Modified Regge Representation

SUPROKASH MUKHERJEE
Saha Institute of Nuclear Physics, Calcutta, India
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A new Regge-type representation for the partial-wave scattering amplitude $A(\lambda,k)$ is proposed, which has correct asymptotic behavior for small k and large λ. The possibility of making the background integral small is also explored.

1. INTRODUCTION

IN practical application of the Regge-type representation of scattering amplitudes, one usually makes pole approximations and neglects the background integral. It is then desirable that, under such an approximation, the pole terms have the correct asymptotic behaviors in momentum and in the angular momentum plane. It has so far been difficult to construct such a representation of the scattering amplitude, where pole terms have these properties.[1–3] For example, in the Regge representation, neither the threshold behavior of the partial-wave scattering amplitude[3] $A(\lambda,k)$ nor its asymptotic behavior in the λ plane is reproduced by the pole terms. On the other hand, the pole terms in the Khuri representation give the correct asymptotic behavior of $A(\lambda,k)$ in the λ plane and the correct threshold behavior for the real part of $A(\lambda,k)$ but not for its imaginary part. This is achieved in the Khuri representation by taking a suitable "part" of the background integral of the Regge representation and lumping it into the pole terms, thereby showing how significant that "part" of the background integral could be for certain purposes. Again the modified Khuri series of Hankins *et al.*[4] shows that one can perhaps do still better by essentially taking a fresh chunk from the background integral of the Khuri representation. In all these representations, however, the questions of the correct asymptotic behavior of the supposedly tractable pole terms in the λ and the k planes are not completely answered, and, as such the neglect of the background integral has remained a questionable procedure in all these cases. In the present paper, we will try to answer some of these questions. We will, in fact, show that one can construct a variety of representations where the pole terms have correct asymptotic behavior, i.e., correct asymptotic behavior in the λ plane and correct threshold behavior for the real and imaginary parts of $A(\lambda,k)$. In spite of these properties of the pole terms, the question of complete negligibility of the background integral remains involved. We will indicate that, except for a few pathological cases, one can in general expect to make the background integral small in a limited energy range, without sacrificing the asymptotic properties of the pole terms.

[1] T. Regge, Nuovo Cimento 14, 951 (1959); 18, 947 (1960); A. Bottino and A. M. Longoni, *ibid.* 24, 353 (1962).
[2] N. N. Khuri, Phys. Rev. **130**, 429 (1963).
[3] See, for example, R. G. Newton, *Complex j-Plane* (W. A. Benjamin, Inc., New York, 1964); S. C. Frautschi, *Regge Poles and S-matrix Theory* (W. A. Benjamin Inc., New York, 1965).
[4] D. Hankins *et al.*, Phys. Rev. **137**, B1034 (1965).

2. FORMALISM

The scattering amplitude of two spinless particles and its Regge and Khuri representations are given,[2,3] respectively, by

$$f(k,z)=\sum_{l=0}^{\infty}(2l+1)A(\lambda,k)P_l(z) \tag{1}$$

$$=\frac{1}{2i}\int_{-i\infty}^{+i\infty}d\lambda'\frac{\lambda' P_{\lambda'-1/2}(-z)}{\cos\pi\lambda'}A(\lambda',k)+\sum_n\frac{2\pi\lambda_n\beta_n P_{\lambda_n-1/2}(-z)}{\cos\pi\lambda_n} \tag{2}$$

$$=\frac{1}{\sqrt{2}}\int_{\xi}^{\infty}dx\frac{\sinh x}{(\cosh x-z)^{3/2}}\left[\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}d\lambda'\, e^{\lambda' x}A(\lambda',k)\right]+\sum_n\beta_n\left[\frac{2\pi\lambda_n P_{\lambda_n-1/2}(-z)}{\cos\pi\lambda_n}-\frac{1}{\sqrt{2}}\int_{-\infty}^{\xi}dx\frac{\sinh x e^{\lambda_n x}}{(\cosh x-z)^{3/2}}\right], \tag{3}$$

where $A(\lambda,k)=[\exp(2i\delta_l)-1]/2ik$, $\lambda=l+\frac{1}{2}$, $z=\cos\theta$, β_n is the residue of the pole of $A(\lambda,k)$ at $\lambda=\lambda_n$ in the right half λ plane, $\xi=\cosh^{-1}(1+\mu^2/2k^2)$, $1/\mu$ is the longest of the ranges appearing in a superposition of Yukawa potentials, and the summations in (2) and (3) are over the Regge poles in the right half λ plane. Regge and Khuri representations, given, respectively, by (2) and (3), are known to be valid[3] at least for Yukawa-like potentials, for which the analytically continued partial-wave amplitude $A(\lambda,k)$ is meromorphic in the right half λ plane, where it behaves asymptotically as[3]

$$A(\lambda,k)\underset{|\lambda|\to\infty}{\sim}\frac{\exp(-\lambda\xi)}{\sqrt{\lambda}}. \tag{4}$$

It is to be noted that, being a meromorphic function in the right half λ plane, $A(\lambda,k)$ may be represented in more than one way in this half plane. For example, $A(\lambda,k)$ for $\mathrm{Re}\lambda\geq 0$ may be written as

$$A(\lambda,k)=\sum_n\frac{\beta_n}{\lambda-\lambda_n}+f_1(\lambda,k), \tag{5}$$

where $f_1(\lambda,k)$ is the analytic part of $A(\lambda,k)$ and, by virtue of (4), may be expressed as

$$f_1(\lambda,k)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\lambda'\frac{A(\lambda',k)}{\lambda-\lambda'}. \tag{6}$$

Again, we may write

$$A(\lambda,k)=\sum_n\frac{\beta_nF(\lambda_n,\lambda)}{\lambda-\lambda_n}+f_2(\lambda,k) \tag{7}$$

$$=\sum_n\frac{\beta_nF(\lambda_n,\lambda)}{\lambda-\lambda_n}+\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{d\lambda'\,F(\lambda',\lambda)A(\lambda',k)}{\lambda-\lambda'}, \tag{8}$$

where

$$F(\lambda,\lambda)=1 \tag{9}$$

and $F(\lambda',\lambda)$ is analytic for $\mathrm{Re}\lambda\geq0$, $\mathrm{Re}\lambda'\geq0$, and where $F(\lambda',\lambda)A(\lambda',k)$ is bounded on the right half λ' plane, so that the integral in (8) could be evaluated by closing the contour in that half-plane. It is easily seen that the representations given in Eqs. (5) and (7) have the same meromorphic structure, i.e., the same set of poles and residues, and one is obtained from the other by adding or subtracting an analytic function from its analytic part [$f_1(\lambda,k)$ or $f_2(\lambda,k)$] and lumping that function into the pole terms; the pole positions and residues are not changed in this process, because of assumption (9). It is readily checked that the Regge representation, given by (2), is obtained from (8) by taking

$$F(\lambda',\lambda)=2\lambda'/(\lambda+\lambda'); \tag{10}$$

and the Khuri representation, given by (3), by taking

$$F(\lambda',\lambda)=\exp\{(\lambda'-\lambda)\xi\}. \tag{11}$$

This is easily verified by making a partial-wave analysis of (2) and (3), and using the identity[5]

$$P_\sigma(-z)=\frac{\sin\pi\sigma}{\pi}\sum_{l=0}^{\infty}\frac{(2l+1)P_l(z)}{(\sigma-l)(\sigma+l+1)}. \tag{12}$$

It should be noted that, if a pole approximation to the partial-wave scattering amplitude is desirable, then the representation in (8) is more useful than that in (5). For, clearly, the partial-wave series (1) is divergent if pole terms alone are used for $A(\lambda,k)$ from (5), as $P_l(z)$ behaves as $1/\sqrt{l}$ for large l. The form (10) for $F(\lambda',\lambda)$, giving the Regge representation, makes the partial-wave series (1) just convergent in the pole approximation; and this convergence rate for the Khuri representation, arising out of the form (11) for $F(\lambda',\lambda)$, is perhaps a little too fast compared to the actual partial-wave series. Whenever the asymptotic form of $A(\lambda,k)$ is different from that of $1/\lambda$, a representation different from that in (5) is desirable.[6]

With the form of $A(\lambda,k)$ given by (8), we have, after some algebra, the expression for the scattering amplitude:

$$f(k,z)=\sum_{l=0}(2l+1)P_l(z)\left[\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{d\lambda'\,F(\lambda',\lambda)A(\lambda',k)}{\lambda-\lambda'}+\sum_n\frac{\beta_nF(\lambda_n,\lambda)}{\lambda-\lambda_n}\right] \tag{13}$$

$$=I+\sum_n\beta_n\left[\frac{2\pi\lambda_nP_{\lambda_n-1/2}(-z)}{\cos\pi\lambda_n}+\frac{1}{\sqrt{2}}\int_{-\infty}^{+\infty}dx\frac{\sinh x}{(\cosh x-z)^{3/2}}\left\{\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{d\lambda'\,e^{\lambda'x}F(\lambda_n,\lambda')}{\lambda'-\lambda_n}\right\}\right], \tag{14}$$

where the new background integral I is given by

$$I=\lim_{d\to0}\frac{1}{(2\pi i)^2\sqrt{2}}\int_{-\infty}^{+\infty}\frac{dx\,\sinh x}{(\cosh x-z)^{3/2}}\int_{d-i\infty}^{d+i\infty}d\lambda''\,e^{\lambda''x}\int_{-i\infty}^{+i\infty}\frac{d\lambda'\,F(\lambda',\lambda'')A(\lambda',k)}{\lambda''-\lambda'}. \tag{15}$$

These results are easily obtained by summing the partial-wave series (13) by the Sommerfield-Watson technique.[3] If $F(\lambda_n,\lambda)$ goes as $e^{-\lambda\xi}$ for large λ in the right half λ plane, then the above representation reduces to

$$f(k,x)=I_1+\sum_n\beta_n\left[\frac{2\pi\lambda_nP_{\lambda_n-1/2}(-z)}{\cos\pi\lambda_n}-\frac{1}{\sqrt{2}}\int_{-\infty}^{\xi}dx\frac{\sinh x e^{\lambda_nx}}{(\cosh x-z)^{3/2}}+\frac{1}{\sqrt{2}}\int_{\xi}^{\infty}\frac{dx\,\sinh x}{(\cosh x-z)^{3/2}}\left\{\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{d\lambda'\,e^{\lambda'x}F(\lambda_n,\lambda')}{\lambda'-\lambda_n}\right\}\right], \tag{16}$$

where I_1 is obtained from I in (15), by replacing $-\infty$, the lower limit of x integration, with ξ. This representation (16) is similar to the Khuri representation (3), from which it differs by the third term in the square bracket in (16), which is itself a part of the background integral of Khuri representation. It may be noted that the representation (16) has the same cut structure in the z plane as that of the Khuri representation. The Khuri representation is obtained from (16), with the help of (11); the last integral in the square bracket of (16) comes out zero after one performs the λ' integration, while for the Regge case, because of (10), the last two integrals in (16) cancel each other.

[5] *Bateman Manuscript Project*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York), Vol. 1.

[6] See, for example, E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, London, 1964), 2nd. ed., Art. 32 and 3.21.

From Eqs. (8) and (13)–(16), it is clear that one can make good use of $F(\lambda',\lambda)$ to ensure the correct asymptotic behaviors of $A(\lambda,k)$, and perhaps also the smallness of the background integral (15). For this purpose, it is enough to use an $F(\lambda',\lambda)$ that damps out the integrand very heavily in the λ' variable, and that at the same time ensures the correct threshold and asymptotic behavior in the λ variable; and this can be assured in many ways. For example, the factor in $F(\lambda',\lambda)$ which guarantees the correct threshold behavior for both real and imaginary parts of $A(\lambda,k)$, and the correct asymptotic behavior in the λ variable, may be taken as $F_1(\lambda',\lambda)$, where

$$F(\lambda',\lambda)=F_1(\lambda',\lambda)F_2(\lambda',\lambda)\,, \tag{17}$$

$$F_1(\lambda',\lambda)=g(\lambda)/g(\lambda')\,, \tag{18}$$

$$g(\lambda)=\exp[-\lambda\xi_1+i\exp(-\lambda\xi_2)]\,, \tag{19}$$

$$\xi_i=\cosh^{-1}\left(1+\frac{\mu_i^2}{2k^2}\right),\quad i=1,2\,, \tag{20}$$

with μ_1 and μ_2 as arbitrary parameters. The factor $F_2(\lambda',\lambda)$ may now be taken so as to damp out the integrand in the λ' variable in (15) or (13). We may, for example, take

$$F_2(\lambda',\lambda)=\left(\frac{2\lambda}{\lambda+\lambda'}\right)^m \tag{21}$$

which, for large positive values of the index m in (21), may serve our purpose, without changing the asymptotic behavior for large λ and small k from that given by the first factor $F_1(\lambda',\lambda)$ in (17). Needless to say, the form in (17) is by no means the only possible choice for $F(\lambda',\lambda)$ that will serve our purpose.

One can, perhaps, improve the situation a little more by subtracting the Born term from $A(\lambda,k)$ first and then following the procedure of Regge, as is done by Ahmadzadeh[7] and by Hankins *et al.*[4] As in Ref. 7, we may write

$$f(k,z)=f_0(k,z)+\frac{g^2}{2k^2}\frac{1}{(1+\mu^2/2k^2-z)}\,, \tag{22}$$

where

$$f_0(k,z)=\sum_{l=0}^{\infty}(2l+1)B(\lambda,k)P_l(z) \tag{23}$$

and

$$B(\lambda,k)=A(\lambda,k)-\frac{g^2}{2k^2}\varphi_{\lambda-1/2}(\cosh\xi)\,. \tag{24}$$

Since, asymptotically, $B(\lambda,k)$ behaves as[7]

$$B(\lambda,k)\underset{|\lambda|\to\infty}{\sim}\frac{e^{-\lambda\xi_3}}{\sqrt{\lambda}} \tag{25}$$

with $\xi_3=\cosh^{-1}(1+\mu^2/k^2)$, we may write, as in (8),

$$B(\lambda,k)=\sum_n\frac{\beta_nF_0(\lambda_n,\lambda)}{\lambda-\lambda_n}+\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{F_0(\lambda',\lambda)B(\lambda',k)d\lambda'}{\lambda-\lambda'}. \tag{26}$$

This time $F_0(\lambda',\lambda)$ may be allowed to go as $e^{\lambda'\xi_3}$ in the right half λ' plane, and the other requirements for the validity of (26) are similar to those of (8). Suppose we use the form (17) for $F_0(\lambda',\lambda)$ with $g(\lambda)$ replaced by $g_0(\lambda)$, where

$$g_0(\lambda)=\exp[-\lambda\xi_4+i\exp(-\lambda\xi_5)]\,, \tag{27}$$

with ξ_4, ξ_5 being of the form in (20). Then for the validity of Eqs. (8) and (26), we must have

$$\xi_1\leq\xi \tag{28}$$

and

$$\xi_4\leq\xi_3 \tag{29}$$

with ξ_2 and ξ_5 arbitrary, except that they must be of the form (20) to guarantee the asymptotic forms. With this information about $B(\lambda,k)$, one obtains an expression for $f_0(k,z)$ similar to that of $f(k,z)$ in (14) and (15), with $A(\lambda,k)$ and $F(\lambda',\lambda)$ replaced by $B(\lambda,k)$ and $F_0(\lambda',\lambda)$, respectively. However, as noted by Ahmadzadeh,[7] one does not gain any extra advantage in this way unless $\xi_4>\xi$. It may be noted that $F(\lambda',\lambda)$ [or $F_0(\lambda',\lambda)$] should have singularities for $\mathrm{Re}\lambda<0$ and $\mathrm{Re}\lambda'<0$. For, otherwise, it would be a constant, by Liouville's theorem,[8] as we assumed $F(\lambda',\lambda)$ [or $F_0(\lambda',\lambda)$] to be analytic in both variables for $\mathrm{Re}\lambda\geq0$, $\mathrm{Re}\lambda'\geq0$, for the validity of (8) [or (26)]. If $F_0(\lambda',\lambda)$ diverges only exponentially in λ in the left half plane without any other singularity, the final form of $f_0(k,z)$ reduces to that obtained by Ahmadzadeh,[7] for in that case the last integral in the square bracket of (16) is identically zero. Further discussion on the singularity structure of $F_0(\lambda',\lambda)$ in the left half plane appears unnecessary at this stage.

3. DISCUSSION

From the above discussion, it is clear that it is possible to construct a "modified Regge representation" where pole terms have all the requisite asymptotic properties possessed by the exact partial-wave scattering amplitude $A(\lambda,k)$. One can, for example, use the representation (16) with the form of $F(\lambda',\lambda)$ given by (17). The first factor $F_1(\lambda',\lambda)$ of $F(\lambda',\lambda)$, given by Eqs. (18)–(20), can by itself guarantee the necessary threshold and asymptotic behavior of $A(\lambda,k)$. From our mode of construction, it is obvious that one can construct a variety of such representations, taking various forms of $F_1(\lambda',\lambda)$, for which one can ensure the existence of all the requisite asymptotic properties of $A(\lambda,k)$ in its pole terms. This still, however, does not guarantee that

[7] A. Ahmadzadeh, Phys. Rev. 133, B1074 (1964).

[8] See Ref. 6, p. 87.

the pole approximation in such representations will be of any use. For the correct asymptotic properties for pole terms do not imply the smallness of the background integral. One can, however, use the factor $F_2(\lambda',\lambda)$ of (17) and (21), which tends to indicate that, at a given energy, the background integral can be made small by taking a sufficiently large positive value for the index m in (21). But this cannot in general be true for all potentials and at all energies. For there are classes of potentials for which there are no poles in the right half λ plane, e.g., Coulomb-like potentials at all energies and Yukawa-like potentials above a certain energy for which all poles recede to the left half λ plane. Therefore, the above statement that the background integral may be made small through the use of the factor $F_2(\lambda',\lambda)$ is valid only in a limited energy range. From the above analysis it may be noted, however, that changing the factor $F(\lambda',\lambda)$ means changing the background integral, and this change is reflected in the pole terms in the right half plane (when they exist) through the changed factors $F(\lambda_n,\lambda)$ at various pole positions. The effect of these factors is to change the behavior of $A(\lambda,k)$ in the neighborhood of its poles without changing the pole positions and residues. Since the background integral, in some sense, represents the effect of the singularities of $A(\lambda,k)$ in the left half λ plane, taking more and more out of the background integral and lumping it into pole terms is equivalent to including more and more of the effects of left-hand singularities. If, finally, in some case of the kind indicated above, the background integral is sufficiently small, then the pole terms should make a reasonably good representation of the partial-wave scattering amplitude in the right half λ plane, to a useful degree of accuracy.

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Spin Correlation Mediated by a Spinless Particle

ASHER PERES*
Department of Physics and Astronomy, University of Maryland, College Park, Maryland
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We consider a process in which a neutral spinless particle decays into a charged spin-1 particle $V^{\pm}$ and a charged spin-0 particle. The latter then decays into a $V^{\mp}$ and another neutral spinless particle. It is shown that the spins of the V^+ and V^- are correlated. This correlation, however, becomes vanishingly small as soon as it is possible to decide whether the V^+ or the V^- was emitted first.

LONG-RANGE correlations between photon polarizations have been known for a long time.[1,2] Recently, in a beautiful experiment, Kocher and Commins[3] were able to observe such correlations for photons emitted at different times.

The process observed by Kocher and Commins can be described by the Feynman diagram of Fig. 1. An excited S^* state decays into a P state, by emission of a photon γ_1. The P state then decays into an S state by emission of another photon γ_2. It can easily be shown that the spins of both photons are correlated.

In the above experiment, the photon spin correlations are trivial because the metastable intermediate particle has spin 1 and carries a coherent superposition of positive and negative helicities. One might thus get the impression that an essential feature of such spin correlations between final particles is that the intermediate particle itself possesses spin. The purpose of this paper is to show that even a spinless intermediate particle can lead to spin correlations,[4] provided that it has some other degree of freedom (such as charge) to allow for interference between different amplitudes.

Consider, e.g., the process illustrated in Fig. 2. Here, a neutral spinless particle S^* decays into either ${}^+S$ and V^-, or S^- and V^+, the $S^{\pm}$ being spinless and the $V^{\pm}$ having spin 1. Later, $S^{\pm}$ decays into $V^{\pm}$ and S

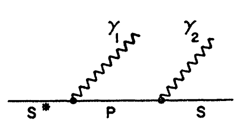

FIG. 1. Successive emission of two photons in the cascade $S^*\to P\to S$

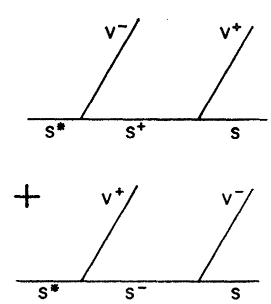

FIG. 2. A V^+ and a V^- are emitted in the successive decays $S^*\to S^{\pm}\to S$.

* On sabbatical leave from Technion–Israel Institute of Technology, Haifa, Israel.

[1] H. S. Snyder, S. Pasternack, and J. Hornbostel, Phys. Rev. **73**, 440 (1948).

[2] C. S. Wu and I. Shaknov, Phys. Rev. **77**, 136 (1950).

[3] C. A. Kocher and E. D. Commins, Phys. Rev. Letters **18**, 575 (1967).

[4] This is, of course, impossible for photon spin correlations, because of angular-momentum conservation. We shall therefore have to consider spin correlations between massive vector particles.