of $B_l(s)$. Again $\alpha(0)$ is the s-channel Regge trajectory intercept.

One of the most immediate consequences of this discussion is that the amplitude $q^{-2l}f_{l\pm}(w)$ which has been used to discuss superconvergence[25] has very strong singularities near $s=(m-\mu)^2$ where q^2 vanishes. It will in fact behave like $[s-(m-\mu)^2]^{-2l+\frac{1}{2}}$. Such a strong singular behavior makes this amplitude extremely sensitive to approximations of low-energy phenomena in the u channel, which are not due to Regge poles.

In discussing the sum rules for $f_{l\pm}(w)$ we must also note that these functions have a kinematical singularity near $w=0$, resulting from Eq. (26). We could get rid of this singularity by considering the function $w^2f_{l\pm}(w)$. For this function we cannot construct superconvergence

[23] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963). A similar discussion was given by E. S. Abers and V. L. Teplitz, Nuovo Cimento **39**, 739 (1965).

[24] D. Z. Freedman and J. M. Wang, Phys. Rev. Letters **17**, 569 (1966); Phys. Rev. **153**, 1596 (1967).

[25] If indeed $\alpha_0<1$ (see Ref. 14) would hold for elastic scattering, $f_{l+}(w)$ would be superconvergent. Even in this case, α_0 is close to unity and the sum rule would converge very slowly.

sum rules because the contribution of the nonforward region (in $\cos\theta$), to this function would be of order $w^{-\epsilon}$ and no longer superconvergent. To overcome this we could consider $w^2[s-(m+\mu)^2]^{-1}f_l(w)$ for $l\geq 1$ where the vanishing of the denominator $s-(m+\mu)^2$ causes no trouble since it is canceled by threshold behavior. This function would, however, be sensitive to all singularities near threshold because of the smallness of the energy denominator. It would also rule out the possibility of getting information about $P_{\frac{1}{2}}$ and S waves. We consider therefore the original function $f_{l\pm}(w)$. The smallness of $A_l(s)$ and $B_l(s)$ given by Eq. (34) will make the kinematic singularity unimportant.

We now turn to the construction of superconvergent functions. Using Eqs. (31), (32), and (33) we find the high-energy behavior of $f_{l+}(w)$:

$$f_{l+}(w)=\frac{1}{16\pi^2}\frac{m}{w}F_0(s)+\frac{w}{16\pi^2}G_0(s)+\text{s.c.} \tag{35}$$

where s.c. are superconvergent terms. Denoting the isospin in the s channel by a superscript I, we find that the functions:

$$f_{l+}{}^I(w)-f_{l'+}{}^I(w)=F_{ll'}{}^I(w) \tag{36}$$

are superconvergent.[25] The function $F_{ll'}{}^I(w)$ should be considered in the complex w plane. The left-hand physical cut will be given by the MacDowell symmetry Eq. (27).

In a similar way we find that $g_l{}^I(s)$ defined by Eq. (28) is superconvergent.[26] We can also show that $H_{ll'}{}^I(s)$ defined by

$$H_{ll'}{}^I(s)=h_l{}^I(s)-h_{l'}{}^I(s) \tag{37}$$

is superconvergent.

The functions $g_l{}^I(s)$ and $H_{ll'}{}^I(s)$ and functions of w^2 and will satisfy superconvergence equations automatically in the w plane. We will therefore consider them in the s plane. Multiplying these functions by w, and considering them in the w plane would give the same result.

We have presented three infinite series of sum rules. The next section will deal with approximate evaluation of these sum rules.

V. APPROXIMATE SATURATION OF SUM RULES

The purpose of this section is an attempt to approximate the sum rules derived in the previous section. Admittedly we must use a rather crude method of saturation for the sum rules in question. The reason for this is twofold. In the physical region of πN scattering, the phase shift are not too well known. For our analysis we need information in the region where phase-shift analysis is still very ambiguous or nonexistent.[27] An even greater difficulty is created by the unphysical singularities which appear in partial-wave dispersion relations. These are even less known then physical phase shifts and thus are the major difficulty in considering our sum rules.

Our approximation scheme is analogous to that of I. We will approximate the amplitude by considering Regge poles in the t and u channels, and a number of resonances in the s channel. These two contributions to any amplitude will be denoted by $A_{\text{Regge}}(s)$ and $A_{\text{Res}}(s)$, respectively.

$A_{\text{Regge}}(s)$ will include a finite number of Regge poles exchanged in the t and u channels. Each of these trajectories will include the exchanges of all single particles on this trajectory. Among these single-particle exchanges the N and N^* exchanges in the u channel and ρ exchange in the t channel are included. Approximate dispersion calculations for partial-wave amplitudes have shown these single-particle exchanges to be the dominant nearby singularities. $A_{\text{Regge}}(s)$ may therefore be a reasonable approximation to unphysical cuts. For A_{Regge} the full contribution of a Regge pole should be taken so that it will satisfy a Mandelstam representation.[19]

In our approximation,

$$A(s)=A_{\text{Regge}}(s)+A_{\text{Res}}(s). \tag{38}$$

It should be emphasized that $A_{\text{Regge}}(s)$ contributes both to the physical and unphysical cuts. The contribution of $A_{\text{Regge}}(s)$ to our sum rules vanishes, since we have essentially constructed our sum rule so that A_{Regge} will satisfy it automatically. This is analogous to the vanishing of the contribution of the first Born approximation discussed in connection with the potential scattering sum rule. The approximate form of our sum rules is therefore given by

$$\int_{\text{phys.}}\text{Im}A_{\text{Res}}(s)ds=0, \tag{39}$$

where the integral extends over physical cuts.

It is this kind of approximation which we had to keep in mind while constructing our sum rules. The main point in the construction being to avoid emphasizing by small denominators unphysical cuts which do not come from Regge-pole exchange.

In this approximation we now turn to discuss the

---

[26] Sakita and Wali (Ref. 7) have considered a combination of $g_l(s)$ for various SU_3 multiplets. Their derivation is, however, incorrect. They do not notice that $g_l(s)$ is superconvergent for each channel separately.

[27] B. H. Bransden, P. J. O'Donnell, and R. G. Moorhouse, Phys. Letters **11**, 339 (1964); Phys. Rev. **139**, B1566 (1965); Phys. Letters **19**, 420 (1965); P. Bareyre, C. Bricman, A. V. Sterling, and G. Villet, *ibid.* **18**, 342 (1965); P. Auvil, A. Donnachie, A. T. Lea, and C. Lovelace, *ibid.* **12**, 76 (1964); **19**, 148 (1965). For a review and further references see C. Lovelace, in *Proceedings of the Thirteenth International Conference on High-Energy Physics* (University of California Press, Berkeley, California, 1967).

sum rules resulting from Eq. (36). We define

$$\bar{f}_l^I=\int_{m+\mu}^{\infty} dw\ \mathrm{Im} f_{l+,\ \mathrm{Res}}^I(w) +\int_{m+\mu}^{\infty} dw\ \mathrm{Im} f_{(l+1)-,\ \mathrm{Res}}^I(w). \quad (40)$$

In the narrow-resonance approximation we have

$$\bar{f}_l^I=\sum_i \frac{1}{k_i}\Gamma_i^{\mathrm{el}}, \quad (41)$$

where k_i and Γ_i^{el} refer to the values of k and the elastic width of the ith resonance having $j=l+\frac{1}{2}$ and isospin I. Our consistency condition is satisfied in this approximation if

$$\bar{f}_l^I=C, \quad (42)$$

where C is a constant independent of l. Table I gives results for various $\bar{f}_l^I$ from S to H waves. In evaluating the width we have used recent phase-shift-analysis results[27] and a compilation of πN resonance data given by Barger and Cline.[28] The large ambiguities in elastic width of resonances is a major source of errors. It is obvious that better data at higher energies would be necessary for a more accurate evaluation of the resonance contributions to $\bar{f}_l^I$. As we see it, the crude agreements as expressed in Table I, are no more than an *indication* that sum rules may be satisfied once better data are available.[29] The data are, so far, not in disagreement with our sum rule.

Now consider the sum rules for $g_l^I(s)$ as defined by Eq. (28). In the same approximation as before,

$$\bar{g}_l^I=\int_{(m+\mu)^2}^{\infty} \mathrm{Im} g_{l\ \mathrm{Res}}^I(s)ds=0. \quad (43)$$

The major contribution to $\bar{g}_1^{3/2}$ would come from the P_{33} N^* resonance; this contribution is equal to 226. The other contributions are small. There is no resonance in P_{31}, and the contribution of the D_{33} and S_{31} resonances are suppressed by factors μ/m. We are therefore forced to conclude that the sum rule for $g_1^{3/2}$ is *badly violated* in our approximation. In reconsidering the reason why these agree much less with experiment than the sum rules for $f_l(w)$, we note that in Eq. (28) which defines $g_l(s)$ the denominators $(w\pm m)^2-\mu^2$ appear. These denominators, as we have argued before, seem to make amplitudes more sensitive to the unphysical thresholds because of their smallness. A similar analysis can be carried out for the superconvergent amplitudes $H_{ll'}^I$. Again, the presence of small denominators may be the reason for the fact that these sum rules are not satisfied in our approximation.

It is obvious from our saturation attempt, that the weakest link in such a saturation is the treatment of the unphysical singularities. We have approximated them by a number of Regge poles, it is clear that some of the sum rules are more sensitive to additional singularities. These are the ones where we have divided by energy denominators. Another sum rule where energy denominators play an important role are the ones which can be derived from unitarity and threshold behavior. Consider

$$\frac{1}{[s-(m+\mu)^2]^l}f_{l+}(w)=J_l(w) \quad (44)$$

for $l\geq 1$ this function is superconvergent as one can easily convince oneself.[30] Saturation of this sum rule in a manner similar to ours would give

$$\int_{(m+\mu)}^{\infty} \mathrm{Im}J_l(w)dw+\int_{-(m+\mu)}^{-\infty} \mathrm{Im}J_l(w)dw=0. \quad (45)$$

These sum rules can obviously not be satisfied since $\mathrm{Im}J_l\geq 0$ on these physical cuts. Again the presence of energy denominators seems to spoil our saturation scheme. If left-hand-cut singularities are treated in a better way these sum rules may yield important information. In cases where $l\geq 2$ a larger number of sum rules may be derived by considering $[s-(m+\mu)^2]J_l$, $[s-(m+\mu)^2]^2J_l$, etc., the number of sum rules depending on l. This approach was advocated by Balachandran.[31]

TABLE I. Values of $\bar{f}_l^I$ for various partial waves. Masses and estimated widths of resonances are taken from Refs. 27 and 28. c denotes a cusp, the contribution of which could not be estimated.

I	l	Partial wave	Mass (MeV)	Γ_{el} (MeV)	Contrib. to $\bar{f}_l^I$	$\bar{f}_l^I$
$\frac{3}{2}$	0	S_{31}	1693	100±30	1.75±0.5	1.7±0.5
$\frac{3}{2}$	1	P_{33}	1238	120	5.1	5.1
$\frac{3}{2}$	2	D_{35}	1812	c	?	?
$\frac{3}{2}$	3	F_{37}	1950	125±50	1.8 ±0.7	1.8±0.7
$\frac{1}{2}$	0	S_{11}	1471	c	?	
		S_{11}	1561	72±40	1.5 ±0.8	
		P_{11}	938		−0.243	
		P_{11}	1471	120±20	2.9 ±0.5	>3.1±1.2
$\frac{1}{2}$	1	P_{13}	1658	c	?	
		D_{13}	1519	75±15	1.6 ±0.3	>1.6±0.3
$\frac{1}{2}$	2	D_{15}	1652	54±20	1.0 ±0.4	2.0±0.9
		F_{15}	1672	70±30	1.0 ±0.5	
$\frac{1}{2}$	3	G_{17}	2190	50±?	0.6	0.6±?
$\frac{1}{2}$	4	H_{19}	2220	100±?	1.1	1.1±?

[28] V. Barger and D. Cline, Ref. 16.

[29] When evaluating physical-cut continuum contribution to our sum rules, the Regge-pole contribution to the continuum should be carefully extracted, and not included in Eq. (40). This again leads to ambiguities.

[30] This has been noticed long ago. It has been analyzed for the P_{33} amplitude by Hyman Goldberg, Phys. Letters **24B**, 71 (1967). In this paper $k^{-2}f_{1+}(w)$ has been used; this introduces unwanted singularities at $W=\pm(m-\mu)$ as discussed (Ref. 23).

[31] A. P. Balachandran, Ann. Phys. (N. Y.) **30**, 476 (1964).

In conclusion we emphasize, that our saturation attempt depends *entirely* on Regge behavior. It is possible that our sum rules are correct and Regge behavior is not. In this case a different approximation scheme should be used.

VI. CONCLUSION

The fact that various partial waves are not independent is evident in both potential and relativistic scattering. In potential scattering the reason underlying this is that the same potential determines all partial-wave amplitudes. One reason for the relations among various partial waves in relativistic scattering can be Regge theory. If indeed it is true that analytic continuation in angular momentum is possible then all physical partial waves are given by one analytic function. It is therefore not surprising that sum rules relating various partial waves can be derived. Our sum rules can be viewed consequences of such behavior. In deriving the set of sum rules we have, however, used much less than the whole Regge theory, or all of potential theory. Our sum rules are therefore less restrictive.

In relativistic theory we have presented a set of high-energy conditions and formulated some of its consequences, superconvergence sum rules for fixed-angle scattering, and superconvergence sum rules for linear combination of partial-wave amplitudes.

We have presented three classes of superconvergent amplitudes for partial waves, $F_{ll'}{}^{I}$, $g_l{}^{I}$, and $H_{ll'}{}^{I}$. The first set of these amplitudes when approximated by resonances in the direct channel and Regge poles in the crossed channel are not in disagreement with experiment. Saturation attempts give very bad results for the second and third sets of sum rules. We have argued that the presence of small denominators in the latter sets of sum rules renders simple approximation schemes invalid.

In view of these facts it is evident that future usefulness of our superconvergence sum rules hinges on the rather hard problem of better treatment of the left-hand-cut singularities. Our approximation scheme, depends crucially on Regge pole theory. It is impossible to tell at the moment how exactly these approximation schemes have to be modified. Some experience in this direction may be gained from the potential scattering sum rules we have presented. This experience might be valuable in spite of the obvious differences between potential scattering and relativistic theory.

We have left our fixed-angle superconvergence sum rules unexploited so far. These share the main problem, of unphysical cuts, with the partial wave sum rules. Careful investigation of these sum rules may, however, improve our understanding of the problems we have to face. If these turn out to be less sensitive to approximations they may serve to increase our knowledge about the consistency of strong-interaction physics.

In conclusion we also mention that the $\pi\pi$ scattering sum rules discussed in I are analogous to the πN sum rules for $f_l(w)$ (because no small denominators appear in both). Since these are not violated by simple saturation schemes we have gained some confidence as to the validity of the former sum rules. In spite of crude approximations they may serve as an indication for the existence of higher resonances.

ACKNOWLEDGMENTS

The author is grateful to N. N. Khuri and V. Singh for helpful discussions and valuable criticism. A Fulbright travel grant from the U. S. Educational Foundation in Israel is gratefully acknowledged.

APPENDIX

In this Appendix we establish the l dependence of $A_l(s)$ and $B_l(s)$. This dependence is analogous to Eq. (7) in potential scattering, and is the basis of our high-energy consistency conditions. For this purpose we proceed in a manner almost identical to that of I. In Eq. (23) which defines $A_l(s)$ and $B_l(s)$ we split the range of integration into three parts: 1, the forward-peak region $t_0<t\leq 0$; 2, the large-momentum-transfer region defined by $-4k^2-u_0\leq t\leq t_0$, and 3, the backward-peak region $-4k^2\leq t\leq -4k^2-u_0$. We define the contribution of these regions to $A_l(s)$ by $A_l{}^1(s)A_l{}^2(s)\times A_l{}^3(s)$, respectively [similar notation will be used for $B_l(s)$]. In region 1 we use the bound in Eq. (14). We also expand the $P_l(\cos\theta)$ around the forward direction

$$P_l\left(1+\frac{t}{2k^2}\right)=\sum_0^l C_{ln}\left(\frac{t}{2k^2}\right)^n. \tag{A1}$$

Using this and Eq. (23) we can write

$$A_l(s)=\sum C_{ln}F_n(s)\,, \tag{A2}$$

$$B_l(s)=\sum C_{ln}G_n(s)\,, \tag{A3}$$

where

$$F_n(s)=\int_0^{t_0}\left(\frac{t}{2k^2}\right)^n A(s,t)\frac{dt}{2k^2}, \tag{A4}$$

$$G_n(s)=\int_0^{t_0}\left(\frac{t}{2k^2}\right)^n B(s,t)\frac{dt}{2k^2}. \tag{A5}$$

Using Eq. (14) we can prove

$$|F_n(s)|<Ct_0{}^n s^{\alpha_0-1-n}\ln^2 s\,, \tag{A6}$$

$$|G_n(s)|<Ct_0{}^n s^{\alpha_0-2-n}\ln^2 s\,. \tag{A7}$$

In considering the region 2 we make use of our high-energy condition expressed in Eq. (15). We use $|P_l(\cos\theta)|\leq 1$ the physical region and find the bounds

$$|A_l{}^2(s)|<Cs^{-1-\epsilon}, \tag{A8}$$

$$|B_l{}^2(s)|<Cs^{-\frac{3}{2}-\epsilon}. \tag{A9}$$

The reason for the different behavior of $A_l^2(s)$ and $B_l^2(s)$ is in the bound on $\alpha+\gamma$ in Eq. (16). The stronger bound on B_l is due to the appearance of $s^{\alpha(t)-1}$ in the Regge behavior of $B(s,t)$.

In region three our treatment will be exactly parallel to that of region 1. We will have, using similar definitions,

$$A_l^3(s)=(-1)^l\sum_0^l C_{ln}F_n'(s)\,, \tag{A10}$$

$$B_l^3(s)=(-1)^l\sum_0^l C_{ln}G_n'(s)\,, \tag{A11}$$

and the bounds

$$|F_n'(s)|<Cu_0{}^n s^{\alpha_B-\frac{3}{2}-n} \tag{A12}$$

and the same bounds for $G_n'(s)$.

We can therefore write

$$A_l(s)=F_0(s)+\tfrac{1}{2}l(l+1)F_1(s)+(-1)^lF_0'(s)+O(s^{\alpha_0-3})+O(s^{\alpha_B-\frac{5}{2}})+O(s^{-1-\epsilon})\,, \tag{A13}$$

$$B_l(s)=G_0(s)+(-1)^lG_0'(s)+O(s^{\alpha_0-3})+O(s^{\alpha_B-\frac{3}{2}})+O(s^{-\frac{3}{2}-\epsilon})\,. \tag{A14}$$

We have used in these equations $C_{l0}=1$ and $C_{l1}=\frac{1}{2}l(l+1)$. The last term whose order we give results from the contribution of $A_l^2(s)$ and $B_l^2(s)$.

Errata

Applications of the Chiral $U(6)\otimes U(6)$ Algebra of Current Densities, J. D. Bjorken [Phys. Rev. **148**, 1467 (1966)]. The conclusions of Sec. IX, which considers the radiative corrections to vector β decay, are incorrect. In addition to the divergent contribution to the corrections calculated there, there is an additional divergent piece [which contributes to $M_{\mu\nu}{}^{(b)}$ in Eq. (9.8)] coming from the equal-time commutator of the space components of isoscalar electromagnetic current with the *axial* current. This contribution is model-dependent. In a model in which the isospin current is carried by $J=\frac{1}{2}$, $I=\frac{1}{2}$ fields of charge $\bar{Q}\pm\frac{1}{2}$, the total divergent correction is [in place of Eq. (9.20)]

$$M\cong GP_\alpha\bar{u}\gamma^\alpha(1-\gamma_5)u\left\{1+\frac{3\alpha}{8\pi}(1+2\bar{Q})\ln\frac{\Lambda^2}{m^2}\right\}.$$

My thanks go to Helen Quinn for finding the mistake. The details of this axial contribution will be given in a forthcoming paper, in collaboration with R. Norton, E. Abers, and D. Dicus.

Kinematic Singularities of Partial-Wave Scattering Amplitudes, Jerrold Franklin [Phys. Rev. **152**, 1437 (1966)]. The discussion of the crossed threshold, $W=m-\mu$ (second paragraph on p. 1440) is incorrect if both particles have spin, because the energy of one of the particles is negative at this threshold. The correct behavior of unitarity corrections at the threshold $W=m-\mu$ if the particle of mass μ has spin s_μ is $k^{\bar{L}+\bar{L}'}$, where $\bar{L}=\mathrm{Max}\{L-2s_\mu;\ \mathrm{Min}|\mathbf{J}-\mathbf{s}_\mu-\mathbf{s}_m|\}$ with the added condition that $\bar{L}+L+2s_\mu$ be even, which might require increasing $\bar{L}$ by one unit. A similar result holds for $\bar{L}'$ and also at the threshold $W=\mu-m$ (with $s_\mu\rightarrow s_m$). The above result for $\bar{L}$ follows from the fact that as a particle's energy, E, approaches $-m$, the relativistic modifications of its spin tensor can contribute a $Y_{2s_m}{}^{\bar{n}}(\theta,\phi)$ to the angular dependence with no k factor. The azimuthal quantum number $\bar{n}$ is related to the Z projection n of the particle spin by $n-s_m\leq\bar{n}\leq n+s_m$. (See, for example, footnote 13 of the paper.) The behavior of the amplitude at the threshold $W=-m-\mu$ remains as given by Eq. (20), that is $k^{\bar{L}+\bar{L}'}$, where $\bar{L}=|J-S|$ with the condition that $\bar{L}$ be increased by one unit if necessary to make $\bar{L}+L+2S$ even. The additional threshold constraints at $W=-\mu-m$ still follow from Eq. (20) with $W\rightarrow -W$ as in the paper, as do threshold constraints at $W=\mu-m$ if $s_\mu<s_m$.

The above discussion requires Eqs. (26) and (27) of the paper to be changed to

$$\bar{A}_{L'S',LS}{}^J(W)=\{W^{2\alpha_{SS'}}(W-m-\mu)^{-(L+L')/2}\times(W-m+\mu)^{-(\bar{L}+\bar{L}')/2}(W+m-\mu)^{-(\bar{L}+\bar{L}')/2}\times(W+m+\mu)^{-(\bar{L}+\bar{L}')/2}\}A_{L'S',LS}{}^J(W), \tag{26}$$

$$\bar{A}_{L'S',LS}{}^J(s)=s^{\alpha_{S'S}}[s-(m+\mu)^2]^{-(L+L')/2}\times[s-(m-\mu)^2]^{-(\bar{L}+\bar{L}')/2}A_{L'S',LS}{}^J(s), \tag{27}$$

with $\bar{L}$ and $\bar{L}'$ given as above for each threshold.

The following corrections should also be noted.

(i) Whenever $A_{S'\lambda',S\lambda}(W)$ is written, the additional argument θ is to be understood.

(ii) The factor in front of Eq. (20) should be $[(2L+1)(2L'+1)]^{1/2}/(2J+1)$.

(iii) The threshold described in the next to last paragraph of Sec. IV as $W=-(m^2-\mu^2)^{1/2}$ should be $W=-m-\mu$.