$O(4)$ Symmetry and Regge-Pole Theory*

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For reactions in which the initial and final states in the t channel contain equal-mass particles (e.g., $N\bar{N}\rightarrow\pi\pi$) of masses m and m', we show, using analytic continuation and Lorentz invariance, that the on-mass-shell helicity amplitudes in the region $t=0$, $(m-m')^2\leqq s\leqq(m+m')^2$ are invariant under the group $O(4)$. Decompositions of the amplitudes in irreducible representations of $O(4)$ (four-dimensional partial-wave expansions) are obtained and related to conventional partial-wave expansions. Poles classified according to the $O(4)$ group are shown to lead to infinite families of Regge poles. The formalism is developed for arbitrary spins, and the case of nucleon-nucleon scattering is studied in detail. Our results for the Regge-pole structure in NN scattering are stronger than those of the conspirator theory.

I. INTRODUCTION

A SURPRISING result of recent work on Regge-pole theory is that Regge trajectories really occur in families with definite requirements on the spacing of members of a family and on the behavior of residue functions at zero values of the invariant mass. Such results have been derived explicitly for unequal-mass spin-zero scattering amplitudes[1,2] and for nucleon-nucleon scattering.[3,4]

Although analyticity properties of scattering amplitudes at zero values of the Mandelstam invariants are the essential ingredients in the arguments of Refs. 1–4, we would like to focus our attention here on an altogether different method by which similar results can be derived. This method involves the association of the Regge-pole structure with a group-invariance property of scattering amplitudes closely connected with the underlying Lorentz invariance of the theory.

Physical theories are required to be manifestly invariant under transformations of the Poincaré group. Partial-wave expansions of scattering amplitudes from which the Regge-pole classification is derived should be regarded as decompositions in irreducible representations of the little group of the Poincaré group which preserves the total energy-momentum vector K^μ in the direct channel. In the physical region, K^μ is positive timelike and the familiar partial-wave expansion is expressed in terms of the representation functions of the corresponding little group $O(3)$. When $K^\mu=0$ the little group is enlarged to a group of four-dimensional transformations isomorphic to the homogeneous Lorentz group $O(3,1)$. In order to incorporate the full symmetry of the amplitude at this point one should really expand in representation functions and classify poles in terms of the four-dimensional group. This is the standpoint adopted in this paper.

Consider the process in which momenta $p_1+p_2\rightarrow p_1'+p_2'$, and define $t=K^2=(p_1+p_2)^2$, $s=(p_1-p_1')^2$. If the masses $p_1^2=p_2^2=m^2$ and $p_1'^2=p_2'^2=m'^2$ are pairwise equal, the point $K^\mu=0$ corresponds to forward scattering in the s channel. In this paper we restrict ourselves to mass configurations of this type.

In more general mass configurations the vector K^μ is lightlike whenever $t=0$ and the four-dimensional symmetry does not strictly apply to the mass-shell amplitude. An off-shell continuation appears necessary in order to formulate the symmetry. It also seems that analyticity arguments similar to those of Refs. 1 and 2 can be used to show directly that the Regge-pole spectrum of the mass-shell amplitude exhibits the symmetry.[5] There is then the curious circumstance that in pairwise equal-mass configurations, as previously defined, group-theoretic assumptions are strictly necessary and lead to stronger results than analyticity arguments,[6] whereas in more general mass configurations analyticity arguments yield as much information as group-theoretic methods.[1]

In our treatment of the four-dimensional symmetry we obtain the compact group $O(4)$ as the invariance group of the mass-shell amplitude in the unphysical region $t=0$, $(m-m')^2\leqq s\leqq(m+m')^2$. Contrary to previous use of the group $O(4)$ in similar connections, we no where make use of the Wick rotation[7] or an off-mass-shell continuation. Our treatment applies to all spins, and the use of a compact group leads to considerable simplification over previous formulations. We are specifically interested in the case of nucleon-nucleon

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[1] D. Z. Freedman and J. M. Wang, Phys. Rev. **153**, 1596 (1967).

[2] D. Z. Freedman, C. E. Jones, and J. M. Wang, Phys. Rev. **155**, 1645 (1967).

[3] D. V. Volkov and V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **44**, 1068 (1963) [English transl.: Soviet Phys.—JETP **17**, 720 (1963)].

[4] M. Gell-Mann, E. Leader, in *Proceedings of the 12th International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967). See also L. Durand, III, Phys. Rev. Letters **18**, 58 (1967).

[5] This has been proven in Refs. 1 and 2 for spin-zero amplitudes. But the question remains whether the analyticity argument based on unequal-mass kinematics determines the spectrum uniquely when spins are present. G. Domokos [Phys. Rev. **159**, 1387 (1967)] has emphasized the difficulty of formulating the symmetry on the mass shell in the general mass case.

[6] For equal-mass spin-zero scattering, analyticity arguments are mute on the question of Regge-pole families, while the group theory predicts the existence of infinite families. In the nucleon-nucleon case, the analyticity argument gives an important constraint, but the group-theoretic results are far stronger as we shall see.

[7] G. C. Wick, Phys. Rev. **96**, 1124 (1954).

scattering and in comparing our results with the conspirator theory of Volkov and Gribov.[3]

In the next section we summarize work on the subject previous to this year, and briefly discuss the group $O(4)$ in Sec. III. Section IV is devoted to the establishment of the $O(4)$ symmetry of the scattering amplitude at $t=0$. The development of general four-dimensional partial-wave decompositions and their relation to conventional three-dimensional partial-wave amplitudes are given in Sec. V, and in Sec. VI we consider explicitly nucleon-nucleon scattering. In Appendix A properties of $O(4)$ representation matrices used in the text are derived, and in Appendix B analytic continuations of the four-dimensional partial-wave amplitudes are obtained using techniques similar to those of Froissart and Gribov.

II. PREVIOUS WORK

A four-dimensional symmetry at the point $K^\mu=0$ has long been known to be associated with amplitudes which satisfy a Bethe-Salpeter equation. This symmetry was discovered and utilized in the early papers on the Bethe-Salpeter equation by Wick[7] and Cutkosky,[8] and its consequences for complex angular momentum in the equal-mass case were first obtained by Domokos and Suranyi[9] and by Nakanishi.[10] For unequal-mass spin-zero Bethe-Salpeter amplitudes it was shown in Ref. 1 that this symmetry implies that Regge trajectories have exactly the properties found via the analyticity argument.

The daughter trajectories correspond to what were called abnormal solutions of the Bethe-Salpeter equation in the older literature.[11] The daughter trajectory results show that the "abnormal solutions" cannot be dismissed as peculiar features of the Bethe-Salpeter equations as often suggested[11]; they are necessary for the analyticity of the unequal-mass amplitudes. Although conclusions about the Bethe-Salpeter equation follow most easily from the Wick rotated form,[7] it would seem that most results concerning the four-dimensional symmetry can also be derived in the original Lorentz metric.[12]

Expansion theorems in terms of the four-dimensional group can be proven in general using the techniques developed in this paper. However, the corresponding classification of poles in the equal-mass case can only be established in the context of specific dynamical models. To date, Bethe-Salpeter equations with simple ladder kernels are the only dynamical models in which this pole classification can be established. More general Bethe-Salpeter kernels are usually too difficult to study mathematically, but in the cases which have been treated, l-plane singularities have the structure suggested by the four-dimensional group. In this paper, we have to assume that the pole structure given by the four-dimensional group is fundamental at $t=0$. This assumption is motivated by the results in the Bethe-Salpeter models and in the unequal-mass case.[1]

Toller[13,14] has given an elegant formulation of the four-dimensional symmetry. He studies the forward scattering amplitude in the crossed s channel and obtains expansion theorems for the amplitude in terms of the continuum of irreducible unitary representations of the noncompact little group $O(3,1)$. He assumes that asymptotic terms corresponding to Regge poles are classified according to this $O(3,1)$ expansion, and explicitly obtains the pole structure of nucleon-nucleon scattering.

Toller's formulation involves new and perhaps very useful ideas. It features a group-theoretic interpretation of the Regge background integral and the association of the signature of a Regge pole with the eigenvalue of the TCP reflection operation. A difficulty of the theory is that the expansion theorem in terms of the noncompact group $O(3,1)$ seems to apply rigorously only to amplitudes which have no Regge poles to the right of $l=-1$ at $t=0$. A difficulty of this type is avoided in our treatment, because the $O(4)$ group is compact. It also seems that the $O(4)$ formulation is considerably simpler than that based on the noncompact $O(3,1)$.

III. THE GROUP $O(4)$

The Lie group $O(4)$ of rotations in a four-dimensional Euclidean space has six infinitesimal generators, a set J_1, J_2, J_3 which generate ordinary rotations in the yz, xz, and xy planes, and a set K_1, K_2, K_3 which generate rotations involving the fourth axis, which we call "boosts" in analogy with standard Lorentz-group terminology. It is convenient to parametrize finite transformations of $O(4)$ in the form

$$\begin{aligned} g&=R(\varphi,\theta,0)L_3(\delta)R(\alpha,\beta,\gamma)\\ &=e^{-i\varphi J_3}e^{-i\theta J_2}e^{-i\delta K_3}e^{-i\alpha J_3}e^{-i\beta J_2}e^{-i\gamma J_3}, \end{aligned} \tag{1}$$

where a standard Euler-angle parametrization has been assumed for rotations. The only boost which need be considered explicitly involves the z axis. In this parametrization the invariant volume element on the group manifold is

$$dg=d\varphi d(\cos\theta)\,\sin^2\delta d\delta d\alpha d(\cos\beta)d\gamma\,. \tag{2}$$

It is well known that the generators $A_i=\frac{1}{2}(J_i+K_i)$ and $B_i=\frac{1}{2}(J_i-K_i)$ satisfy independently the commutation relations of ordinary angular momentum and that there therefore exists a correspondence between $O(4)$ and $SU(2)\times SU(2)$. In the direct product group, pure rotations take the form (U,U) and pure boosts the form

[8] R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954).

[9] G. Domokos and P. Suranyi, Nucl. Phys. **54**, 529 (1964).

[10] N. Nakanishi, Phys. Rev. **136**, B1830 (1964).

[11] For a list of references about abnormal solutions see N. Nakanishi, Phys. Rev. **138**, B1182 (1965).

[12] S. Nussinov and J. Rosner, J. Math, Phys. **7**, 1670 (1966).

[13] M. Toller, Nuovo Cimento **37**, 631 (1965); University of Rome Reports No. 76, 1965 and No. 84, 1966 (unpublished).

[14] A. Sciarrino and M. Toller, University of Rome Report No. 108, 1967 (unpublished).

(V,V^{-1}), where U and V are arbitrary elements of $SU(2)$.

Matrices of the four-component representation can be constructed from the expression

$$\begin{pmatrix} t'+z' & x'-iy' \\ x'+iy' & t'-z' \end{pmatrix} = U\begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} V^\dagger. \quad (3)$$

For a boost along the z axis we take

$$U=\begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{+i\delta/2} \end{pmatrix}, \quad V=U^{-1} \quad (4)$$

and obtain

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\delta & 0 & 0 & i\sin\delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\sin\delta & 0 & 0 & \cos\delta \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}. \quad (5)$$

The boost matrix obtained in this way differs from the standard form by a unitary transformation. However, the form (5) is best adopted to our configuration where spatial components of four-vectors are imaginary.

Irreducible representations (I.R.) of $O(4)$ are denoted by the pair of numbers (a,b), where a and b are eigenvalues of the Casimir operators $A^2=a(a+1)$, $B^2=b(b+1)$. There are two convenient basis sets for an irreducible representation, one in which the operators A_3 and B_3 are diagonal and the other in which the total angular momentum J^2 and its third component J_3 are diagonal. Transformation between the two bases is simply done by adding angular momenta **A** and **B** to make **J**, and it is easy to see that the I.R. (a,b) contains ordinary angular momenta j in integer intervals from $j_{\min}=|a-b|$ to $j_{\max}=a+b$.

In the basis $|(ab)jm\rangle$ the representation matrix of the transformation (1) can be written as

$$D_{jm;j'm'}{}^{(a,b)}(g)=\sum_{m''} D_{mm''}{}^{j}(\varphi,\theta,0) \times d_{jj'm''}{}^{(ab)}(\delta) D_{m''m'}{}^{j}(\alpha,\beta,\gamma), \quad (6)$$

where the D^j are the ordinary representation matrices of $SU(2)$, and the boost matrix is found by transformation from the A_3,B_3 basis to be given by the trigonometric polynomial

$$d_{jj'm}{}^{(a,b)}(\delta)=\sum_{\mu} C(a,b,j;\mu,m-\mu) \times C(a,b,j';\mu,m-\mu)e^{-i(2\mu-m)\delta} \quad (7)$$

in which ordinary Clebsch-Gordan coefficients appear as coefficients. The polynomial (7) can be expressed in terms of Gegenbauer functions (see Appendix A). The normalization of the boost matrices is

$$\sum_m \int_0^\pi d\delta \sin^2\delta\, d_{jj'm}{}^{(ab)*}(\delta) d_{jj'm}{}^{(a'b')}(\delta) = \delta_{aa'}\delta_{bb'} \frac{\pi(2j+1)(2j+1)}{2(2a+1)(2b+1)}. \quad (8)$$

It will be convenient at a later stage to introduce the quantities $n=a+b$ and $M=a-b$ and to label I.R.'s and representation matrices by the pairing (n,M) instead of (a,b).

IV. $O(4)$ SYMMETRY OF SCATTERING AMPLITUDES

We start by considering t-channel center-of-mass-frame helicity amplitudes[15] $T_{\lambda_1'\lambda_2';\lambda_1\lambda_2}(p_1'p_2';p_1p_2)$ for which we assume conventional analyticity properties[16] in the variables $t=(p_1+p_2)^2$ and $s=(p_1'-p_1)^2$. The Lorentz transformation law

$$\begin{aligned} T_{\lambda_1'\lambda_2';\lambda_1\lambda_2}(p_1'p_2';p_1p_2) &= \sum_{\mu's} D_{\lambda_1'\mu_1'}{}^{s_1}[R_w^{-1}(\Lambda,p_1')] D_{-\lambda_2',-\mu_2'}{}^{s_2}[R_w^{-1}(\Lambda,p_2')] \\ &\quad \times T_{\mu_1'\mu_2',\mu_1\mu_2}(\Lambda p_1'\Lambda p_2';\Lambda p_1\Lambda p_2) \\ &\quad \times D_{\mu_1\lambda_1}{}^{s_1}[R_w(\Lambda,p_1)] D_{-\mu_2-\lambda_2}{}^{s_2}[R_w(\Lambda,p_2)], \quad (9) \end{aligned}$$

tells how to transform to an arbitrary Lorentz frame. The Wigner rotations are given by

$$\begin{aligned} R_w(\Lambda,p) &= L^{-1}(\Lambda p)\Lambda L(p), \\ L(p) &= e^{-i\varphi J_3} e^{-i\theta J_2} \exp(-i\bar{\delta}\bar{K}_3), \end{aligned} \quad (10)$$

where the J_i generate rotations and the $\bar{K}_i$ generate boosts. The angles $\bar{\delta}$, θ, φ specify the orientation of momentum four-vectors p according to

$$p^0=m\cosh\bar{\delta}, \quad \mathbf{p}=m\sinh\bar{\delta}\,\hat{r}(\theta,\varphi), \quad (11)$$

where $\hat{r}(\theta,\varphi)$ is a spatial unit vector of polar angles (θ,φ). For physical (positive timelike) momentum vectors we adopt the convention that $\bar{\delta}\geqq 0$ for particle 1 and $\bar{\delta}\leqq 0$ for particle 2. In the c.m. frame in the physical region of the t channel,

$$\varphi_1=\varphi_2, \quad \theta_1=\theta_2, \quad \bar{\delta}_1=-\bar{\delta}_2=\sinh^{-1}\{[(t/4m^2)-1]^{1/2}\}.$$

These conventions are completely equivalent to those of the second paper of Ref. 15.

By the Hall-Wightman theorem,[17] Eq. (9) can be extended analytically to any transformation Λ of the complex Lorentz group. Further, Eq. (9) still specifies the Lorentz transformation law in unphysical regions of the variables s and t. Continuation in t and s is done via the c.m. frame helicity amplitudes, and Eq. (9) then specifies the transformation law to any frame connected to the c.m. frame by the transformation Λ of the complex Lorentz group.

The first step in obtaining the $O(4)$ symmetry is to continue the t-channel helicity amplitudes from the

[15] M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) **7**, 404 (1959); G. C. Wick, *ibid.* **18**, 65 (1962).

[16] Y. Hara, Phys. Rev. **136**, B507 (1964); L. L. Wang, *ibid.* **142**, 1187 (1964); H. P. Stapp, *ibid.* **160**, 1251 (1967).

[17] D. Hall and A. S. Wightman, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **31**, No. 5 (1957); H. P. Stapp, University of California Radiation Laboratory Report No. UCRL-10843 (unpublished).

physical region to $t=0$, defining

$$\bar{\delta}_1=-\bar{\delta}_2=i\delta=i\sin^{-1}[(1-t/4m^2)^{1/2}]$$

so that[18] $\delta=\pi/2$ at $t=0$. We see from (11) that the vector K^μ vanishes at $t=0$, and that the amplitude there depends only on the relative momenta $p=\frac{1}{2}(p_1-p_2)$ and $p'=\frac{1}{2}(p_1'-p_2')$. After the continuation process just described the relative momentum vector has components

$$p^0=0\,,\quad \mathbf{p}=im\hat{r}(\theta,\varphi)\,, \tag{12}$$

with arbitrary spatial orientation at $t=0$. This form defines the center-of-mass frame at $t=0$.

The covering group of the complex Lorentz group is $SL(2,C)\times SL(2,C)$ and we restrict ourselves now to transformations g of its compact subgroup $O(4)$ or rather $SU(2)\times SU(2)$. Such a transformation takes the relative momentum vector (12) into a vector of the form

$$p^0=m\cos\delta\,,\quad \mathbf{p}=im\sin\delta\,\hat{r}(\theta',\varphi')\,, \tag{13}$$

where δ, θ', φ' specify the polar coordinates of an arbitrary point on the surface of a four-dimensional Euclidean sphere.

By starting from center-of-mass–frame relative momentum vectors (for initial and final states) of variable spatial orientation, we apply transformations g of $O(4)$ and use Eq. (9) to obtain scattering amplitudes for relative momentum vectors p' and p with arbitrary orientations on the sphere. This set of on-mass-shell amplitudes is obviously invariant under the group $O(4)$.

Variation of p and p' over the sphere corresponds to variation of s in the interval $(m-m')^2\leqq s\leqq(m+m')^2$, where m and m' are the masses of particles in the initial and the final state of the t channel, respectively. In this interval, the amplitude is analytic except for possible poles due to bound states in the s and u channels and possible annihilation cuts at which the amplitude is bounded. For example, in NN scattering there is the deuteron pole in the s channel and a pion pole and multipion and kaon annihilation cuts in the u channel.

We subtract out the pole terms and treat them explicitly later. This subtraction is done in a Lorentz-invariant manner so that the underlying group theory is not destroyed. The amplitude $T_{\lambda_1'\lambda_2'\lambda_1\lambda_2}(p',-p',p,-p)$ remaining after the subtraction is square integrable as a function of the orientation of p and p' on the sphere. It is therefore a bounded integral operator in search of a Hilbert space, and we now define the Hilbert space as the set of functions $f_{\lambda_1\lambda_2}(p)$ defined on the sphere through Eq. (13) with norm

$$\sum_{\lambda_1\lambda_2}\int d\Omega\,|f_{\lambda_1\lambda_2}(p)|^2<\infty\,, \tag{14}$$

where $d\Omega=\sin^2\delta d\delta d(\cos\theta)d\varphi$ is the surface element of the four-dimensional sphere. The introduction of a Hilbert space makes it very easy to obtain the group-theoretic decomposition of the amplitude.

We change to bra-ket notation and write

$$T_{\lambda_1'\lambda_2'\lambda_1\lambda_2}(p'-p';p-p)=\langle p'\lambda_1'\lambda_2'|T|p\lambda_1\lambda_2\rangle. \tag{15}$$

The ket $|p\lambda_1\lambda_2\rangle\equiv|\hat{e}(\delta\theta\varphi)\lambda_1\lambda_2\rangle$ is an improper basis ket of the Hilbert space. It is defined in terms of the ket $|\bar{p}\lambda_1\lambda_2\rangle$ with momentum $\bar{p}$ in the direction of the north pole of the sphere by

$$|p\lambda_1\lambda_2\rangle=e^{-i\varphi J_3}e^{-i\theta J_2}e^{-i\delta K_3}|\bar{p}\lambda_1\lambda_2\rangle. \tag{16}$$

We have introduced the unit vector $\hat{e}(\delta\theta\varphi)$ of polar orientation $(\delta\theta\varphi)$ in the four-dimensional space, and our kets are normalized by

$$\langle p'\lambda_1'\lambda_2'|p\lambda_1\lambda_2\rangle=\delta^3(\hat{e}'-\hat{e})\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}. \tag{17}$$

The kets transform under $O(4)$ according to

$$U(g)|p,\lambda_1\lambda_2\rangle=\sum_{\mu_1\mu_2}D_{\mu_1\lambda_1}{}^{s_1}(R_w(g,p))\times D_{-\mu_2-\lambda_2}{}^{s_2}(R_w(g,-p))|gp,\mu_1,\mu_2\rangle\,, \tag{18}$$

where $U(g)$ is the unitary operator in the Hilbert space corresponding to the transformation g of $O(4)$. Invariance under $O(4)$ is expressed simply by the equation

$$U(g^{-1})TU(g)=T. \tag{19}$$

V. $O(4)$ DECOMPOSITION OF THE AMPLITUDE

Our goal is to obtain the four-dimensional partial-wave expansion of the amplitudes (15), and we do this by decomposing the Hilbert space into finite-dimensional subspaces whose basis states transform according to definite irredicible representations of $O(4)$. The invariance (19) ensures a corresponding decomposition of the matrix elements of the operator T, and this decomposition is the desired partial-wave expansion. Our procedure is well known in ordinary quantum mechanics.[19]

We study first the behavior under ordinary rotations of the north-pole helicity ket $|\bar{p}\lambda_1\lambda_2\rangle$. For an ordinary rotation $R(\alpha,\beta,\gamma)$ the Wigner rotations (10) are given by

$$\begin{aligned}R_w(R(\alpha,\beta,\gamma),\bar{p})&=R(\alpha,\beta,\gamma)\,,\\ R_w(R(\alpha,\beta,\gamma),-\bar{p})&=e^{-i\pi K_3}R(\alpha,\beta,\gamma)e^{+i\pi K_3}\\ &=R(\alpha,-\beta,\gamma)\,.\end{aligned} \tag{20}$$

The relation $-\bar{p}=e^{+i\pi K_3}\bar{p}$ is implied by previous conventions.[18] The last equality in (20) follows from the group structure of $O(4)$.

[18] The choice of phase in the square-root argument of the inverse sine is immaterial if it is adhered to consistently. See. T. L. Trueman and G. C. Wick, Ann. Phys. (N.Y.) **26**, 322 (1964).

[19] E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* (Academic Press Inc., New York, 1959).

Using (18) and (20) we obtain

$$R(\alpha,\beta,\gamma)|\bar{p}\lambda_1\lambda_2\rangle=\sum_{\mu_1\mu_2}(-1)^{\lambda_2-\mu_2}D_{\mu_1\lambda_1}{}^{s_1}(\alpha,\beta,\gamma)\times D_{-\mu_2,-\lambda_2}{}^{s_2}(\alpha,\beta,\gamma)|\bar{p}\mu_1\mu_2\rangle. \quad (21)$$

We define a new set of kets by the equation

$$|ps\lambda\rangle=\sum_{\lambda_1\lambda_2}(-1)^{s_2-\lambda_2}C(s_1s_2s;\lambda_1,-\lambda_2)|p,\lambda_1\lambda_2\rangle, \quad (22)$$

of which *the north-pole ket* $|\bar{p}s\lambda\rangle$ transforms under rotations according to the I.R.s of $SU(2)$,

$$U(R(\alpha\beta\gamma))|\bar{p}s\lambda\rangle=\sum_{\lambda'}D_{\lambda'\lambda}{}^{s}(\alpha\beta\gamma)|\bar{p}s\lambda'\rangle. \quad (23)$$

The phase factor $(-1)^{s_2-\lambda_2}$ is very important to the final results. The quantum number s, although a purely rotational quantum number, differs from the total spin normally defined for two-particle systems. Total spin is usually defined by coupling the two spins in the center-of-mass frame, whereas our spin s is defined in the "north-pole frame." These frames are connected by the boost $e^{-i\pi K_3/2}$ which does not commute with rotations. Hence the relation between the two total spins is not simple.

The integral over the group

$$\int dg\, D_{jm,j'm'}{}^{(ab)*}(g)U(g)|\bar{p}s\lambda\rangle \quad (24)$$

either vanishes or defines a state transforming according to the I.R. (a,b). Using (1), (6), and (23), one can easily integrate over the angles α, β, and γ, and find that the integral vanishes unless $j'=s$, $m'=\lambda$ and that it is independent of λ. It also vanishes unless the angular momenta j and s are contained in the representation (a,b), so that $|a-b|\leqq j,s\leqq a+b$. Therefore, (24) reduces to the set of states

$$|a,b,j,ms\rangle=N_s{}^{ab}\sum_{\lambda}\int d\Omega\times D_{m\lambda}{}^{j}(\varphi\theta 0)^*d_{js\lambda}{}^{*(a,b)}(\delta)|ps\lambda\rangle, \quad (25)$$

for which a, b, j, and s satisfy the inequality above, and the set of states (25) is complete in the Hilbert space. For fixed a, b, and s, the states $|a,b,j,m,s\rangle$ transform according to the I.R. (a,b). The spin index s plays a role similar to the helicity in the treatment of Jacob and Wick.[15] A state with a given value of s contains $O(4)$ representations with $|M|=|a-b|\leqq s$.

The normalization constant, which is fixed by the requirement

$$\langle a'b'j'm's'|abjms\rangle=\delta_{a'a}\delta_{b'b}\delta_{j'j}\delta_{m'm}\delta_{s's}, \quad (26)$$

is given by

$$(N_s{}^{ab})^2=(2\pi^2)^{-1}(2s+1)^{-1}(2a+1)(2b+1). \quad (27)$$

The unitary transformation matrices

$$\langle p'\lambda_1\lambda_2|ps\lambda\rangle=\delta^3(\hat{e}'-\hat{e})\delta_{\lambda,\lambda_1-\lambda_2}\times C(s_1s_2s;\lambda_1,-\lambda_2)(-1)^{s_2-\lambda_2}, \quad (28)$$

$$\langle ps\lambda|abjms'\rangle=\delta_{ss'}N_s{}^{ab}D_{m\lambda}{}^{j}(\varphi,\theta,0)^*d_{js\lambda}{}^{(a,b)*}(\delta),$$

which can easily be obtained from (22) and (25), are very useful in deriving the decomposition theorems we need.

Before obtaining these expansion theorems it is useful to study the behavior of the states (25) under the discrete transformations corresponding to parity and charge conjugation. To do so it is simplest to introduce the direct product notation

$$|\bar{p}\lambda_1\lambda_2\rangle=|\lambda_1\rangle\otimes e^{+i\pi K_3}|-\lambda_2\rangle \quad (29)$$

in which the different transformation properties of particles 1 and 2 are manifest. The individual kets $|\lambda_1\rangle$ and $|\lambda_2\rangle$ transform in the same way under the group.

The parity operator commutes with rotations and anticommutes with boost generators. Therefore

$$P|\bar{p}\lambda_1\lambda_2\rangle=\eta_1\eta_2|\lambda_1\rangle\otimes e^{-i\pi K_3}|-\lambda_2\rangle=(-1)^{2s_2}\eta_1\eta_2|\bar{p}\lambda_1\lambda_2\rangle, \quad (30)$$

where η_1 and η_2 are the intrinsic parities of particles 1 and 2.

Charge conjugation is useful only for particle-antiparticle channels, so we take $s_1=s_2$. It commutes both with parity and with transformations of the proper group $O(4)$, and is equivalent to a factor $(-1)^{2s_2}$ times the exchange operator of the two particles. Therefore

$$C|\bar{p}\lambda_1\lambda_2\rangle=(-1)^{2s_2}\{e^{i\pi K_3}|-\lambda_2\rangle\}\otimes|\lambda_1\rangle=e^{+i\pi K_3}|\bar{p},-\lambda_2,-\lambda_1\rangle. \quad (31)$$

From (22), we easily obtain

$$P|\hat{e}(\delta\theta\varphi)s\lambda\rangle=(-1)^{2s_2}\eta_1\eta_2|\hat{e}(-\delta,\theta,\varphi)s\lambda\rangle,$$
$$C|ps\lambda\rangle=(-1)^{s+\lambda}e^{+i\pi K_3}|ps\lambda\rangle. \quad (32)$$

Using the symmetry properties (A2) and (A3) of the boost matrices we finally find

$$P|abjms\rangle=\eta_1\eta_2(-1)^{2(a+b+s_2)-j-s}|bajms\rangle, \quad (33)$$

$$C|abjms\rangle=(-1)^{2a-s}|abjms\rangle. \quad (34)$$

The reduced matrix elements $\langle abjms'|T|abjms\rangle$ are diagonal in a, b, j, and m, and independent of j and m, because of Schur's lemma and $O(4)$ invariance. We introduce the notation

$$T_{s's}{}^{nM}=\langle abjms'|T|abjms\rangle, \quad (35)$$

where $n=a+b$, $M=a-b$. Parity conservation (33) implies that

$$T_{s's}{}^{n,M}=\eta_1'\eta_2'\eta_1\eta_2(-1)^{2s_2+2s_2'-s-s'}T_{s's}{}^{n,-M}. \quad (36)$$

For identical particles or for particle-antiparticle channels, this implies that

$$T_{s's}{}^{n,M}=(-1)^{s+s'}T_{s's}{}^{n,-M},$$
$$T_{s's}{}^{n,0}=0 \quad \text{if} \quad (-1)^{s+s'}=-1. \tag{37}$$

These relations restrict the number of independent reduced matrix elements.

The transformation matrices (28) can be used to construct expansion theorems in terms of the $T_{s's}{}^{nM}$ for amplitudes $\langle p's'\lambda'|T|ps\lambda\rangle$ or $\langle p'\lambda_1'\lambda_2'|T|p\lambda_1\lambda_2\rangle$ with arbitrary orientation of p and p' on the four-dimensional sphere. These expansions converge in the L_2 topology on the sphere. We write explicit expansions only for the amplitudes necessary for the further development of the theory.

The conventional c.m.-frame helicity amplitude at $t=0$ is given by

$$T_{\lambda_1'\lambda_1'\lambda_1\lambda_2}(\theta)=\left\langle \hat{e}\left(\frac{\pi}{2},\theta,0\right)\lambda_1'\lambda_2'|T|\hat{e}\left(\frac{\pi}{2},0,0\right)\lambda_1\lambda_2\right\rangle. \tag{38}$$

Inserting complete sets of states (25) and using (27) and (28), we obtain

$$T_{\lambda_1'\lambda_2'\lambda_1\lambda_2}(\theta)=(2\pi^2)^{-1}(-1)^{s_2'-\lambda_2'+s_2-\lambda_2}\sum_{s=|s_1-s_2|}^{s_1+s_2}\sum_{s'=|s_1'-s_2'|}^{s_1'+s_2'}\sum_{M=-\min(s,s')}^{\min(s,s')}\sum_{n=\min(s,s')}^{\infty}\sum_{j=|M|}^{n}[(2s'+1)(2s+1)]^{-1/2}$$
$$\times C(s_1',s_2',s';\lambda_1',-\lambda_2')C(s_1,s_2,s;\lambda_1,-\lambda_2)[(n+1)^2-M^2]T_{s's}{}^{n,M}d_{js'\lambda'}{}^{(n,M)*}(\pi/2)d_{js\lambda}{}^{(n,M)}(\pi/2)d_{\lambda\lambda'}{}^{j}(\theta), \tag{39}$$

where $\lambda=\lambda_1-\lambda_2$, $\lambda'=\lambda_1'-\lambda_2'$. The partial-wave helicity amplitudes[20] $\langle\lambda_1'\lambda_2'|T^j|\lambda_1\lambda_2\rangle$ can be identified as the coefficients of $d_{\lambda\lambda'}{}^{j}(\theta)$ in Eq. (39). Parity-conserving helicity amplitudes, the amplitudes which are Reggeized, can be obtained by forming suitable linear combinations.

To project out the $O(4)$ partial-wave amplitudes $T_{s's}{}^{nM}$ we define the amplitudes

$$T_{s's\lambda}(\delta)=\langle\hat{e}(\delta,0,0),s'\lambda|T|\hat{e}(0,0,0),s\lambda\rangle, \tag{40}$$

observing that invariance under rotations about the z axis implies that these amplitudes are diagonal in λ, and that invariance under the mirror reflection $Pe^{-i\pi J_2}$ implies

$$T_{s's\lambda}(\delta)=\eta_1'\eta_2'\eta_1\eta_2(-1)^{2s_2+2s_2'+s'+s-2\lambda}T_{s's-\lambda}(\delta), \tag{41}$$

which reduces to

$$T_{s's\lambda}(\delta)=(-1)^{s'+s}T_{s's-\lambda}(\delta) \tag{42}$$

for identical particles or particle and antiparticle. These amplitudes have the simple decomposition

$$T_{s's\lambda}(\delta)=(2\pi^2)^{-1}[(2s+1)(2s'+1)]^{-1/2}$$
$$\times\sum_M\sum_n[(n+1)^2-M^2]d_{s's\lambda}{}^{n,M*}(\delta)T_{s's}{}^{nM}, \tag{43}$$

where the sums have the same limits as in (39). $T_{s's}{}^{nM}$ can be projected out using Eq. (8), and we obtain

$$T_{s's}{}^{nM}=\sum_{\lambda=-\min(s',s)}^{\min(s',s)}\int_0^{\pi}d\delta\,\sin^2\delta\,T_{s's\lambda}(\delta)d_{s's\lambda}{}^{nM}(\delta). \tag{44}$$

The $O(4)$ partial-wave amplitudes, defined for physical n by Eq. (44) can be continued to complex n using techniques[21] similar to those of Froissart and Gribov.

VI. NUCLEON-NUCLEON SCATTERING

At this point we restrict ourselves explicitly to the process $N\bar{N}\to N\bar{N}$, the simplest spin configuration in which the $O(4)$ symmetry leads to interesting results. In this process s and therefore M are restricted to the values 0 and 1. Parity conservation, (37) and (42), implies that $s=0$ and $s=1$ states do not couple. Therefore we can simplify the notation by setting $T_{s's}{}^{nM}\equiv\delta_{s's}T_s{}^{n,M}$. There are three independent amplitudes $T_0{}^{n0}$, $T_1{}^{n,0}$, and $T_1{}^{n,1}$ for a given n.

We write expressions for the parity-conserving helicity amplitudes of GGMW [22]

$$\begin{aligned}
f_0{}^j&=\langle\tfrac12,\tfrac12|T^j|\tfrac12,\tfrac12\rangle-\langle\tfrac12,\tfrac12|T^j|-\tfrac12,-\tfrac12\rangle,\\
f_1{}^j&=\langle\tfrac12,-\tfrac12|T^j|\tfrac12,-\tfrac12\rangle-\langle\tfrac12,-\tfrac12|T^j|-\tfrac12,\tfrac12\rangle,\\
f_{11}{}^j&=\langle\tfrac12,\tfrac12|T^j|\tfrac12,\tfrac12\rangle+\langle\tfrac12,\tfrac12|T^j|-\tfrac12,-\tfrac12\rangle,\\
f_{22}{}^j&=\langle\tfrac12,-\tfrac12|T^j|\tfrac12,-\tfrac12\rangle+\langle\tfrac12,-\tfrac12|T^j|-\tfrac12,\tfrac12\rangle,\\
f_{12}{}^j&=2\langle\tfrac12,\tfrac12|T|\tfrac12,-\tfrac12\rangle.
\end{aligned} \tag{45}$$

We identify the partial-wave helicity amplitudes in (39) and use (37) and (A11) and (A12) to write

$$f_0{}^{j\pm}=[3\pi^2(2j+1)]^{-1}\Big[\sum_{\substack{\kappa=1\\ \text{odd}}}^{\infty}(j+\kappa+1)^2|d_{j10}{}^{j+\kappa,0}(\pi/2)|^2T_1{}^{j+\kappa,0\mp}$$
$$+\sum_{\substack{\kappa=0\\ \text{even}}}^{\infty}2(j+\kappa)(j+\kappa+2)|d_{j10}{}^{j+\kappa,1}(\pi/2)|^2T_1{}^{j+\kappa,1\pm}\Big], \tag{46}$$

[20] The normalization of these amplitudes is given by

$$T_{\lambda_1'\lambda_2'\lambda_1\lambda}(\theta)=\sum(2j+1)d_{\lambda\lambda'}{}^{j}(\theta)\langle\lambda_1'\lambda_2'|T^3|\lambda_1\lambda_2\rangle.$$

[21] This is shown in Appendix B for $N\bar{N}$ scattering.

[22] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960). Our amplitudes differ from GGMW by a factor of $\sqrt{t}$.

$$f_1^{j\pm}=2[3\pi^2(2j+1)]^{-1}[\sum_{\substack{\kappa=0\\ \text{even}}}^{\infty}(j+\kappa+1)^2|d_{j11}{}^{j+\kappa,0}(\pi/2)|^2T_1{}^{j+\kappa,0\pm}+\sum_{\substack{\kappa=1\\ \text{odd}}}^{\infty}2(j+\kappa)(j+\kappa+2)|d_{j11}{}^{j+\kappa,1}(\pi/2)|^2T_1{}^{j+\kappa,1\mp}], \quad (47)$$

$$f_{11}^{j\pm}=[\pi^2(2j+1)]^{-1}\sum_{\substack{\kappa=0\\ \text{even}}}^{\infty}(j+\kappa+1)^2|d_{j00}{}^{j+\kappa,0}(\pi/2)|^2T_0{}^{j+\kappa,0\pm}, \quad (48)$$

$$f_{22}^{j\pm}=[3\pi^2(2j+1)]^{-1}\sum_{\substack{\kappa=0\\ \text{even}}}^{\infty}2(j+\kappa)(j+\kappa+2)|d_{j11}{}^{j+\kappa,1}(\pi/2)|^2T_1{}^{j+\kappa,1\pm}. \quad (49)$$

The $\pm$ superscripts refer to signatures and should be disregarded until we discuss them explicitly below. Amplitudes $T_1{}^{j+\kappa,0}$ with κ even do not contribute in (46) because the corresponding $d_{j10}{}^{j+\kappa,0}(\pi/2)$ vanish. Similar remarks apply to the other terms. From (48) and (49) we see that only $M=0$ couples to the parity-conserving helicity state 1, and only $M=1$ couples to the helicity state 2. Therefore the amplitude $f_{12}{}^j$ which couples these two states must vanish, a result which is also obtained in the conventional theory.[22]

Formulas (46)–(49) relate the conventional parity-conserving partial-wave helicity amplitudes to the $O(4)$ partial-wave amplitudes for integer values of j. In Appendix B, we study the problem of obtaining the appropriate continuation of these relations to complex j. We summarize here the results of that study, and then go on to examine the structure of Regge-pole families. There are subtle points involved in the treatment of Appendix B, and the reader interested in the detailed implementation of these ideas is urged to read it.

For convenience, we use the generic symbols T^n and f^j to denote any one of the set of corresponding amplitudes. In Appendix B it is shown that there exist separate continuations $T^{n\pm}$ away from even and odd integer n, respectively, which are holomorphic in the half-plane $\mathrm{Re}n>N$, where N is the number of subtraction necessary in the forward dispersion relations in nucleon-nucleon scattering. A simple multiple of $T^{n\pm}$ satisfies the requirements of Carlson's theorem and therefore $T^{n\pm}$ is the unique holomorphic continuation in $\mathrm{Re}n>N$ with reasonable asymptotic behavior. In particular, asymptotic behavior in $\mathrm{Re}n$ assures the convergence of (46)–(49) uniformly in j. We refer to the $T^{n\pm}$ as the amplitudes of even and odd *Lorentz signature*, respectively.

When the continuations $T^{n\pm}$ are inserted in Eqs. (46)–(49), these equations may be used to define continued partial-wave amplitudes[23] $f^{j\pm}$ away from even and odd integral j which are holomorphic in $\mathrm{Re}j>N$. A simple multiple of the $f^{j\pm}$ defined in this way satisfies the hypotheses of Carlson's theorem, and the theorem assures us that these $f^{j\pm}$ coincide with the conventional continuations away from even and odd integers with the same analyticity and asymptotic properties.

In the region $\mathrm{Re}n<N$ very little has been proven about the analyticity properties of the $T^{n\pm}$. We adopt as a working hypothesis the assumption that the $T^{n\pm}$ have only simple poles in the region $\mathrm{Re}n<N$. This corresponds to the assumption of j-plane meromorphy usually made in phenomenological applications of S-matrix theory. Other kinds of singularities—branch points, for example—would yield families of similar singularities equally spaced in the j plane at $t=0$. In this paper we study only poles.

A pole in $T^{n\pm}$ is called a Lorentz pole, following Toller. Since the three independent amplitudes $T_0{}^{n0\pm}$, $T_1{}^{n0\pm}$, and $T_1{}^{n1\pm}$ correspond to transitions from states which transform according to different I.R.'s of the invariance group $O(4)$ combined with parity, the Lorentz poles of each amplitude will be independent. Each Lorentz pole, at $n=\alpha^\pm$, gives rise to an infinite family of Regge poles whose structure can be deduced from Eqs. (46)–(49).

Type I. Lorentz pole of $T_0{}^{n,0\pm}$ ($M=0$, $s=0$). Near such a pole the amplitude behaves like

$$T_0{}^{j+\kappa,0\pm}\approx\gamma^\pm/(j+\kappa-\alpha^\pm). \quad (50)$$

This amplitude couples only to $f_{11}{}^{j\pm}$, and from (48) the Lorentz pole yields a series of Regge poles spaced by two units of angular momentum at $j=\alpha^\pm, \alpha^\pm-2, \cdots$, with residues $\beta^{0\pm}, \beta^{2\pm}, \cdots$. The ratio of the residues of the parent and daughter poles can be computed from Eq. (48). We obtain, for example,

$$\beta^2/\beta^0=(2\alpha+1)|d_{\alpha00}{}^{\alpha,0}(\pi/2)|^2/(2\alpha-3)|d_{\alpha-2,00}{}^{(\alpha,0)}(\pi/2)|^2=(2\alpha+1)/4\alpha. \quad (51)$$

All Regge poles in this family have signature $(-1)^j$ equal to Lorentz signature and $P=(-1)^j=C$. An additional series of poles with opposite signature and parity but the same charge conjugation would appear at $j=\alpha^\pm-1, \alpha^\pm-3, \cdots$ in the unequal-mass case.[1]

Type II. Lorentz pole of $T_1{}^{n,0\pm}$ ($M=0$, $s=1$). This

[23] The absolute-value signs and complex conjugates in (46)–(49) refer specifically to integral j. It is a simple matter to remove them and define continuations of products of two $d_{j\lambda\lambda}{}^{(j+\mu,M)}(\pi/2)$ which are analytic in j.

Lorentz pole leads to Regge poles of $f_1{}^{j\pm}$ at $j=\alpha^\pm$, $\alpha^\pm-2$, $\cdots$ with residues $\beta_1{}^{0\pm}$, $\beta_1{}^{2\pm}$, $\cdots$ and to Regge poles of $f_0{}^{j\mp}$ at $j=\alpha^\pm-1$, $\alpha^\pm-3$, $\cdots$ with residues $\beta_0{}^{1\mp}$, $\beta_0{}^{3\mp}$, $\cdots$. From Eqs. (46)–(47) and Eqs. (A7)–(A8) we calculate the ratio

$$\beta_0{}^{1\mp}/\beta_1{}^{0\pm}=2(2\alpha+1)\,|d_{\alpha-1,1,0}{}^{\alpha,0}(\pi/2)|^2/ (2\alpha-1)\,|d_{\alpha,11}{}^{\alpha,0}(\pi/2)|^2=(2\alpha+1)/\alpha(\alpha+1)\,. \quad (52)$$

All Regge poles in this family have a common charge-conjugation value opposite in sign to the Lorentz signature. The Regge poles of $f_1{}^{j\pm}$ have $P=-(-1)^j=C$ and those of $f_0{}^{j\mp}$ have $P=-(-1)^j=-C$.

Type III. Lorentz pole of $T_1{}^{n,1\pm}$ ($M=1$). This Lorentz pole yields Regge poles of $f_0{}^{j\pm}$ and $f_{22}{}^{j\pm}$ at $j=\alpha^\pm$, $\alpha^\pm-2$, $\cdots$, and poles of $f_1{}^{j\mp}$ at $j=\alpha^\pm-1$, $\alpha^\pm-3$, $\cdots$. The ratio of residues of the leading members of the singlet and coupled triplet series is, from Eq. (46), (49), and (A10),

$$\beta_{22}/\beta_0=(\alpha+1)/\alpha\,. \quad (53)$$

All poles of this family have charge conjugation equal to Lorentz signature. The poles of $f_0{}^{j\pm}$ have $P=-(-1)^j=-C$, those of $f_{22}{}^{j\pm}$ have $P=(-1)^j=C$, and those of $f_1{}^{j\mp}$ have $P=-(-1)^j=C$.

The variation of the discrete quantum numbers C and P within a Regge family can be understood easily. The charge-conjugation operation commutes with all the transformations of the $O(4)$ group, and its eigenvalue within a given irreducible representation depends only on the Casimir operators of $O(4)$ and not, for example, on the j value of the individual state in the representation. For this reason a Lorentz pole and all the Regge poles to which it gives rise have the same charge conjugation. The same remarks apply to all internal quantum numbers such as isospin.

Parity does not commute with $O(4)$ transformations containing boosts, and a Lorentz pole does not therefore have a definite parity. In an $M=0$ representation there is a definite correlation between the parity of a basis state and its j value, Eq. (33). The parity of a Regge pole in an $M=0$ family is therefore correlated with the j value of possible physical states on the trajectory, and this structure is exhibited in families I and II. The basis states of $M\neq0$ representations are not parity eigenstates, and in general have nonvanishing coupling to angular-momentum helicity states of both parities. Parity doubling, the occurrence of opposite-parity states of the same j value, should be expected in Regge families corresponding to $M\neq0$ Lorentz poles; this phenomenon is exhibited in family III.

A given Lorentz pole can couple to many different channels at $t=0$. Because of $O(4)$ invariance it couples to states of the same M value in all the different channels.[24] Therefore M is a universal quantum number of trajectories at $t=0$. If $M=0$, then the quantity $P\times(-1)^j$, the product of parity and signature, is the same in all channels to which the trajectories of the corresponding Regge family couple.

All Regge trajectories with nonvanishing residues at $t=0$ must be classified in $O(4)$ families. Trajectories which couple to pseudoscalar-meson pairs at $t=0$ must correspond to Lorentz poles with $M=0$ and $s=0$, since these are the only $O(4)$ states which contribute in the t-channel process $N+\bar{N}\to PS+PS$ because of the final-state spins.[25] Therefore P, P', and ρ which couple to $\pi\pi$ and ω, φ and A_2 which couple to $K\bar{K}$ must be the leading trajectories of $O(4)$ families of type I.

The quantum numbers of the A_1 trajectory suggest that it be identified with the leading member of an $O(4)$ family of type II.[26,27] The next leading member is a trajectory of the singlet $f_0{}^{j+}$ amplitude. This trajectory lies one unit below the A_1 at $t=0$, and its first physical manifestation would be a $J^{PG}=0^{--}$ particle. The $O(4)$ symmetry at $t=0$, of course, does not require that the subsidiary trajectories of a family rise high enough to make physical particles. We merely observe here that the $\pi(1640)$ meson[28] is a possible candidate for the 0^{--} meson just described.

The π, η, and B trajectories couple to the $f_0{}^j$ amplitude. If their residues are nonvanishing at $t=0$, the trajectories are presumably associated with Regge families of type III. This assignment requires the existence of a trajectory of the $f_{22}{}^j$ amplitude with intercept at $t=0$ equal to that of the original trajectory.

Since the 0^+ parity partner of the π meson is not observed at low mass, the $f_{22}{}^j$ trajectory of the pion family would have to have an extremely shallow or even a negative slope. The ratio $\beta_{22}(0)/\beta_\pi(0)$ of the residues of triplet and singlet trajectories is negative if $\alpha_\pi(0)<0$, and this may be connected with the behavior of the triplet trajectory suggested above. The $\pi_v(1030)$ meson[28] is a possible candidate for the 0^+ state on this trajectory.

The $M=1$ classification of the π meson is consistent with the PCAC hypothesis which requires that the π-meson mass extrapolate smoothly to zero. The $N\bar{N}\pi$ coupling vanishes at $t=0$ for a zero-mass pion. This vanishing is automatically incorporated into the group-theoretic coefficients in Eq. (46) since an $M=1$ representation of $O(4)$ does not contain states with $j=0$.

The $O(4)$ classification of the π, η, A_1, and B trajectories can in principle be verified in detailed Regge-pole analyses of processes to which these trajectories couple.

[24] As discussed earlier, the $O(4)$ symmetry does not strictly apply to two-body unequal-mass channels. However, the pole structure in the unequal-mass case would still exhibit the symmetry (see Ref. 5), so that the quantum number M would be meaningful even in unequal-mass channels.

[25] This argument should be compared to the conventional argument based on angular-momentum conservation, crossing, and factorization which shows that the $N\bar{N}$ residue β_{22} of any trajectory which couples to pseudoscalar-meson pairs must vanish at $t=0$ (see Ref. 2).

[26] Assignment to a type-III family is possible, but unlikely, since it requires the existence of high-lying trajectories of the $f_0{}^j$ and $f_{22}{}^j$ amplitudes.

[27] L. Durand (Ref. 4) has suggested an A_1 conspiracy of this type.

[28] A Rosenfeld *et al.*, Rev. Mod. Phys. **39**, 1 (1957).

[29] This argument was first suggested by G. F. Chew (private communication).

Simultaneous analysis of the processes $np \to pn$, $\gamma p \to \pi^+ n$, and $\pi^- p \to \rho^0 n$ would be important for a better understanding of this subject. It is only for the first process that reasonably successful analyses have been obtained.[30]

In the conspirator theory[3,4] of NN scattering roughly similar results are obtained from the identity[22]

$$f_0{}^{j-1} - f_0{}^{j+1} - \frac{j-1}{j} f_{22}{}^{j-1} + \frac{j+2}{j+1} f_{22}{}^{j+1} - \frac{2j+1}{j(j+1)} f_1{}^j = 0 \quad (54)$$

which must be satisfied at $t=0$ in order to avoid a kinematic singularity of the pseudoscalar invariant amplitude. Near a Regge pole the identity can be satisfied either by a vanishing residue or by correlation in position and residue of the Regge poles of the three partial waves which participate in Eq. (54). Such correlations are called conspiracies.[4] There are an infinite number of conspiring families which can satisfy the identity (54). Gribov and Volkov[1] admit only two possible conspiracies. Because they reject the possibility of daughter trajectories, their solution must be regarded as inadequate.

Since the analyticity requirements are built into our theory the partial-wave identity is automatically satisfied. The predictive power of the $O(4)$ symmetry is much stronger than that of Eq. (54). Indeed we find on the basis of group-theoretic requirements that families II and III are the only possible conspiracies.

Although calculation of ratios of Regge-pole residues within a given $O(4)$ family usually requires explicit evaluation of the coefficients in Eqs. (46)–(49), the ratio of residues of the leading members can be obtained very simply from Eq. (54). In fact the moment of truth in our calculation came when the requirement of agreement between both methods of calculating the ratio of residues was checked. It is easy to see that the values (51) and (52), calculated group-theoretically, are exactly the values required by Eq. (54).

In both the conspirator and $O(4)$ theories, there is no requirement (contrary to older theories) that residues of trajectories of the $f_0{}^j$ and $f_1{}^j$ amplitudes vanish at $t=0$. From our point of view the vanishing of such residues would be purely accidental.

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APPENDIX A: PROPERTIES OF $d_{jj'm}{}^{(n,M)}(\delta)$

All properties of the boost representation functions needed in the text can be derived from the defining equation

$$d_{js\lambda}{}^{(n,M)}(\delta) = \sum_\mu C\left(\frac{n+M}{2}, \frac{n-M}{2}, j, \mu, \lambda-\mu\right) \times C\left(\frac{n+M}{2}, \frac{n-M}{2}, s, \mu, \lambda-\mu\right) e^{-i(2\mu-\lambda)\delta} \quad (A1)$$

by using known properties of the Clebsch-Gordan coefficients. We note first the properties

$$d_{js\lambda}{}^{(n,M)*}(\delta) = d_{js\lambda}{}^{(n,M)}(-\delta) = (-1)^{j-s} d_{js\lambda}{}^{(n-M)}(\delta), \quad (A2)$$

$$d_{js-\lambda}{}^{(n,M)}(\delta) = d_{js\lambda}{}^{(n,-M)}(\delta), \quad d_{js\lambda}{}^{(n,M)}(\delta) = (-1)^{n+M-\lambda} d_{js\lambda}{}^{(n,M)}(\delta-\pi). \quad (A3)$$

The boost functions can be expressed in terms of Gegenbauer functions by using recursion relations for the Clebsch-Gordan coefficients and the basic relations

$$C_n{}^1(\cos\delta) = \sin(n+1)\delta/\sin\delta, \quad (d^k/dx^k)C_n{}^\lambda(x) = 2^k[\Gamma(\lambda+k)/\Gamma(\lambda)]C_{n-k}{}^{\lambda+k}(x). \quad (A4)$$

The case $M=s=\lambda=0$ has been treated by Bander and Itzykson.[31] Their result can be written as

$$d_{j00}{}^{(n,0)}(\delta) = \left\{\frac{(2j+1)\Gamma(n-j+1)}{(n+1)\Gamma(n+j+2)}\right\}^{1/2} \times (2i)^j \Gamma(1+j) \sin^j\delta\, C_{n-j}{}^{1+j}(\cos\delta). \quad (A5)$$

More general cases can be treated using the recursion relations of Bander and Itzykson,[31] Eq. (3.7.13) of Edmonds,[32] and the relation

$$[(J-\mu)(J+\mu+1)]^{1/2} C(j, j', J; m, \mu-m+1) = [(j'-\mu+m)(j'+\mu-m+1)]^{1/2} C(j, j', J; m, \mu-m) \times [(j-m+1)(j+m)]^{1/2} C(jj'J, m-1, \mu-m+1). \quad (A6)$$

We list the results

$$d_{j0}{}^{(n,0)}(\delta) = \left(\frac{3}{n(n+2)}\right)^{1/2} i\frac{d}{d\delta} d_{j00}{}^{(n,0)}(\delta), \quad (A7)$$

$$d_{j11}{}^{(n,0)}(\delta) = \left(\frac{3}{2n(n+1)(n+2)j(j+1)}\right)^{1/2} \left[n(n+1)+1+\frac{d^2}{d\delta^2}\right] [\sin\delta\, d_{j00}{}^{(n,0)}(\delta)], \quad (A8)$$

[30] R. J. N. Phillips (to be published); F. Arbab & J. Dash, University of California Radiation Laboratory Report UCRL-17585, 1967 (unpublished). We are grateful to the latter authors for helpful discussions.

[31] M. Bander and C. Itzykson, Rev. Mod. Phys. 38, 330 (1966) (see Appendix). Our $d_{j00}{}^{n0}(\delta)$ differ from the $T_{nj}(\delta)$ in this paper by an over-all factor.

[32] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

$$d_{j10}^{(n,1)}(\delta)=\left(\frac{n-1}{(n+2)(n+j+1)(n-j)}\right)^{1/2}\left[(n+2)\cos\delta+\sin\delta\frac{d}{d\delta}\right]d_{j11}^{n-1,0}(\delta)\,, \tag{A9}$$

$$d_{j11}^{(n,1)}(\delta)=[8j(j+1)]^{-1/2}\left[3e^{-i\delta}+e^{i\delta}+2\sin\delta\left(\frac{d}{d\delta}-in\right)\right]d_{j10}^{(n,1)}(\delta)\,, \tag{A10}$$

for the functions which contribute in NN scattering.

To derive Eqs. (46)–(49) one needs the following symmetry properties of boost functions of argument $\pi/2$ which can be obtained from (A5) and (A7)–(A10):

$$d_{js\lambda}^{(n,0)}(\pi/2)=(-1)^{n-j+s+\lambda}d_{js\lambda}^{(n,0)}(\pi/2)\,, \tag{A11}$$

$$\begin{aligned}d_{j1\lambda}^{(n,1)}(\pi/2)&=(-1)^{n-j+\lambda}d_{j1\lambda}^{(n,-1)}(\pi/2)\\&=(-1)^{n-j+\lambda}d_{j1-\lambda}^{(n,1)}(\pi/2)\,.\end{aligned} \tag{A12}$$

These relations are valid for complex j and n with $n-j=\kappa$ a non-negative integer. These relations can be derived more simply by taking $\delta=\pi/2$ in (A2) and (A3) and using Carlson's theorem to extend the result to complex j, n with $n-j=\kappa$.

For use in Appendix B we list the boost functions with $j=s$:

$$\begin{aligned}d_{000}^{(n,0)}(\delta)&=(n+1)^{-1}C_n^1(\cos\delta)\,,\\
d_{110}^{(n,0)}(\delta)&=3[n(n+1)(n+2)]^{-1}\\&\quad\times[n^2C_n^1(\cos\delta)-4C_{n-2}^2(\cos\delta)]\,,\\
d_{111}^{(n,0)}(\delta)&=6[n(n+1)(n+2)]^{-1}C_{n-1}^2(\cos\delta)\\&=d_{11-1}^{(n,0)}(\delta)\\&=d_{110}^{(n,1)}(\delta)\,,\\
d_{111}^{(n,1)}(\delta)+d_{11-1}^{(n,1)}(\delta)&=3[n(n+1)(n+2)]^{-1}\\&\quad\times\{n(n+1)C_n^1(\cos\delta)-2C_{n-2}^2(\cos\delta)\}\,.\end{aligned} \tag{A13}$$

These formulas can be derived from (A5) and (A7)–(A10) using recursion relations for the Gegenbauer functions.

APPENDIX B: COMPLEX $O(4)$ ANGULAR MOMENTUM

In $N\bar{N}$ scattering Eq. (42) implies

$$T_{s's\lambda}(\delta)=\delta_{s's}T_{s\lambda}(x)\,, \tag{B1}$$

$$T_{1\lambda}(x)=T_{1-\lambda}(x)\,, \tag{B2}$$

where $x=\cos\delta$, so that there are three independent amplitudes of the form (40). $O(3)$ invariance gives

$$\begin{aligned}T_{s\lambda+1}(x=1)&=\langle\hat{e}(000)s\lambda|J_-TJ_+|\hat{e}(000)s\lambda\rangle\\&\quad\times[s(s+1)-\lambda(\lambda+1)]^{-1}=T_{s\lambda}(1)\,.\end{aligned} \tag{B3}$$

Using $J_+e^{-i\pi K_3}=-e^{-i\pi K_3}J_+$, the relation

$$T_{s\lambda+1}(-1)=-T_{s\lambda}(-1) \tag{B4}$$

can be similarly obtained. Relations (B3) and (B4) which restrict to two the number of independent amplitudes at threshold in the s and u channels reflect the well-known fact that the scattering of two spin-$\frac{1}{2}$ particles at threshold is characterized by two parameters, the scattering lengths.

The projection formulas (44) which define the $O(4)$ partial-wave amplitudes for integer n can be written

$$T_s^{nM}=\sum_{\lambda=-s}^{s}\int_{-1}^{1}dx(1-x^2)^{1/2}T_{s\lambda}(x)d_{ss\lambda}^{n,M}(\delta)\,. \tag{B5}$$

Using an explicit expression for the $N\bar{N}$ scattering amplitudes in terms of Fermi amplitudes,[22] one can show that the amplitudes $T_{s\lambda}(x)$ are free of kinematical singularities. They therefore satisfy dispersion relations which may be written, ignoring subtractions, as

$$T_{s\lambda}(x)=\frac{1}{\pi}\int_{-1}^{-\infty}dx'\frac{A_{s\lambda}(x')}{x'-x}+\frac{1}{\pi}\int_{x_0}^{\infty}dx'\frac{B_{s\lambda}(x')}{x'-x}\,, \tag{B6}$$

where[33]

$$x_0=\frac{2m_\pi^2}{m_N^2}-1\,,\quad x=1-\frac{s}{2m_N^2}=-1+\frac{u}{2m_N^2}\,.$$

$A_{s\lambda}(x')$ is the s-channel discontinuity and $B_{s\lambda}(x')$ is the u-channel discontinuity to which pion annihilation contributes.

We wish to obtain a continuation in n of T_s^{nM} by substituting the dispersion relation (B6) in (B5) and using arguments similar to those of Froissart and Gribov. This substitution leads to the result

$$T_s^{nM}=\sum_{\lambda=-s}^{s}\left[-\int_{-\infty}^{-1}dx\,A_{s\lambda}(x)E_{ss\lambda}^{nM}(x)+\int_{x_0}^{\infty}dx\,B_{s\lambda}(x)E_{ss\lambda}^{nM}(x)\right]\,, \tag{B7}$$

where

$$E_{s's\lambda}^{nM}(x)\equiv\frac{1}{\pi}\int_{-1}^{1}\frac{dx'(1-x'^2)^{1/2}}{x-x'}d_{s's\lambda}^{n0}(x')\,. \tag{B8}$$

$E_{ss\lambda}^{nM}(x)$ can be evaluated using (A13) and the following formulas:

$$\frac{1}{\pi}\int_{-1}^{1}dx'(1-x'^2)^{1/2}\frac{C_n^1(x')}{x-x'}\equiv D_n^1(x)=[x-(x^2-1)^{1/2}]^{n+1}\,, \tag{B9}$$

[33] Actually because only $M=0$ $s=0$ states couple to pseudoscalar pairs at $t=0$, only $T_{00}(x)$ will contain the $\pi\pi$ and $K\bar{K}$ branch points. The annihilation threshold for $T_{10}(x)$ and $T_{11}(x)$ is actually $x_0=(9m_\pi^2/4m_N^2)-1$.

$$\frac{1}{\pi}\int_{-1}^{1}dx'(1-x'^2)^{1/2}\frac{C_{n-1}^2(x')}{x-x'}=\frac{1}{4(x^2-1)}[nD_{n+1}^1(x)-(n+2)D_{n-1}^1(x)]+\frac{1}{4}\left[\frac{1}{x-1}-\frac{(-1)^n}{x+1}\right]. \tag{B10}$$

Using (A13) and (B7)–(B10), we can write explicitly

$$T_0^{n0}=\frac{(-1)^n}{n+1}\int_1^\infty dx\,A_{00}(-x)D_n^1(x)+\frac{1}{n+1}\int_{x_0}^\infty dx\,B_{00}(x)D_n(x)\,, \tag{B11}$$

$$\begin{aligned}T_1^{n0}=&\frac{3}{n(n+1)(n+2)}(-1)^n\int_1^\infty\frac{dx}{x^2-1}[A_{10}(-x)\{(x^2-1)n^2D_n^1(x)-[(n-1)D_n^1(x)-(n+1)D_{n-2}^1(x)]\}\\&\qquad-A_{11}(-x)[nD_{n+1}^1(x)-(n+2)D_{n-1}^1(x)]]\\&+\frac{3}{n(n+1)(n+2)}\int_{x_0}^\infty\frac{dx}{x^2-1}[B_{10}(x)\{(x^2-1)n^2D_n^1(x)-[(n-1)D_n^1(x)-(n+1)D_{n-2}^1(x)]\}\\&\qquad+B_{11}(x)[nD_{n+1}^1(x)-(n+2)D_{n-1}^1(x)]],\end{aligned} \tag{B12}$$

$$\begin{aligned}T_1^{n1}=&\frac{3}{2n(n+1)(n+2)}(-1)^n\int_1^\infty\frac{dx}{x^2-1}[-A_{10}(-x)\{nD_{n+1}^1(x)-(n+2)D_{n-1}^1(x)\}\\&\qquad+A_{11}(-x)\{(x^2-1)2n(n+1)D_n^1(x)-[(n-1)D_n^1(x)-(n+1)D_{n-2}^1(x)]\}]\\&+\frac{3}{2n(n+1)(n+2)}\int_{x_0}^\infty\frac{dx}{x^2-1}[B_{10}(x)\{nD_{n+1}^1(x')-(n+2)D_{n-1}^1(x')\}\\&\qquad+B_{11}(x)\{2n(n+1)(x^2-1)D_n^1(x)-[(n-1)D_n^1(x')-(n+1)D_{n-2}^1(x)]\}].\end{aligned} \tag{B13}$$

The (non-Carlsonian) contributions of the terms of the second bracket in (B10) to (B12) and (B13) vanish because of the symmetry properties (B3) and (B4). The same symmetry properties, applied to the absorptive parts, ensure that the integrals (B12) and (B13) converge in spite of the singular factor $(x^2-1)^{-1}$. The role of the symmetry properties (B3) and (B4) in ensuring the success of the complex n continuation of the T_s^{nM} is very interesting.

The $D_n^1(x)$ are entire functions of n, and Eqs. (B11)–(B13) therefore converge and define amplitudes T_s^{nM} analytic in the half-plane $\mathrm{Re}n>N$, where N is the number of subtractions necessary in the dispersion relations (B6). To obtain amplitudes with reasonable asymptotic behavior in n, we replace the factor $(-1)^n$ by $\pm$, in this way defining amplitudes $T_s^{nM\pm}$, of definite "Lorentz signature," which interpolate from even and odd integers, respectively.

We now investigate the asymptotic behavior in n of the $T_s^{n,M\pm}$, with explicit reference to the amplitude $T_0^{n,0\pm}$ of simplext structure. Introducing the variables $x=\cosh\bar{\delta}$ and $x=\cos\delta$ we can write (B11) as

$$\begin{aligned}(n+1)&T_0^{n0\pm}\\&=\int_0^\infty d\bar{\delta}\sinh\bar{\delta}[\pm A_{00}(-\cosh\bar{\delta})+B_{00}(\cosh\bar{\delta})]\\&\quad\times\exp[-\bar{\delta}(n+1)]+\int_0^{\varphi_0}d\delta\sin\delta\,B_{00}(\cos\delta)e^{+i\delta(n+1)},\end{aligned} \tag{B14}$$

where $\varphi_0=\cos^{-1}x_0$. We divide the interval of integration of the first term in (B14) into two intervals (0,1) and $(1,\infty)$. The infinite integral is bounded in magnitude by $De^{-\mathrm{Re}n}$, where D is a constant. The integral over (0,1) can be estimated very simply; letting

$$C^\pm(\bar{\delta})=\pm A_{00}(\cosh\bar{\delta})+B_{00}(\cosh\bar{\delta})\,,$$

we obtain

$$\begin{aligned}&\left|\int_0^1 d\bar{\delta}\sinh\bar{\delta}\,C^\pm(\bar{\delta})e^{-\bar{\delta}(n+1)}\right|\\&\qquad\leq\frac{1}{(\mathrm{Re}n)^2}\max_{0\leq\bar{\delta}\leq1}|C^\pm(\delta)|[1+O(1/\mathrm{Re}n)]\end{aligned} \tag{B15}$$

as $\mathrm{Re}n\to\infty$ and a constant bound as $\mathrm{Im}n\to\infty$.

The second term in (B14) is only slightly more complicated. We integrate by parts and easily obtain the estimate

$$\begin{aligned}&\left|\int_0^{\varphi_0}d\delta\sin\delta\,B_{00}(\cos\delta)e^{+i\delta(n+1)}\right|\\&\qquad\leq\frac{A}{|n|}+\frac{B}{|n|}e^{-\varphi_0\,\mathrm{Im}n},\end{aligned} \tag{B16}$$

where A and B are constants.

The angle φ_0 is greater than $\pi/2$ and we must multiply $T_0^{n0\pm}$ by the factor $e^{-i(\varphi_0-\pi/2-\epsilon)n}$, where ϵ is positive infinitesimal, in order to obtain amplitudes which satisfy the requirements of Carlson's theorem in the form pertaining to continuation from alternate integers. Similar asymptotic estimates can be derived for the amplitudes $T_1^{n,0\pm}$ and $T_1^{n,1\pm}$.

By setting $n=j+\kappa$ and $T_s^{n,M\pm}=T_s^{j+\kappa,M\pm}$, it is a straightforward matter to use the asymptotic bounds just derived to show that the infinite series (46)–(49) converge uniformly in j and therefore define functions $f^{j\pm}$ analytic in $\mathrm{Re}\,j>N$ which coincide with the physical partial-wave amplitudes at even- and odd-integral values of j.

The amplitudes $e^{-i(\varphi_0-\pi/2-\epsilon)}f^{j\pm}$ satisfy the asymptotic requirements of Carlson's theorem and therefore coincide for all j with the conventional continuations away from even and odd integers with the same asymptotic properties. Amplitudes $a^{j\pm}(t)$ defined by Froissart-Gribov integrals fail to have the required asymptotic behavior near $t=0$, but closely related amplitudes do,[34] and one can show that

$$f^{j\pm}=\pm e^{-i\pi j}a^{j\pm}(t=0). \tag{B17}$$

Conditionally convergent expansions of the form of Eqs. (46)–(49) can be written for the bound-state pole terms subtracted from the amplitudes in the group-theoretic analysis. The pole contributions can therefore be included in the final form of the $O(4)$ decompositions.

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Production of 2^+ Mesons and $SU(6)$ Symmetry

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The sum rules for production amplitudes of 2^+ mesons near threshold in pseudoscalar meson-baryon scattering have been studied within the framework of $SU(6)$ symmetry. It is concluded that the nonet of 2^+ mesons should be assigned to the **405** multiplet in view of the fact that some of the well-observed processes are forbidden if these mesons belong to **189**. The sum rules based on the assignment of 2^+ mesons to the **405** multiplet of $SU(6)$ are found to be consistent with the experimental data available at present.

I. INTRODUCTION

During recent years there have been numerous experimental evidences indicating the existence of meson resonances (f',f,A_2,K^{**}) with spin 2 and positive parity. Glashow and Socolow[1] calculated the partial-decay widths of the 2^+ mesons based on the assignment of these mesons to the reducible nonet representation of $SU(3)$ with considerable octet-singlet mixing. In this way the predicted decay widths were found to agree well with the observed values. In the present paper, we study the production of these 2^+ resonances near threshold in pseudoscalar meson-baryon scattering within the framework of $SU(6)$ symmetry. Although the lowest-possible multiplet of $SU(6)$ containing the nonet of 2^+ mesons is **189**, there are a number of arguments[2] in favor of assigning these mesons to **405**. In the present calculations we oberves the following important points.

If the observed 2^+ mesons are assigned to **189**, we find that although all the production amplitudes can be expressed in terms of a single parameter, some of the experimentally observed processes with known cross sections become forbidden. A reasonable conclusion, therefore, is to assume that the observed 2^+ mesons should rather belong to the **405** multiplet of $SU(6)$, in which case the relevant production amplitudes can be expressed in terms of four independent parameters. The predicted $SU(6)$ sum rules in this case, however, are not amenable to accurate verification because of the lack of extensive experimental data; but in some cases a crude estimation is still possible and our results are found to be quite consistent with experiment.

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II. THE SUM RULES

As already mentioned in the preceding section, we shall consider the processes

$$(0^-\text{ meson})+(\tfrac{1}{2}^+\text{ baryon}) \rightarrow (2^+\text{ meson})+(\tfrac{1}{2}^+\text{ baryon}), \tag{1}$$

where the 0^- mesons and the $\frac{1}{2}^+$ baryons belong to the **35** and the **56** multiplets of $SU(6)$, respectively. The 2^+ meson nonet, namely (8,5) and (1,5), consisting of