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Relativistic, 3-Dimensional, 2-Body Integral Equations. On-Shell and Off-Shell Formalisms*

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Covariant on-shell and off-shell formalisms are presented on the basis of the Bethe-Salpeter equation and the 2-particle unitarity condition. The 2-body theory is formulated in terms of the relativistic 3-dimensional integral equations of the Low type. Three approaches are presented which take into account differently the left-hand cuts, besides including the 2-particle right-hand cut. The partial-wave expansion and some methods of solving the reduced equations are pointed out. A covariant formulation and a generalization for arbitrary masses are included in full detail.

1. INTRODUCTION

A MANAGEABLE scheme of relativistic, unitary on-shell and off-shell 2-body theory is highly desired for several purposes, such as providing a covariant and unitary input to a 3-particle theory and perhaps later providing such an input to a many-particle theory. The necessity of having a covariant off-shell 2-particle theory as a starting point for building up a 3-particle or multiparticle theory is obvious. The 2-particle subsystem has to be tied up with the rest, which means that some of the particles, coming out from a 2-particle blob, become virtual, and consequently we will have to deal with an off-shell 2-particle theory.

The relativistic, covariant formulation of the 2-particle theory also plays an important role if we want to use such a theory in solving 3-particle problems, for the reason that when we deal with a 3-particle system we have to make several Lorentz transformations between, for example, the 2-particle rest frame and the total 3-particle center-of-mass system. In addition the Lorentz transformations between three different 2-particle rest frames turn out to be very important in several practical problems of the 3-body system, the $\pi\pi N$ system being an example.

Besides the necessity of having a 2-body off-shell theory as an input for a 3-particle system, one may think of applications of a 2-body off-shell scheme itself in checking several dynamical assumptions about 2-body interactions. The present investigation is then motivated by these two purposes: to provide a practical input to the 3-body theory and to formulate an off-shell 2-body theory. To get the 2-body theory in a simple form we have approximated the full unitarity condition with the 2-particle unitarity. This is a reasonable approximation from the point of view of building up a 3-particle and multiparticle theories, where the higher-unitarity cuts will be preserved. Based on this assumption, we have reduced the Bethe-Salpeter equation from the 4-dimensional to a 3-dimensional integral equation, which can be decomposed into a 1-dimensional integral equation for the partial-wave amplitudes. The same problem has been discussed previously, from a different point of view, by several authors.[1] In spite of having 3-dimensional integrals, we have formulated the 2-body theory in a manifestly covariant way, having all relativistic kinematical factors built in and using invariant variables.

We consider three slightly different approaches of formulating the 2-body theory in an approximate 2-particle unitary form. In one of the approaches only the right-hand cut is included, while in the other two both left- and right-hand cuts are taken into account. Several of our results have already been established,[1,2] or are a straightforward generalizations of known equations. The structure of our equations is that of a Low-type equation.

The ladder approximation is used to show explicitly the difference between the three approaches and to explain several details. The box graph is an especially good example for comparing our approximate schemes, based on 2-particle unitarity, because in the appropriate variables the whole analytic structure of the box graph is given solely by the 2-particle unitarity cut.

From our discussion of the different approaches it will be seen that one should include both left- and right-hand cuts. The importance of the left-hand structure of the 2-body T matrix has also been shown by the arguments of Basdevant and Omnes.[3] They pointed out that in the Faddeev equations one needs the 2-body T matrix in the range of energy squared from a positive value to minus infinity. Of course, none of our three approaches includes all left-hand cuts, but there is shown how to incorporate some of them without making additional singularities for negative energies squared.

In Sec. 2 we start from the full Bethe-Salpeter equation and, with the help of the 2-particle unitarity condi-

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[1] A. A. Lagunov and A. N. Tavkhelidze, Nuovo Cimento **29**, 380 (1963); G. Tiktopoulos (unpublished); B. Lee and R. Sawyer (unpublished); M. K. Polivanov and S. S. Khoruzhi, Zh. Eksperim. i Teor. Fiz. **46**, 339 (1963) [English transl.: Soviet Phys.—JETP **19**, 232 (1964)]; R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).
[2] H. P. Noyes and D. Y. Wong, Phys. Rev. Letters **3**, 191 (1959); G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960); G. Marchesini and D. Y. Wong, University of California, San Diego, Report No. UCSD-10P10-9, 1966 (unpublished).
[3] J. L. Basdevant and R. L. Omnes, Phys. Rev. Letters **17**, 775 (1966).

tion, derive a formula for the discontinuity of the whole homogeneous term of the Bethe-Salpeter equation. That formula is used in building up a reconstructed, reduced Bethe-Salpeter equation in the 3-dimensional space. This equation is discussed in Sec. 3, where we present the three different approaches. The differences among these approaches lie in the particular arrangement of variables and amount to different incorporations of the left-hand cuts. In Sec. 3 we point out several good and bad properties of each approach.

Section 4 provides an example of the ladder graphs which serve as a tool for explaining some details and for sharpening the differences among the three approaches.

In Sec. 5 and in the Appendix we present a covariant formulation and deal with arbitrary masses. It turns out that for handling such a problem it is best to use the Wightman-Gårding relative momentum. One of its advantages is that the time-reversal invariance is automatically maintained after reduction.

Section 6 is concerned with the partial-wave expansion, and each approach is discussed separately. These considerations are extended in Sec. 7, where a few methods of solving the reduced equations are sketched.

The last section contains a brief summary and several proposals for further investigations.

2. BETHE-SALPETER EQUATION AND THE 2-PARTICLE UNITARITY

We start from the full Bethe-Salpeter equation written in the following forms:

$$T(\hat{K},\hat{p},\hat{p}')=I(\hat{K},\hat{p},\hat{p}')-\frac{i}{(2\pi)^4}\int d^4\hat{p}'' \times I(\hat{K},\hat{p},\hat{p}'')G^0(\hat{K},\hat{p}'')T(\hat{K},\hat{p},''\hat{p}')\,, \quad \text{(1a)}$$

$$T(\hat{K},\hat{p},\hat{p}')=I(\hat{K},\hat{p},\hat{p}')-\frac{i}{(2\pi)^4}\int d^4\hat{p}'' \times T(\hat{K},\hat{p},\hat{p}'')G^0(\hat{K},\hat{p}'')I(\hat{K},\hat{p}'',\hat{p}')\,, \quad \text{(1b)}$$

where $\hat{K}$ is the total 4-momentum; $\hat{p}$, $\hat{p}'$, $\hat{p}''$ are the usual

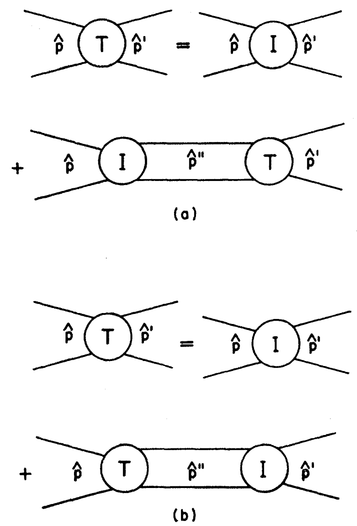

FIG. 1. Diagrams representing the Bethe-Salpeter equation (1).

relative 4-momenta in the initial, final, and intermediate states respectively; $I(\hat{K},\hat{p},\hat{p}')$ is the interaction: a simple example of it, for the ladder graph, is

$$I(\hat{p},\hat{p}')=\frac{g^2}{(\hat{p}-\hat{p}')^2-\mu^2}\,,$$

where g is the coupling constant and μ the mass of the exchanged particle; $G^0(\hat{K},\hat{p}'')$ is the 2-particle free Green's function, given by

$$G^0(\hat{K},\hat{p}'')=[(\tfrac{1}{2}\hat{K}+\hat{p}'')^2-1]^{-1}[(\tfrac{1}{2}\hat{K}-\hat{p}'')^2-1]^{-1},$$

where, for simplicity, we have assumed that the masses of the initial, final, and propagating particles are set equal to unity (the general mass case is presented in Sec. 5); The signature is $+\,-\,-\,-$. The above equations (1a) and (1b) are represented by graphs in Fig. 1.

We shall now use the 2-particle unitarity condition to calculate the right-hand discontinuity of the *whole* homogeneous term in the total energy squared $s=\hat{K}^2$. First we write a trivial relation obtained from (1):

$$\mathrm{Disc}_s[T(\hat{K},\hat{p},\hat{p}')-I(\hat{K},\hat{p},\hat{p}')]=\mathrm{Disc}_s\left[-\frac{i}{(2\pi)^4}\int d^4\hat{p}''I(\hat{K},\hat{p},\hat{p}'')G^0(\hat{K},\hat{p}'')T(\hat{K},\hat{p}'',\hat{p}')\right]. \quad (2)$$

Then we make an assumption that $I(\hat{K},\hat{p},\hat{p}')$ does not have any right-hand singularity in s, so that the left-hand side of (2) is simply $\mathrm{Disc}_s[T(\hat{K},\hat{p},\hat{p}')]$. If $I(\hat{K},\hat{p},\hat{p}')$ had a right-hand cut in s, then we could include it explicitly.

Next, we write the usual 2-particle unitarity condition for T in the form

$$\mathrm{Disc}_s[T(\hat{K},\hat{p},\hat{p}')]=\frac{i}{(2\pi)^2}\int d^4\hat{p}''T(\hat{K},\hat{p},\hat{p}'')\delta_p[(\tfrac{1}{2}\hat{K}+\hat{p}'')^2-1]\delta_p[(\tfrac{1}{2}\hat{K}-\hat{p}'')^2-1]T^{\dagger}(\hat{K},\hat{p}'',\hat{p}')\,, \quad (3)$$

where δ_p means that only the contributions of the "proper" roots of $(\frac{1}{2}\hat{K}\pm\hat{p}'')^2=1$ are to be taken.

From (2) and (3) we obtain the following relation, keeping in mind the assumption about $I(\hat{K},\hat{p},\hat{p}')$:

$$\mathrm{Disc}_s\left[-\frac{i}{(2\pi)^4}\int d^4\hat{p}''I(\hat{K},\hat{p},\hat{p}'')G^0(\hat{K},\hat{p}'')T(\hat{K},\hat{p}'',\hat{p}')\right]$$
$$=\frac{i}{(2\pi)^2}\int d^4\hat{p}''T(\hat{K},\hat{p},\hat{p}'')\delta_p[(\tfrac{1}{2}\hat{K}+\hat{p}'')^2-1]\delta_p[(\tfrac{1}{2}\hat{K}-\hat{p}'')^2-1]T^\dagger(\hat{K},\hat{p}'',\hat{p}'). \quad (4)$$

The relation (4) is an exact statement about the homogeneous term on the right-hand cut, if one assumes only 2-particle unitarity. Another way of obtaining (4) is by applying Cutkosky's[4] rules to the graphs representing the homogeneous term. The result is exactly the same as before, if one cuts in all possible ways only two intermediate lines which correspond to the 2-particle cut in s. We can simply illustrate this result in the example of ladder graphs, as shown in Fig. 2. This figure is only an illustration, and the derivation of (4) is not based on a perturbation theory. However, Fig. 2 explains why we have T in the two places on the right-hand side of the relation (4). In Fig. 3 we show that the higher-unitarity cuts, which correspond to cutting more than two-particle intermediate lines (for example, in the 6th-order graph), are ignored by (4).

3. REDUCED EQUATIONS

A. Reconstruction Rule

The purpose of reconstructing the Bethe-Salpeter equation is to obtain an equation in a smaller number of variables. This means that we have to redefine the homogeneous term of the integral equation (1). We shall use the 2-particle unitarity condition (4) for obtaining a new homogeneous term. Schematically, we write our reconstruction rule in the following way:

$$(IG^0T)_r=\frac{1}{2\pi i}\int_4^\infty ds''\frac{\mathrm{Disc}_{s''}(IG^0T)}{s''-s}, \quad (5)$$

where r stands for "reconstructed." The s'' dependence in the right-hand side of (5) plays an essential role and it will be discussed in detail.

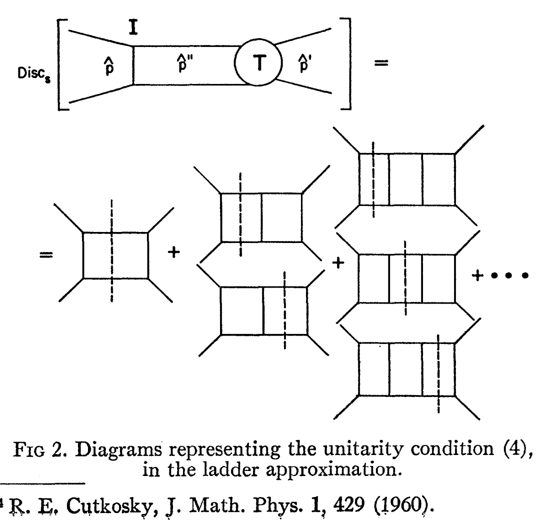

FIG 2. Diagrams representing the unitarity condition (4), in the ladder approximation.

[4] R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).

From (4) and (5) we see that the reconstructed homogeneous term will be given by a 3-dimensional integral, because of the two δ functions in (4). The three integrals will be in one radial and two angular variables. Comparing these integrals with the original ones in the Bethe-Salpeter equation (1), we see that we have reduced the relative energy to only one value, $p^{0\prime\prime}=0$, and that the magnitude of the relative momentum is related to the energy s'' by $p''^2=\frac{1}{4}(s''-4)$. Clarifying equation (7).

Considering the inhomogeneous term and the left-hand side of the Bethe-Salpeter equation (1), we see that they are functions of the initial and final variables, and that, *a priori*, these variables can take any values, obeying only the constraint of total–4-momentum conservation.

Leaving the left-hand side of (1) and the inhomogeneous term unchanged, except for setting some variables to zero, we use (5) to define the approximate homogeneous term.[5] Schematically, we write the reconstructed Bethe-Salpeter equation in the form

$$T=I+(IG^0T)_r. \quad (6)$$

The variables in (6) will be specified explicitly later, at which time we will see that the relation (6) will be either an equation for the half-on-shell or completely on-shell amplitude, or the right-hand side of (6) will give the fully off-shell continuation of the amplitude in terms of the half-on-shell amplitudes.

B. Expressions for Disc$_{s''}$ (IG^0T)

There are several ways of writing an expression for $\mathrm{Disc}_{s''}(IG^0T)$, depending on the choice of variables in the right-hand side of (4); the problem lies in identifying the s'' dependence and distinguishing it from the s dependence. This question is essential in Eq. (5) and will lead to different relations of the form (6).

Let us first consider the external variables, i.e., initial and/or final momenta. Having satisfied total-momentum conservation, we are left with three 4-vectors $\hat{K}$, $\hat{p}$, and $\hat{p}'$. Considering the s dependence, we distinguish the following four cases:

(a) *Completely off-shell:*
$\hat{K}$, $\hat{p}$, $\hat{p}'$ are independent.

[5] If there are any bound states, then we add the appropriate poles to (5). These will take care of the point spectrum which is not included in (5).

(b) *Half-on-shell. Initial particles on-shell:*
$p^0=0,\ |\mathbf{p}|^2=\frac{1}{4}s-1,$
where $s=\hat{K}^2$. We shall denote such $\hat{p}$ as $\hat{p}(s)$; $\hat{K}$ and $\hat{p}'$ are independent.

(c) *Half-on shell. Final particles on-shell:*
$\hat{p}'=\hat{p}'(s)$;
$\hat{K}$ and $\hat{p}$ are independent.

(d) *Fully on-shell:*
$\hat{p}=\hat{p}(s),\ \hat{p}'=\hat{p}'(s).$

This list of (a), (b), (c), and (d) indicates the different s dependences, but does not yet answer our question, because we have yet to distinguish between s and s'' dependences, keeping in mind Eq. (5). Our problem is an off-energy-shell continuation, and three different continuations can be proposed. In one of them, the s dependence in formula (5) will be given only through the denominator $(s''-s)$. This implies that the reconstructed homogeneous term $(IG^0T)_r$ will have only a right-hand cut in s. The other two approaches will allow a left-hand cut in s by preserving some s dependence in the expression for $\mathrm{Disc}_{s''}(IG^0T)$. In these cases it is essential to distinguish between s'' and s dependences.

The three different off-energy-shell continuations will be defined by specifying the *external* variables in the Eq. (4). Let us denote the *whole* right-hand side of (4) by F; then we can indicate the three continuations in the following way:

Continuation (α): $\mathrm{Disc}_{s''}(IG^0T)=F_1(s'',|\mathbf{p}|^2,|\mathbf{p}'|^2,\cos(\mathbf{p},\mathbf{p}'),p^0,p^{0\prime})$, where $|\mathbf{p}|^2$ and/or $|\mathbf{p}'|^2$ are equal to $(s''-4)/4$, depending on the case, (a), (b), (c), or (d). In this method we are working with fixed $\cos(\mathbf{p},\mathbf{p}')$.

Continuation (β): $\mathrm{Disc}_{s''}(IG^0T)=F_2(s'',t_{ie},t_{i0})$, where $t_{ie}=(\hat{p}-\hat{p}'')^2$, $t_{i0}=(\hat{p}'-\hat{p}'')^2$.

Continuation (γ): $\mathrm{Disc}_{s''}(IG^0T)=F_3(s'',t)$, where $t=(\hat{p}-\hat{p}')^2$.

By writing F_1, F_2, F_3, we have pointed out different expressions for F, considered as a function of s''. Each of these approaches has some advantages and disadvantages. These will be presented in the next section, where we shall also describe results of adopting either continuation (α), or (β), or (γ), together with Eqs. (5) and (6).

C. Reduced Relations

Let us start by showing the common features of all relations which will be derived from continuation (α), or (β), or (γ), by using Eqs. (5) and (6). Each of them will contain a 3-dimensional integral with the same kinematical factor as a weighting factor. That factor can be found from the following relation:

$$\int\int ds''dp^{0\prime\prime}f(s'',p^{0\prime\prime},\cdots)\delta_p[(\tfrac{1}{2}\hat{K}+\hat{p}'')^2-1]\times\delta_p[(\tfrac{1}{2}\hat{K}-\hat{p}'')^2-1]=(|\mathbf{p}''|^2+1)^{-\frac{1}{2}}\times f(s''=4(|\mathbf{p}''|^2+1),\ p^{0\prime\prime}=0,\ \cdots),\quad(7)$$

where dots "..." in f mark the place for other variables. All relations will have the following form:

$$T=I+\frac{1}{(2\pi)^3}\int d^3\mathbf{p}''\frac{TT^\dagger}{(|\mathbf{p}''|^2+1)^{1/2}[4(|\mathbf{p}''|^2+1)-s]}.\quad(8)$$

The variables of T and I will be specified later in full detail. Relation (8) is of the form of the Low equation and it will be either a Low-type equation[6] for the half-on-shell or the fully on-shell amplitude, or it will be an expression for the completely off-shell amplitude in terms of the half-on-shell amplitudes.

Now, we discuss the variables of T and I, according to the continuations (α), (β), or (γ) and show the different aspects of each of them separately.

(α) To present clearly the result of applying the first method, we use the following notation:

$T\equiv T(s,\hat{p},\hat{p}')$ —completely off-shell;
$\cdot\bar{T}\equiv T(s,\hat{p}(s),\hat{p}')$ —half-on-shell, initial on-shell;
$\bar{T}\cdot\equiv T(s,\hat{p},\hat{p}'(s))$ —half-on-shell, final on-shell;
$\bar{T}\equiv T(s,\hat{p}(s);\hat{p}'(s))$—completely on-shell,

where we have used the center-of-mass (c.m.) system, i.e., $\mathbf{K}=0$, and $\hat{p}(s)$ means $p^0=0$, $|\mathbf{p}|^2=\frac{1}{4}s-1$. In that notation we get the following relations of the form (8) [We shall write them only schematically because all kinematical factors are the same as in (8).]:

$$T=I+\bar{T}\cdot\ \cdot\bar{T}^\dagger,\quad(9a)$$

$$\cdot\bar{T}=\cdot\bar{I}+\bar{T}^\dagger\ \cdot\bar{T},\quad(9b)$$

$$\bar{T}\cdot=\bar{I}\cdot+\bar{T}\cdot\ \bar{T}^\dagger,\quad(9c)$$

$$\bar{T}=\bar{I}+\bar{T}\ \bar{T}^\dagger.\quad(9d)$$

The nice properties of relations (9a), (9b), (9c), and (9d) are:

(i) Only one equation (9d) is quadratic and it describes the completely on-shell amplitude. Two equations (9b) and (9c) are linear integral equations for $\cdot\bar{T}$ and $\bar{T}\cdot$, correspondingly, with the kernel *given* by the completely on-shell amplitude. The relation (9a) is a formula for finding the fully off-shell amplitude in terms of the solutions of (9b) and (9c).

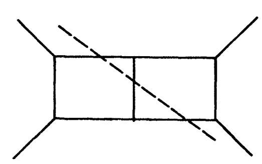

FIG. 3. The 6th-order graph in the ladder approximation and the 3-particle cut.

[6] We use the expressions: "of the form of the Low equation", or "Low-type equation" to distinguish from the original Low equation [F. E. Low, Phys. Rev. **97**, 1392 (1953)]. It should be noticed that the quantity $\langle p',q,j|0_i(x)|p\rangle$, for which Low has written an integral equation, is not the conventional T matrix (clarifying Low's remark on p. 1395). The amplitude T which we are dealing with originates from the conventional T matrix which satisfies Eq. (1).

(ii) In the case of the ladder approximation the completely on-shell amplitude, corresponding to the box graph, is given exactly in the forward direction for all energies.

(iii) The partial-wave projection can be made in a standard way.

The bad properties of (9a), (9b), (9c), and (9d) are:

(i) The appropriate on-shell limits, taken in the homogeneous terms, do not give the homogeneous terms of the lower equations. This is due to the fact that only the expression for $\mathrm{Disc}_s(IG^0T)$ is a continuous function of s, for different on-shell cases. However, $(IG^0T)_r$ does not have this property, because by taking the on-shell limits we get some left-hand cuts which are completely ignored in the first approach.

(ii) The box graph is given incorrectly for directions other than the forward one.

Both of these bad properties are caused by the lack of the left-hand cut, and they indicate that it is not a good enough approximation to take only the 2-particle right-hand cut plus the inhomogeneous term. In some particular cases one could correct the inhomogeneous term by adding an appropriate function with the left-hand cut, but this has to be done by hand and it will vary from case to case.

(β) In the second method we can remove the first bad property by restricting the class of off-shell amplitudes to such, that the dependence on $|\mathbf{p}|$ and $|\mathbf{p}'|$ will be only through t variables, which for $p^0=p^{0\prime}=p^{0\prime\prime}=0$ are given by

$$\begin{aligned}t&=(\hat{p}-\hat{p}')^2=-|\mathbf{p}|^2-|\mathbf{p}'|^2+2|\mathbf{p}||\mathbf{p}'|\cos(\mathbf{p},\mathbf{p}')\,,\\ t_{ie}&=(\hat{p}-\hat{p}'')^2=-|\mathbf{p}|^2-|\mathbf{p}''|^2\\&\qquad+2|\mathbf{p}||\mathbf{p}''|\cos(\mathbf{p},\mathbf{p}'')\,,\\ t_{i0}&=(\hat{p}'-\hat{p}'')^2=-|\mathbf{p}'|^2-|\mathbf{p}''|^2\\&\qquad+2|\mathbf{p}'||\mathbf{p}''|\cos(\mathbf{p}',\mathbf{p}'')\,.\end{aligned}\tag{10}$$

We have used $p^0=p^{0\prime}=p^{0\prime\prime}=0$ because it simplifies (10) and does not produce any new s, s'' dependence through the variables p^0, $p^{0\prime}$, $p^{0\prime\prime}$. This is the case for equal masses and it can be generalized for arbitrary masses by working with the Wightman-Gårding relative momentum. This is discussed in Sec. 5.

For amplitudes T which have the following dependence on variables

$$\begin{aligned}&T(s,|\mathbf{p}|,|\mathbf{p}'|,\cos(\mathbf{p},\mathbf{p}'),p^0,p^{0\prime}))\\&\quad=T(s,t=-|\mathbf{p}|^2-|\mathbf{p}'|^2+2|\mathbf{p}||\mathbf{p}'|\cos(\mathbf{p},\mathbf{p}'),p^0,p^{0\prime})\,,\end{aligned}$$

we get only *one* relation of the form

$$T=I+\bar{T}\cdot\cdot\bar{T}^\dagger.\tag{11}$$

Relation (11) describes all four cases (a), (b), (c), and (d), which can be obtained by a different on-shell limits taken in (11). The reason for getting only one relation is that in the second approach we have hidden some s dependence in the variables t_{ie}, t_{i0}, and we have used exactly that place for making the off-energy-shell continuation.

The nice properties of (11) are:

(i) There is only one relation which describes all cases corresponding to the different on-shell, off-shell limits.

(ii) There are included some left-hand cuts through the t variables.

(iii) The partial-wave projection can be made in a standard way.

The bad properties of (11) are:

(i) Beside the correct left-hand cuts, there are generated some additional left-hand cuts due to the pinching singularities. See Sec. 4.

(ii) The expression for the box graph, even at the forward direction, at threshold [i.e., for $\cos(\mathbf{p},\mathbf{p}')=1$ and $(s=4)$] differs from the exact answer by about 10%. Both of these difficulties are caused by some additional left-hand cuts due to the pinch between the singularities introduced by t_{ie} or t_{i0} and the fixed singularity coming from the weighting factor $(|\mathbf{p}''|^2+1)^{-1/2}$. For more details see Sec. 4.

(γ) In the third approach we try to avoid both difficulties of the second method. Thus, we have to remove the above-mentioned pinch of singularities. This can be done by making the off-energy-shell continuation only through the t variable, and not separately through t_{ie} and t_{i0}. We have no problem in the left-hand side of (8) or in the inhomogeneous term. The problem arises in the homogeneous term. According to the third approach, we have to write $\mathrm{Disc}_{s''}(IG^0T)$ as a function of s'' and t. To do it let us start from the following expression:

$$\begin{aligned}&\mathrm{Disc}_{s''}\left(-\frac{i}{(2\pi)^4}\int d^4\hat{p}''IG^0T\right)=\frac{i}{2(4\pi)^2}\frac{p''}{(p''^2+1)^{1/2}}\\&\quad\times\int\int d\cos\theta''d\varphi''T(s'',\cos(\mathbf{p},\mathbf{p}'))\\&\qquad\times T^\dagger(s'',\cos(\mathbf{p}'',\mathbf{p}'))\equiv F(s'',\cos(\mathbf{p},\mathbf{p}')).\end{aligned}\tag{12}$$

In this expression we have to make a rearrangement of variables to get $F(s'',t)$ from $F(s'',\cos(\mathbf{p},\mathbf{p}'))$, which can be achieved by the following substitution:

$$\begin{aligned}\cos(\mathbf{p},\mathbf{p}')\to1+t/2p^2(s'')=1+[&-|\mathbf{p}|^2-|\mathbf{p}'|^2\\&+2|\mathbf{p}||\mathbf{p}'|\cos(\mathbf{p},\mathbf{p}')]/2p^2(s'')\,,\end{aligned}\tag{13}$$

where $p^2(s'')\equiv(s''-4)/4$, and we have taken $p^0=p^{0\prime}=0$.

Finally, we get the relation (8) in the following form:

$$T(s,t=-|\mathbf{p}|^2-|\mathbf{p}'|^2+2|\mathbf{p}|\,|\mathbf{p}'|\cos(\mathbf{p},\mathbf{p}'),p^0,p^{0\prime})=I(s,t=-|\mathbf{p}|^2-|\mathbf{p}'|^2+2|\mathbf{p}|\,|\mathbf{p}'|\cos(\mathbf{p},\mathbf{p}'),p^0,p^{0\prime})$$
$$+\frac{1}{(2\pi)^3}\int dp''\frac{p''^2}{(p''^2+1)^{1/2}[4(p''^2+1)-s]}\biggl[\int\int d\cos\theta''d\varphi''T(s'',t_{ie}=\tfrac{1}{2}(s''-4)[\cos(\mathbf{p},\mathbf{p}'')-1],\ p^0,\ p^{0\prime\prime}=0)$$
$$\times T^{\dagger}(s'',\ t_{io}=\tfrac{1}{2}(s''-4)[\cos(\mathbf{p}'',\mathbf{p}')-1],\ p^{0\prime\prime}=0,\ p^{0\prime})\biggr]\biggr|_{\cos(\mathbf{p},\mathbf{p}')\to 1+[-|\mathbf{p}|^2-|\mathbf{p}'|^2+2|\mathbf{p}||\mathbf{p}'|\cos(\mathbf{p},\mathbf{p}')]/2p''^2}. \quad (14)$$

The function T depends on six variables, s, $|\mathbf{p}|$, $|\mathbf{p}'|$, $\cos(\mathbf{p},\mathbf{p}')$, p^0, $p^{0\prime}$, but the dependence on $|\mathbf{p}|$, $|\mathbf{p}'|$, and $\cos(\mathbf{p},\mathbf{p}')$ is allowed only through the variable t. The dependence on p^0 and $p^{0\prime}$ does not cause any problem in considering the s dependence, because the on-shell limit of p^0 and $p^{0\prime}$ is equal to zero. In the covariant description, the variables p^0, $p^{0\prime}$ are replaced by $\hat{q}\cdot\hat{K}$, $\hat{q}'\cdot\hat{K}$, where $\hat{q}$, $\hat{q}'$ are the Wightman-Gårding relative momenta.

The relation (14) is an integral equation, and in Sec. 7 we shall discuss some methods of solving it. Now let us state its properties.

The nice properties of (14) are:

(i) There is only one relation which describes all on-shell, off-shell limits.

(ii) There are included only the correct left-hand cuts through the t variable.

(iii) The box graph calculated from (14) is given *exactly* in any direction and arbitrary energy.

The bad properties of (14) are:

(i) The relation (14) has a rather complicated form before the $\cos\theta''$, φ'' integration is made explicitly.

(ii) The partial-wave projection has to be considered with special care.

The above difficulties are induced by the special procedure of properly including the left-hand cut, together with the 2-particle right-hand cut. These difficulties can be lessened by presenting some methods of solving (14), which we shall do in Sec. 7.

Comparing the three approaches, we see that the first, which included only the right-hand cut, suffered from the most serious diseases. The second and third approaches are quite similar, and both include the left-hand cut. The argument of simplicity is certainly in favor of the second approach. However, in the third approach the left-hand cuts are treated more correctly than in the second. This is important if we are to use the 2-pody T matrix as an input in a 3-body problem. In the Faddeev equation we need the 2-body T matrix in the range of energy squared from a positive value to minus infinity. See a discussion of this point in Ref. 3.

4. LADDER GRAPHS

We discuss ladder graphs to show explicitly several details of the three different approaches presented in the previous section. For simplicity, we shall retain the assumption of equal masses ($m_1=m_2=1$) and carry out calculations in the c.m. system. We denote by μ the mass of the exchanged particle and write the Yukawa potential in the form

$$I(\hat{p},\hat{p}')=\frac{g^2}{(\hat{p}-\hat{p}')^2-\mu^2}. \quad (15)$$

It is obvious that in all three approaches the inhomogeneous term is the same. We can write (15) in two other forms:

$$I=\frac{g^2}{t-\mu^2}=g^2[(p^0-p^{0\prime})^2-|\mathbf{p}|^2-|\mathbf{p}'|^2+2|\mathbf{p}|\,|\mathbf{p}'|\cos(\mathbf{p},\mathbf{p}')-\mu^2]^{-1}. \quad (16)$$

Let us now consider the following problem. Having $I(s,\cos(\mathbf{p},\mathbf{p}'))$, i.e., on-shell,

$$I(s,\cos(\mathbf{p},\mathbf{p}'))=\frac{g^2}{\frac{1}{2}(s-4)[\cos(\mathbf{p},\mathbf{p}')-1]-\mu^2}, \quad (17)$$

how do we continue it off the energy shell? To continue $I(s,\cos(\mathbf{p},\mathbf{p}'))$ off the energy shell, i.e., when the magnitudes of the relative momenta $|\mathbf{p}|$ and $|\mathbf{p}'|$ are not determined by s, we use the following replacement:

$$\cos(\mathbf{p},\mathbf{p}')\to 1+[-|\mathbf{p}|^2-|\mathbf{p}'|^2+2|\mathbf{p}|\,|\mathbf{p}'|\cos(\mathbf{p},\mathbf{p}')]/2p^2(s), \quad (18)$$

where

$$p^2(s)\equiv\tfrac{1}{4}(s-4).$$

Substituting (18) in (17), we get for the off-energy-shell Yukawa potential

$$I(s,|\mathbf{p}|,|\mathbf{p}'|,\cos(\mathbf{p},\mathbf{p}'))=g^2[-|\mathbf{p}|^2-|\mathbf{p}'|^2+2|\mathbf{p}|\,|\mathbf{p}'|\cos(\mathbf{p},\mathbf{p}')-\mu^2]^{-1}. \quad (19)$$

That result coincides with (16) or (15) if we consider $p^0=p^{0\prime}=0$. The last condition for unequal masses reads $q^0=q^{0\prime}=0$, where $\hat{q}$ is the Wightman-Gårding relative momentum. From (19), (16), and (18) we see that the prescription used in the third approach gives us the *correct* off-shell continuation of the Yukawa potential.

The next problem to consider is the box graph. Let us start by looking at the $\mathrm{Disc}_{s''}$(Box). According to the three approaches of Sec. 3 B, they are:

$$(1)\quad \mathrm{Disc}_{s''}(\mathrm{Box})=\frac{ig^4p''}{2(4\pi)^2(p''^2+1)^{1/2}}\int\int d\cos\theta'' d\varphi'' \times[-|\mathbf{p}|^2-|\mathbf{p}''|^2+2|\mathbf{p}||\mathbf{p}''|\cos(\mathbf{p},\mathbf{p}'')-\mu^2]^{-1}[-|\mathbf{p}''|^2-|\mathbf{p}'|^2+2|\mathbf{p}''||\mathbf{p}'|\cos(\mathbf{p}'',\mathbf{p}')-\mu^2]^{-1},$$

where $|\mathbf{p}|^2$, $|\mathbf{p}'|^2$ are equal to $\frac{1}{4}(s''-4)$, depending on the case [(a), (b), (c), or (d)].

$$(2)\quad \mathrm{Disc}_{s''}(\mathrm{Box})=\frac{ig^4p''}{2(4\pi)^2(p''^2+1)^{1/2}}\int\int d\cos\theta'' d\varphi''(t_{ie}-\mu^2)^{-1}(t_{io}-\mu^2)^{-1},$$

where

$$t_{ie}=(\hat{p}-\hat{p}'')^2,\quad t_{io}=(\hat{p}''-\hat{p}')^2.$$

$$(3)\quad \mathrm{Disc}_{s''}(\mathrm{Box})=\frac{ig^4p''}{2(4\pi)^2(p''^2+1)^{1/2}}\left[\int\int d\cos\theta'' d\varphi''\{\tfrac{1}{2}(s''-4)[\cos(\mathbf{p},\mathbf{p}'')-1]-\mu^2\}^{-1} \times\{\tfrac{1}{2}(s''-4)[\cos(\mathbf{p}'',\mathbf{p}')-1]-\mu^2\}^{-1}\right]\Bigg|_{\cos(\mathbf{p},\mathbf{p}')\to 1+[-|\mathbf{p}|^2-|\mathbf{p}'|^2+2|\mathbf{p}||\mathbf{p}'|\cos(\mathbf{p},\mathbf{p}')]/[\frac{1}{2}(s''-4)]}.$$

To get the expression for the Box, we take a dispersion integral, i.e., use our formula (5). We shall consider the external momenta (initial and final) on shell, and fix the z axis along $\mathbf{p}$, to get simple formulas and to show explicitly the differences between them. If we denote

$$\cos(\mathbf{p}'',\mathbf{p}')\equiv\cos\theta''\cos\theta'+\sin\theta''\sin\theta'\cos(\varphi''-\varphi'),\tag{20}$$

then we get the following results:

$$(1)\quad \mathrm{Box}(s,\cos\theta')=\frac{g^4}{4(2\pi)^3}\int dp''p''^2(p''^2+1)^{-1/2}(1-\tfrac{1}{4}s+p''^2)^{-1} \times\int\int d\cos\theta'' d\varphi''[2p''^2(\cos\theta''-1)-\mu^2]^{-1}\{2p''^2[\cos(\mathbf{p}'',\mathbf{p}')-1]-\mu^2\}^{-1},\tag{21}$$

$$(2)\quad \mathrm{Box}(s,\cos\theta')=\frac{g^4}{4(2\pi)^3}\int dp''p''^2(p''^2+1)^{-1/2}(1-\tfrac{1}{4}s+p''^2)^{-1}\int\int d\cos\theta'' d\varphi'' \times[1-\tfrac{1}{4}s-p''^2+(s-4)^{1/2}p''\cos\theta''-\mu^2]^{-1}[1-\tfrac{1}{4}s-p''^2+(s-4)^{1/2}p''\cos(\mathbf{p}'',\mathbf{p}')-\mu^2]^{-1},\tag{22}$$

$$(3)\quad \mathrm{Box}(s,\cos\theta')=\frac{g^4}{4(2\pi)^3}\int dp''p''^2(p''^2+1)^{-1/2}(1-\tfrac{1}{4}s+p''^2)^{-1} \times\left[\int\int d\cos\theta'' d\varphi''[2p''^2(\cos\theta''-1)-\mu^2]^{-1}\{2p''^2[\cos(\mathbf{p}'',\mathbf{p}')-1]-\mu^2\}^{-1}\right]\Bigg|_{\cos\theta'\to 1+[(s-4)/4p''^2](\cos\theta'-1)}.\tag{23}$$

The first expression (21) has only a right-hand cut in s, and if we calculate the amplitude from it, we get the correct answer only in the forward direction. One can see from (23) that in the forward direction, i.e., $\cos\theta'=1$, the two expressions [(21) and (23)] coincide.

In (22) we get both right- and left-hand cuts in s; however, the structure of the left-hand singularities is more complicated than in the correct expression for the box graph. That is, besides the two-particle exchange cut starting at $s=4-4\mu^2$, we get an additional, incorrect cut starting at $s=4-4(1+\mu)^2$. The last cut is due to the pinch of the fixed singularity at $p''^2=-1$ and the moving singularity at $(\hat{p}''-\hat{p})^2=\mu^2$. The presence of the additional left-hand cut in (22) causes about a 10% error in the amplitude calculated from the box graph in the forward direction and at threshold, i.e., for $\cos\theta'=1$, $s=4$.

The third expression for $\mathrm{Box}(s,\cos\theta')$, given by (23) provides the *exact* answer for the box graph as a function of s and $\cos\theta'$. The reason for this is that the box graph considered as a function of s and t has, for any fixed t, only the 2-particle right-hand discontinuity in s. This analytic property of the $\mathrm{Box}(s,t)$ is explicitly presented in the Mandelstam[7] paper. Our prescription of replacing $\cos\theta'$ by the expression on the right-hand side of the arrow just means that we rewrite a function of s'' and $\cos\theta'$ as a function of s'' and t. In the t variable, given on-shell by

$$t=\tfrac{1}{2}(s-4)(\cos\theta'-1),\tag{24}$$

there is hidden the s dependence which has to be distinguished from the s'' dependence. The s dependence

[7] S. Mandelstam, Phys. Rev. 115, 1741 (1959).

coming through the t variable gives the correct left-hand cut in s of the Box, considered as a function of s and $\cos\theta'$.

5. COVARIANT FORMULATION, ARBITRARY MASSES

This section has two aims: firstly, we would like to set up a covariant framework of invariant variables; secondly, we would like to rederive some of the results from the previous sections in a manifestly covariant way and for arbitrary masses.

A. Invariants and Accessible Region

We are dealing with a relativistic problem, so it will be appropriate to have our relations written explicitly in a covariant form. To do this we introduce several invariants which we shall use as variables in our formulas.

Let us first define several 4-momenta which we shall need to construct invariants. We take a graph corresponding to the homogeneous term in the Bethe-Salpeter equation and denote in it the 4-momenta $\hat{k}_1, \cdots \hat{k}_6$ (see Fig. 4).

In terms of $\hat{k}_1, \cdots, \hat{k}_6$, there are defined the usual total and relative momenta

$$\hat{K}=\hat{k}_1+\hat{k}_2=\hat{k}_3+\hat{k}_4=\hat{k}_5+\hat{k}_6, \tag{25}$$

$$\hat{p}=\frac{m_2\hat{k}_1-m_1\hat{k}_2}{m_1+m_2}, \quad \hat{p}'=\frac{m_2\hat{k}_3-m_1\hat{k}_4}{m_1+m_2}, \quad \hat{p}''=\frac{m_2\hat{k}_5-m_1\hat{k}_6}{m_1+m_2}.$$

Besides these momenta we shall also need the Wightman-Gårding relative momentum $\hat{q}$, defined by[8]

$$\hat{q}=\tfrac{1}{2}(\hat{k}_1-\hat{k}_2)-\frac{m_1^2-m_2^2}{2\hat{K}^2}\hat{K}; \tag{26}$$

similarly, for $\hat{q}'$ and $\hat{q}''$.

Using these momenta, we introduce several sets of invariants. Invariants, corresponding to the intermediate particles, are grouped in the following sets:

$$s=\hat{K}^2, \quad s_5=\hat{k}_5^2, \quad s_6=\hat{k}_6^2, \quad s_{15}=(\hat{k}_1-\hat{k}_5)^2, \quad s_{35}=(\hat{k}_3-\hat{k}_5)^2, \tag{27a}$$

$$s=\hat{K}^2, \quad \hat{p}''\cdot\hat{K}, \quad \hat{p}''^2, \quad \hat{p}''\cdot\hat{p}, \quad \hat{p}''\cdot\hat{p}', \tag{27b}$$

$$s=\hat{K}^2, \quad \hat{q}''\cdot\hat{K}, \quad \hat{q}''^2, \quad \hat{q}''\cdot\hat{q}, \quad \hat{q}''\cdot\hat{q}', \tag{27c}$$

where $\hat{p}''\cdot\hat{K}$, for example, denotes the scalar product of two 4-vectors $\hat{p}''$ and $\hat{K}$. For initial and/or final particles we can easily define sets of invariants similar to those given by (27), except for one of the last two invariants in each set, which are connected with angles. The reason for this is that we need an additional 4-vector which is not the one corresponding to the intermediate particles. The lacking 4-vector can be either one of the previously introduced 4-vectors, (in which case we get two of the invariants being identical, a condition that corresponds to choosing a special axis to define the angle), or, if we use the 2-particle theory as an input in a 3-, or many-particle system, we can introduce a completely new 4-vector, (for example, the relative momentum of 2-particle subsystem with respect to another particle).

We have introduced three sets of invariants (27a), (27b), and (27c), because the first two sets are commonly used, while the third one, with the Wightman-Gårding relative momentum, turns out to be the most appropriate one both for making the reduction from 4- to 3-dimensional space and for performing partial-wave decomposition in a manifestly covariant way. The advantage of using the set (27c) over the others in making

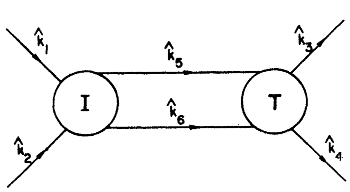

FIG. 4. Diagram representing the homogeneous term in the Bethe-Salpeter equation.

the reduction will be explained in this section, while the discussion of partial-wave decomposition will be presented in the next section.

The invariants (27a) can be expressed in terms of (27b) or (27c) and, for further discussion, we need the following relations:

$$\begin{aligned} s_5&=\left(\frac{m_1}{m_1+m_2}\right)^2 s+\frac{2m_1}{m_1+m_2}\hat{p}''\cdot\hat{K}+\hat{p}''^2, \\ s_6&=\left(\frac{m_1}{m_1+m_2}\right)^2 s-\frac{2m_2}{m_1+m_2}\hat{p}''\cdot\hat{K}+\hat{p}''^2, \end{aligned} \tag{28}$$

$$\begin{aligned} s_5&=\frac{1}{4s}(s+m_1^2-m_2^2)^2+\frac{s+m_1^2-m_2^2}{s}\hat{q}''\cdot\hat{K}+\hat{q}''^2, \\ s_6&=\frac{1}{4s}(s-m_1^2+m_2^2)^2-\frac{s-m_1^2+m_2^2}{s}\hat{q}''\cdot\hat{K}+\hat{q}''^2. \end{aligned} \tag{29}$$

Having defined 3 sets of invariants which will be used as variables in our formulas, we have to find out the appropriate domain of these variables, hereafter called the accessible region in the space of invariants. The accessible region is the image of the 4-dimensional space of a *real* four-vector $\hat{k}_5$. For the first set of invariants (27a), an extensive discussion of the accessible region

[8] A. S. Wightman, "Lectures on Invariance in Relativistic Quantum Mechanics" (Les Houches, 1960), in the book by C. De Witt and R. Omnes, *Dispersion Relations and Elementary Particles* (Hermann & Cie., Paris, 1960). A reduction procedure using Wightman-Gårding relative momentum was proposed in the summer of 1965 by R. Stora (private communication).

can be found in Taylor's[9] paper and we shall use several of his results. Let us illustrate the accessible region in a very simple case by considering two subspaces, namely, $(|\mathbf{k}_5|,k_5^0)$ and (s_5,s_6). Using the definitions of s_5, s_6, it is easy to prove that the right-hand half-plane in variables $(|\mathbf{k}_5|,k_5^0)$ is mapped into an accessible region in the (s_5,s_6) plane, bounded by a parabola

$$(s_5-s_6)^2=s[2(s_5+s_6)-s]. \tag{30a}$$

This is illustrated in Fig. 5. In the subspaces $(\hat{p}''\cdot\hat{K},\hat{p}''^2)$, and $(\hat{q}''\cdot\hat{K},\hat{q}''^2)$, corresponding to sets (27b) and (27c), the accessible region is also bounded by parabolas, given by equations

$$s\hat{p}''^2=(\hat{p}''\cdot\hat{K})^2, \tag{30b}$$

$$s\hat{q}''^2=(\hat{q}''\cdot\hat{K})^2, \tag{30c}$$

and shown in Fig. 6. Comparing Fig. 6 with Fig. 5, we see that the sets of invariants (27b) or (27c) would be more appropriate candidates for our variables than (27a), because for these sets the boundary of the accessible region is given by a more simple and symmetrical expression. Two other properties of the set (27c) will be shown which make it the most appropriate system of invariant variables. These properties are connected with reduction (see the Subsec. 5 B) and with partial-wave analysis (see Sec. 6).

To complete the discussion of accessible region in the whole 4-dimensional space of invariants we have to include the remaining two invariants which are connected with angles. We present this problem in the Appendix. There we repeat some of the results of Taylor[9] for sets (27a) and (27c). It turns out that conditions giving the boundary of the accessible region in the 4-dimensional space corresponding to (27c) look much simpler than those for the set (27a); in the Appendix compare Eq. (A3) with (A6).

B. Mass-Shell Conditions and Reduction Hyperplane

The mass-shell conditions corresponding to sets (27a), (27b), and (27c) have the following forms:

$$s_5=m_1^2,\quad s_6=m_2^2, \tag{31a}$$

$$\hat{p}''\cdot\hat{K}=\frac{m_1^2-m_2^2}{2}\left[1-\frac{s}{(m_1+m_2)^2}\right],\qquad \hat{p}''^2=m_1m_2\left[1-\frac{s}{(m_1+m_2)^2}\right] \tag{31b}$$

$$\hat{q}''\cdot\hat{K}=0,\qquad \hat{q}''^2=\frac{m_1^2+m_2^2}{2}-\frac{s}{4}-\frac{(m_1^2-m_2^2)^2}{4s}. \tag{31c}$$

Formally, the first set (31a) looks simplest, but it will not be suitable for the reduction procedure which explores the s dependence, which is hidden in the (s_5,s_6) variables. The other two sets [31b) and (31c)] give us explicit expressions in terms of s.

The reduction procedure can be looked at as an approximation of the integrals over a 4-dimensional manifold by an integral over a 3-dimensional subspace. To explain some details we shall simplify the problem of reduction by discussing only a 2-dimensional subspace of the whole 4-dimensional space of invariants. In that simplified picture, the reduction procedure means an approximation of the integrals over a 2-dimensional manifold by an integral along a line, called hereafter the reduction line. Generally, i.e., in the whole 4-dimensional space, the "reduction line" is actually a 3-dimensional manifold, which we will call the reduction hyperplane.

In our simplified picture of 2-dimensional subpsace we get from (31b) the following equation for the reduction line in the subspace of $(\hat{p}''\cdot\hat{K},\hat{p}''^2)$ variables:

$$\hat{p}''\cdot\hat{K}=\frac{m_1^2-m_2^2}{2m_1m_2}\hat{p}''^2. \tag{32}$$

This is a straight line in these variables passing through the origin. We show a few examples of this reduction line in Fig. 7(a). The dashed line depicts that part of the reduction line which passes through the forbidden region.

The reduction line as well as the reduction hyperplane look simplest in the variables of the set (27c). We have a very nice and simple equation, namely,

$$\hat{q}''\cdot\hat{K}=0. \tag{33}$$

In the picture of 2-dimensional subspace of $(\hat{q}''\cdot\hat{K},\hat{q}''^2)$ variables, the reduction line is simply the negative half the $\hat{q}''^2$ axis[10] [see Fig. 7(b)] and in the whole 4-dimensional space, the reduction hyperplane is a 3-dimensional manifold, given by the intersection of the hyperplane (33) with the accessible region defined in the Appendix.

It is worthwhile to notice that the reduction line in the subspace $(\hat{q}''\cdot\hat{K},\hat{q}''^2)$ always stays in the region where $\hat{q}''^2<0$, while in the subspace of $(\hat{p}''\cdot\hat{K},\hat{p}''^2)$ we get

[9] J. G. Taylor, Nuovo Cimento, Suppl. 1, 988 (1963).

[10] On the basis of the following relation among invariants:

$$\hat{p}''\cdot\hat{K}=\frac{m_1^2-m_2^2}{2}\left[1-\frac{s}{(m_1+m_2)^2}\right]+\hat{q}''\cdot\hat{K},$$

the image of the reduction line $\hat{q}''\cdot\hat{K}=0$, in the plane $(\hat{p}''\cdot\hat{K},\hat{p}''^2)$, is a straight line parallel to the $\hat{p}''^2$ axis. This line is given by

$$\hat{p}''\cdot\hat{K}=\frac{m_1^2-m_2^2}{2}\left[1-\frac{s}{(m_1+m_2)^2}\right].$$

Thus, the reduction condition given by (33) differs from that given by (32). A comparison between these two covariant reduction conditions will be discussed separately in connection with the Ida-Maki-Wanders representation of the Bethe-Salpeter amplitudes. [See G. Wanders, Phys. Rev. 104, 1782 (1956); M. Ida and K. Maki, Progr Theoret. Phys. (Kyoto) 26, 470 (1961)].

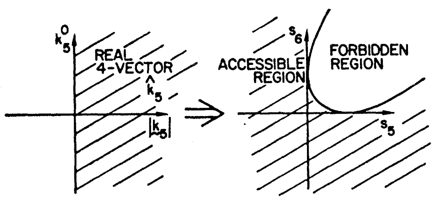

FIG. 5. Accessible regions in the planes $(|\mathbf{k}_5|,k_5^0)$ and (s_5,s_6).

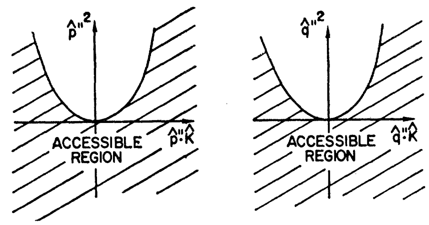

FIG. 6. Accessible regions in the planes $(p''\cdot\hat{K},p''^2)$ and $(\hat{q}''\cdot\hat{K},\hat{q}''^2)$.

a piece of the reduction line in the region $\hat{p}''^2>0$, except in the case of equal masses, when the momenta $\hat{p}$ and $\hat{q}$ coincide and the reduction line is the negative half of the $\hat{p}''^2$ axis. We shall use this fact in the section dealing with partial-wave decomposition.

Considering the initial and/or final particles, one can also draw the boundary of the accessible region in the $(\hat{q}\cdot\hat{K},\hat{q}^2)$ and/or $(\hat{q}'\cdot\hat{K},\hat{q}'^2)$ planes. On such figures the reduction line corresponds to the set of physical points, i.e., particles being put on-mass-shells. Depending on the energy s, the physical points will be staying at different points, but always at the negative half of $\hat{q}^2$ and/or $\hat{q}'^2$ axes.

If one uses the usual relative momentum $\hat{p}$, then, by looking at Fig. 7(a) and at the mass-shell conditions (31b), one sees that the physical points $s>(m_1+m_2)^2$ are only on those parts of reduction lines which are in the lower half-plane, so that $\hat{p}^2$ and/or $\hat{p}'^2$ are less than zero.

C. Covariant Bethe-Salpeter Equation and On-Shell Equation

In this section we shall sketch, as an example, a few steps for obtaining a manifestly covariant formulation of the previously derived relations (7) and (9d). Here, we shall only emphasize those points which are new in the covariant formulation and make our considerations very schematic, because the main ideas are exactly the same as those presented in Secs. 2 and 3.

We have to start from the full Bethe-Salpeter equation, i.e., before reduction, written in terms of the variables corresponding to the set (27c). The Bethe-Salpeter equation takes the form

$$T(\hat{K}^2,\hat{q}^2,\hat{q}'^2,\hat{q}\cdot\hat{q}',\hat{q}\cdot\hat{K},\hat{q}'\cdot\hat{K})=I(\hat{K}^2,\hat{q}^2,\hat{q}'^2,\hat{q}\cdot\hat{q}',\hat{q}\cdot\hat{K},\hat{q}'\cdot\hat{K})-\frac{i}{(2\pi)^4}\int_{-\infty}^{\infty}d\hat{q}''\cdot\hat{K}\int_{-\infty}^{(\hat{q}''\cdot\hat{K})^2/s}d\hat{q}''^2$$
$$\times\int_{-\infty}^{f_1(\hat{q}''\cdot\hat{K},\hat{q}\cdot\hat{K},\hat{q}^2,\hat{K}^2)}d\hat{q}''\cdot\hat{q}\int_{-\infty}^{f_2(\hat{q}''\cdot\hat{K},\hat{q}'\cdot\hat{K},\hat{q}'^2,\hat{K}^2)}d\hat{q}''\cdot\hat{q}'\frac{\partial(q_x'',q_y'',q_z'',q''^0)}{\partial(\hat{q}''\cdot\hat{K},\hat{q}''^2,\hat{q}''\cdot\hat{q},\hat{q}''\cdot\hat{q}')}$$
$$\times\frac{I(\hat{K}^2,\hat{q}^2,\hat{q}''^2,\hat{q}\cdot\hat{q}'',\hat{q}\cdot\hat{K},\hat{q}''\cdot\hat{K})T(\hat{K}^2,\hat{q}''^2,\hat{q}'^2,\hat{q}''\cdot\hat{q}',\hat{q}''\cdot\hat{K},\hat{q}'\cdot\hat{K})}{[s_5(\hat{q}''\cdot\hat{K},\hat{q}''^2)-m_1^2][s_6(\hat{q}''\cdot\hat{K},\hat{q}''^2)-m_2^2]},\quad(34)$$

where the upper limits $f_1(\hat{q}''\cdot\hat{K},\hat{q}\cdot\hat{K},\hat{q}^2,\hat{K}^2)$ and $f_2(\hat{q}''\cdot\hat{K},\hat{q}'\cdot\hat{K},\hat{q}'^2)$, as well as $(\hat{q}''\cdot\hat{K})^2/s$, take care of the accessible region; they are explicitly defined in the Appendix. In the Appendix there is also an expression for the appropriate Jacobian. The propagators, marked in (34) as $(s_5-m_1^2)(s_6-m_2^2)$, can be written explicitly in terms of the appropriate variables by using the right-hand sides of formula (29). The Bethe-Salpeter equation, written in the form (34) (with all necessary remarks being postponed to the Appendix) looks rather lengthy and it would be difficult to present clearly the main ideas of covariant reduction while retaining all details. Therefore we shall show our scheme of covariant reduction in a slightly simplified way, by dealing in a manifestly covariant way only with the energy and radial variables and considering, instead of the variables $\hat{q}''\cdot\hat{q}$ and $\hat{q}''\cdot\hat{q}'$, the angles θ'', φ''. As it concerns the method of reduction, the use of angles is solely a notational simplification, while in partial-wave analysis it deserves more attention; we shall come back to this point in the next section.

In the simplified, semicovariant variables, the full Bethe-Salpeter equation takes the form

$$T(s,\hat{q}^2,\hat{q}'^2,\Omega,\Omega',\hat{q}\cdot\hat{K},\hat{q}\cdot'\hat{K})=I(s,\hat{q}^2,\hat{q}'^2,\Omega,\Omega',\hat{q}\cdot\hat{K},\hat{q}'\cdot\hat{K})-\frac{i}{2(2\pi)^4\sqrt{s}}\int_{-\infty}^{\infty}d\hat{q}''\cdot\hat{K}\int_{-\infty}^{(\hat{q}''\cdot\hat{K})^2/s}d\hat{q}''^2$$
$$\times\int_{-1}^{1}d\cos\theta''\int_0^{2\pi}d\varphi''\frac{I(s,\hat{q}^2,\hat{q}''^2,\Omega,\Omega'',\hat{q}\cdot\hat{K},\hat{q}\cdot\hat{K}'')T(s,\hat{q}''^2,\hat{q}'^2,\Omega'',\Omega',\hat{q}''\cdot\hat{K},\hat{q}'\cdot\hat{K})}{[s_5(\hat{q}''\cdot\hat{K},\hat{q}''^2)-m_1^2][s_6(\hat{q}''\cdot\hat{K},\hat{q}''^2)-m_2^2]},\quad(35)$$

where Ω, Ω', Ω'' stand for the appropriate pairs of angles θ, φ, θ', φ', θ'', φ'' of the vectors $\mathbf{q}$, $\mathbf{q}'$, $\mathbf{q}''$. Comparing Eqs. (35) and (34) with Eq. (1), we see that the essential novelties of the covariant formulation, which are: The new factor which varies as $1/\sqrt{s}$ in the front of the homogeneous term (coming from the Jacobian) and the s dependence of some of the upper limits of integration, are common both in (34) and (35), so we can safely present the covariant reduction by using (35) without loosing the main points.

From (35) and the 2-particle unitarity condition, written in the appropriate variables, we get a covariant analog of the basic formula (4). It is

$$\mathrm{Disc}_s\left[-\frac{i}{(2\pi)^4}\int d^4\hat{q}''IG^0T\right]=\frac{i}{2(2\pi)^2\sqrt{s}}$$
$$\times\int_{-\infty}^{\infty}d\hat{q}''\cdot\hat{K}\int_{-\infty}^{(\hat{q}''\hat{K})^2/s}d\hat{q}''^2\int_{-1}^{1}d\cos\theta''\int_0^{2\pi}d\varphi''$$
$$\times T(s,\hat{q}^2,\hat{q}''^2,\Omega,\Omega'',\hat{q}\cdot\hat{K},\hat{q}''\cdot\hat{K})\delta_p(s_5-m_1^2)$$
$$\times\delta_p(s_6-m_2^2)T^\dagger(s,\hat{q}''^2,\hat{q}'^2,\Omega'',\Omega',\hat{q}''\cdot\hat{K},\hat{q}'\cdot\hat{K}).\quad(36)$$

There should be added a few remarks concerning the relation (36). This relation can be obtained either exactly [in the same way as (4) was obtained, i.e., by calculating the discontinuity of the homogeneous term via the discontinuity of T itself, from whence we get that (36) is simply (4) rewritten in the covariant variables], or we can get (36) by applying the Cutkosky rules directly to the homogeneous term of Eq. (35) (taking care of the limits and the Jacobian being s dependent). The last problem is not very difficult since the s dependence is given by analytic functions in the region we will be concerned with, in (39), i.e., that above the 2-particle threshold. The results obtained in both ways coincide and give the relation (36).

The next step in the reduction procedure is to consider the appropriate on-shell limits of the off-shell variables. Here we only mention that the notation of Sec. 3 has to be replaced by $\hat{q}''\cdot\hat{K}$ and $\hat{q}''^2$, depending on s through the formulas

$$\hat{q}''\cdot\hat{K}(s)=0,\quad(37)$$
$$\hat{q}''^2(s)=\tfrac{1}{2}(m_1^2+m_2^2)-\tfrac{1}{4}s-(1/4s)(m_1^2-m_1^2)^2,$$

which are simply the mass-shell conditions (31c) in the appropriate variables. The second relation of (37) can be solved for s, and if one takes the "proper root," corresponding to the physical solution, one gets

$$s=[(-\hat{q}''^2+m_1^2)^{1/2}+(-\hat{q}''^2+m_2^2)^{1/2}]^2.\quad(38)$$

The reconstruction rule (5) is the same as before except that now we integrate from $(m_1+m_2)^2$, so we have

$$(IG^0T)_r=\frac{1}{2\pi i}\int_{(m_1+m_2)^2}^{\infty}ds''\frac{\mathrm{Disc}_{s''}(IG^0T)}{s''-s}.\quad(39)$$

In connection with our previous remarks about the new s dependence, it should be noted that in (39) s'' has been inserted in the expression for the discontinuity of the homogeneous term everywhere s had been used [Eq. (36)]. In particular, the factor $1/\sqrt{s}$ which stands in front of the right-hand side of (36) becomes now $1/\sqrt{s''}$. Also the s-dependent limits in (36) become s''-dependent limits in (39).

Now we could write down covariant analogs of all relations of Sec. 3. After all of the above remarks; it is rather obvious how to do this therefore we shall only consider covariant analogs of (7) and (9d). Equation (7) can be replaced by the following relation among δ functions:

$$\delta(s_5(\hat{q}''\cdot\hat{K},\hat{q}''^2)-m_1^2)\delta(s_6(\hat{q}''\cdot\hat{K},\hat{q}''^2)-m_2^2)=\left[\frac{\partial(s_5,s_6)}{\partial(s'',\hat{q}''\cdot\hat{K})}\bigg|_{\text{on-shell}}\right]^{-1}$$
$$\times\delta(s''-[(-\hat{q}''^2+m_1^2)^{1/2}+(-\hat{q}''^2+m_2^2)^{1/2}]^2)\delta(\hat{q}''\cdot\hat{K})=-\frac{s''}{(-\hat{q}''^2+m_1^2)^{1/2}(-\hat{q}''^2+m_2^2)^{1/2}}$$
$$\times\delta(s''-[(-\hat{q}''^2+m_1^2)^{1/2}+(-\hat{q}''^2+m_2^2)^{1/2}]^2)\delta(\hat{q}''\cdot\hat{K}).\quad(40)$$

Using (40) we can trivially perform two integrations over s'' and $\hat{q}''\cdot\hat{K}$ and be left only with a 3-dimensional integral over $\hat{q}''^2$ and $\Omega,''$. It should be noticed that the upper limit of the integration over $\hat{q}''^2$ now becomes zero, and this simply means that we are staying on the reduction hyperplane, or, in the 2-dimensional subspace of $(\hat{q}''\cdot\hat{K},\hat{q}''^2)$, that we integrate along the reduction line, i.e., the negative half of the $\hat{q}''^2$ axis.

A covariant analog of the completely on-shell equation (9d) is

$$\bar{T}(s,\,\hat{q}^2=\hat{q}'^2=\hat{q}^2(s),\,\Omega,\,\Omega',\,\hat{q}\cdot\hat{K}=\hat{q}'\cdot\hat{K}=0)=\bar{I}-\frac{1}{4(2\pi)^3}\int_{-\infty}^{0}d\hat{q}''^2\int_{-1}^{1}d\cos\theta''\int_0^{2\pi}d\varphi''$$
$$\times\frac{[(-\hat{q}''^2+m_1^2)^{1/2}+(-\hat{q}''^2+m_2^2)^{1/2}](-\hat{q}''^2)^{1/2}\bar{T}(s'',\,\hat{q}^2=\hat{q}''^2=\hat{q}^2(s''),\,\Omega,\,\Omega'',\,\hat{q}\cdot\hat{K}=\hat{q}''\cdot\hat{K}=0)}{(-\hat{q}''^2+m_1^2)^{1/2}(-\hat{q}''^2+m_2^2)^{1/2}\{[(-\hat{q}''^2+m_1^2)^{1/2}+(-\hat{q}''^2+m_2^2)^{1/2}]^2-s\}}$$
$$\times\bar{T}^\dagger(s'',\,\hat{q}''^2=\hat{q}'^2=\hat{q}^2(s''),\,\Omega'',\,\Omega',\,\hat{q}''\cdot\hat{K}=\hat{q}'\cdot\hat{K}=0),\quad(41)$$

where

$$\hat{q}^2(s)=\tfrac{1}{2}(m_1^2+m_2^2)-\tfrac{1}{4}s-(1/4s)(m_1^2-m_2^2)^2,\quad s(\hat{q}^2)=[(-\hat{q}^2+m_1^2)^{1/2}+(-\hat{q}^2+m_2^2)^{1/2}]^2.$$

From Eq. (41) we can easily obtain in the c.m. system a general form of Eq. (9d) for arbitrary masses. The reduction condition $\hat{q}''\cdot\hat{K}=0$ takes the form (in the c.m. system)

$$q^{0\prime\prime}[(-\hat{q}''^2+m_1^2)^{1/2}+(-\hat{q}''^2+m_2^2)^{1/2}]=0\,, \tag{42}$$

which is equivalent to $q^{0\prime\prime}=0$. We can also replace the variable $\hat{q}''^2$ by its value on the reduction line, which is $-|\mathbf{q}''|^2$ because of the condition (42). We shall denote $q''^2\equiv|\mathbf{q}''|^2$ and write the generalization of (9d) for arbitrary masses in the form

$$\begin{aligned}\bar{T}(s,\, q^2=q'^2=q^2(s),\, \Omega,\, \Omega',\, q^0=q^{0\prime}=0)=\bar{I}\\ +\frac{1}{2(2\pi)^3}\int_0^\infty dq''\int_{-1}^{1} d\cos\theta''\int_0^{2\pi} d\varphi''\frac{q''^2[(q''^2+m_1^2)^{1/2}+(q''^2+m_2^2)^{1/2}]}{[(q''^2+m_1^2)(q''^2+m_2^2)]^{1/2}}\\ \times\frac{\bar{T}(s'',\, q^2=q''^2=q^2(s''),\, \Omega,\, \Omega'',\, q^0=q^{0\prime\prime}=0)\bar{T}^\dagger(s'',\, q''^2=q'^2=q^2(s''),\, \Omega'',\, \Omega',\, q^{0\prime\prime}=q^{0\prime}=0)}{\{[(q''^2+m_1^2)^{1/2}+(q''^2+m_2^2)^{1/2}]^2-s\}}\,,\end{aligned} \tag{43}$$

where

$$q^2(s)=-\tfrac{1}{2}(m_1^2+m_2^2)+\tfrac{1}{4}s+(1/4s)(m_1^2-m_2^2)^2\,,$$
$$s(q^2)=[(q^2+m_1^2)^{1/2}+(q^2+m_2^2)^{1/2}]^2.$$

We have sketched only a few steps of making a covariant formulation of some of the relations of Sec. 3. In a similar way one can proceed with all other relations. The t variable should be defined as a square of the difference of two Wightman-Gårding's relative momenta. This will enable us to consider arbitrary masses.

6. PARTIAL-WAVE DECOMPOSITION

To make the partial-wave decomposition in a relativistically correct way it is necessary to have two 4-vectors $\hat{q}$ and $\hat{q}'$ which are spacelike, i.e., $\hat{q}^2<0$ and $\hat{q}'^2<0$. Having two spacelike 4-vectors, one for incoming and another for outgoing particles, we can define a scalar variable which provides a relativistic generalization of the barycentric scattering angle to a reference system wherein $|\mathbf{K}|\neq 0$. We refer here to a paper by Macfarlane.[11] The necessity of having a spacelike 4-vector in the partial-wave analysis is rather obvious since we would like to extract the *rotational* invariance of the full T matrix. Having a spacelike 4-vector, we can always get to the frame of reference where its time component is zero, thus in that frame we can build a most general function of the three space-components by using the eigenfunctions of the rotation group.

We have already shown several advantages of using the Wightman-Gårding relative momentum $\hat{q}$; here we shall use the property that on the reduction line, as well as in the whole lower half-plane $(\hat{q}\cdot\hat{K},\hat{q}^2)$, that vector is spacelike. Our momentum $\hat{q}$, defined by (26), differs from that of Macfarlane[11,12] only by a normalization factor.

The Macfarlane results of the relativistic partial-wave analysis were obtained for the completely on-shell amplitude. We can generalize them for cases corresponding to points on the lower half-plane $(\hat{q}\cdot\hat{K},\hat{q}^2)$, but we cannot use the same method to obtain the partial-wave decomposition in the full Bethe-Salpeter equation before reduction, since the accessible region contains domains where $\hat{q}''$ is timelike, i.e., $\hat{q}''^2>0$ (see Fig. 6). It should be noticed that we can restrict the external momenta to stay within the lower half-planes $(\hat{q}\cdot\hat{K},\hat{q}^2)$ and $(\hat{q}'\cdot\hat{K},\hat{q}'^2)$, but we cannot put such a restriction in the full Bethe-Salpeter equation on the *internal* momenta which can be anywhere in the accessible region.

If we first make the reduction and thus have the internal momenta restricted to the reduction hyperplane, then we can use Macfarlane's[11] method of relativistic partial-wave expansion, allowing the external momenta to be only in the lower half-planes $(\hat{q}\cdot\hat{K},\hat{q}^2)$ and $(\hat{q}'\cdot\hat{K},\hat{q}'^2)$. This is why we have postponed the partial-wave decomposition until after we had established the reconstructed (reduced) Bethe-Salpeter equation.

We shall now consider some details of the partial-wave expansion and present them in the three approaches separately. In the first two approaches one can make the partial-wave expansion in a standard way. To illustrate it, let us start from the first approach and concentrate on the completely on-shell amplitude. We define the following decomposition:

$$\begin{aligned}&\bar{T}(s,\hat{q}^2=\hat{q}'^2=\hat{q}^2(s),\, \hat{q}\cdot\hat{q}',\, \hat{q}\cdot\hat{K}=\hat{q}'\cdot\hat{K}=0)\\ &=\sum_{l=0}^{\infty}(2l+1)P_l(-\hat{q}\cdot\hat{q}'/(\hat{q}^2\hat{q}'^2)^{1/2})\bar{T}_l(s)\,,\end{aligned} \tag{44}$$

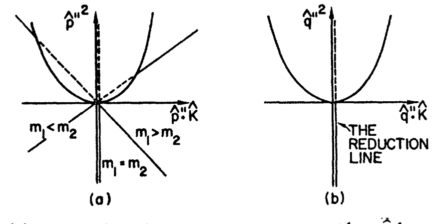

FIG. 7 (a) Reduction lines in the subspace $(\hat{p}''\cdot\hat{K}\hat{p},''^2)$. (b) The reduction line in the subspace $(\hat{q}''\cdot\hat{K},\hat{q}''^2)$, for arbitrary masses. This line also represents the set of physical points, i.e., when both particles are on their mass shells.

[11] A. J. Macfarlane, Rev. Mod. Phys. 34, 41 (1962).
[12] A. J. Macfarlane, J. Math. Phys. 4, 490 (1963).

and use the same formula to define $\bar{I}(s)_l$. The argument of the Legendre polynomials is the relativistic generalization of the barycentric scattering angle, which is

$$\text{“}\cos\theta\text{”}=-\hat{q}\cdot\hat{q}/(\hat{q}^2\hat{q}'^2)^{1/2}. \tag{45}$$

It is easy to check that in the c.m. system the right-hand side of (45) gives the cosine of the scattering angle if we have $\hat{q}\hat{K}=\hat{q}'\hat{K}=0$.

The partial-wave equation for the completely on-shell amplitude in the first approach takes the following form:

$$\bar{T}_l(s)=\bar{I}_l(s)-\frac{1}{2(2\pi)^2}\int_{-\infty}^{0}d\hat{q}''^2 \times\frac{(-\hat{q}''^2s'')^{1/2}\bar{T}_l(s'')\bar{T}_l^{\dagger}(s'')}{(-\hat{q}''^2+m_1^2)^{1/2}(-\hat{q}''^2+m_2^2)^{1/2}(s''-s)}, \tag{46}$$

where

$$s''=[(-\hat{q}''^2+m_1^2)^{1/2}+(-\hat{q}''^2+m_2^2)^{1/2}]^2.$$

For the equal-mass case ($m_1=m_2=1$) we get

$$\bar{T}_l(s)=\bar{I}_l(s)+\frac{1}{(4\pi)^2}\int_4^{\infty}ds'' \times\frac{(s''-4)^{1/2}\bar{T}_l(s'')\bar{T}_l^{\dagger}(s'')}{(s'')^{1/2}(s''-s)}. \tag{47}$$

Equation (47) is a well-known equation,[2] obtained from the 2-particle unitarity condition. In order to solve it we can repeat the standard N/D method and obtain a linearized integral equation. Thus, if we are concerned with the partial-wave amplitudes, we are able to linearize the only quadratic equation of the first approach.

In the second approach we do the partial-wave expansion in the same way as in the first one. We can handle at once all on-shell, off-shell cases because we have only one equation, written schematically in (11). Let us define the off-energy-shell partial-wave amplitudes by the following expansion:

$$\begin{aligned}T(s,\hat{q}^2,\hat{q}'^2,\hat{q}\cdot\hat{q}',\hat{q}\cdot\hat{K}=\hat{q}'\cdot\hat{K}=0)&=\sum_{l=0}^{\infty}(2l+1)P_l(-\hat{q}\cdot\hat{q}'/(\hat{q}^2\hat{q}'^2)^{1/2})T_l(s,\hat{q}^2,\hat{q}'^2)\\&=\sum_{l=0}^{\infty}(2l+1)P_l(\cos(\mathbf{q},\mathbf{q}'))T_l(s,|\mathbf{q}|^2,|\mathbf{q}'|^2)_{\text{c.m.s.}}.\end{aligned} \tag{48}$$

Using (48) we get the following partial-wave equation in the second approach:

$$T_l(s,\hat{q}^2,\hat{q}'^2)=I_l(s,\hat{q}^2,\hat{q}'^2)-\frac{1}{2(2\pi)^2}\int_{-\infty}^{0}d\hat{q}''^2 \times\frac{(-\hat{q}''^2s'')^{1/2}T_l(s'',\hat{q}^2,\hat{q}''^2(s''))T_l^{\dagger}(s'',\hat{q}''^2(s''),\hat{q}'^2)}{(-\hat{q}''^2+m_1^2)^{1/2}(-\hat{q}''^2+m_2^2)^{1/2}(s''-s)}. \tag{49}$$

We would also obtain Eq. (49) from the first approach for the completely off-shell case, if we imposed the condition $\hat{q}\cdot\hat{K}=\hat{q}'\cdot\hat{K}=0$ on the external momenta. However, the on-shell limit of (49) does not coincide with (46). Looking at the homogeneous terms of (46) and (49), we see that the first one does not have any left-hand cut in s, while in the second one we can obtain such a cut, for example, in the ladder approximation, (see Sec. 4).

Finally, let us discuss the partial-wave expansion within the third approach. *We have to make the partial-wave decomposition of the already continued off-shell function.* If we start from a given expression for an on-shell amplitude $T(s,\cos(\mathbf{q},\mathbf{q}'))$, we define its off-shell continuation by

$$T(s,|\mathbf{q}|,|\mathbf{q}'|,\cos(\mathbf{q},\mathbf{q}'))\equiv T(s,\cos(\mathbf{q},\mathbf{q}')\to 1+[-|\mathbf{q}|^2-|\mathbf{q}'|^2+2|\mathbf{q}||\mathbf{q}'|\cos(\mathbf{q},\mathbf{q}')]/2q^2(s)), \tag{50}$$

assuming that

$$\hat{q}\cdot\hat{K}=\hat{q}'\cdot\hat{K}=0,$$

and then make the following partial-wave decomposition:

$$T(s,|\mathbf{q}|,|\mathbf{q}'|,\cos(\mathbf{q},\mathbf{q}'))=\sum_{l=0}^{\infty}(2l+1)\times P_l(\cos(\mathbf{q},\mathbf{q}'))T_l(s,|\mathbf{q}|,|\mathbf{q}'|). \tag{51}$$

The above procedure does not cause any trouble because the series (51) is convergent for any values of s, $|\mathbf{q}|$, and $|\mathbf{q}'|$, since $|\cos(\mathbf{q},\mathbf{q}')|<1$ is always satisfied because we consider $\hat{q}\cdot\hat{K}=\hat{q}'\cdot\hat{K}=0$. We can define $T_l(s,|\mathbf{q}|,|\mathbf{q}'|)$ using (50) and (51), and write it in the form

$$T_l(s,|\mathbf{q}|,|\mathbf{q}'|)=\frac{1}{2}\int_{-1}^{1}d\cos(\mathbf{q},\mathbf{q}')P_l(\cos(\mathbf{q},\mathbf{q}')) \times T(s,\cos(\mathbf{q},\mathbf{q}')\to 1+[-|\mathbf{q}|^2-|\mathbf{q}'|^2+2|\mathbf{q}||\mathbf{q}'|\cos(\mathbf{q},\mathbf{q}')]/2q^2(s)). \tag{52}$$

The substitution for $\cos(\mathbf{q},\mathbf{q}')$ in (50) and (52) defines the off-energy-shell continuation.

We would get a quite different result if we had started from the partial-wave expansion of the on-shell amplitude, and in this expansion had made the substitution for $\cos(\mathbf{q},\mathbf{q}')$. In this case we would have gotten the following series:

$$\sum_{l=0}^{\infty}(2l+1)P_l(1+[-|\mathbf{q}|^2-|\mathbf{q}'|^2+2|\mathbf{q}||\mathbf{q}'| \times\cos(\mathbf{q},\mathbf{q}')]/2q^2(s))T_l(s). \tag{53}$$

This series (53) can diverge for such values of $s,|\mathbf{q}|,|\mathbf{q}'|$, and $\cos(\mathbf{q},\mathbf{q}')$ for which the argument of P_l gets out of the Lehmann ellipse. It should be noticed, however, that the incorrect series (53) is completely different from the series (51). We can see immediately that in (51) we have the radial part of the amplitude continued off-shell, while in (53) the radial part remains the same as it was in the expansion of the on-shell amplitude.

Looking at the definition (52) of the radial part of an off-shell amplitude, and having in mind an expansion of the on-shell amplitude, we see that a given $T_l(s,|\mathbf{q}|,|\mathbf{q}'|)$ is obtained from many $T_l(s)$ which have to be summed before we make the off-shell continuation. However, the on-shell limit of $T_l(s,|\mathbf{q}|,|\mathbf{q}'|)$ coincides with the only *one* $T_l(s)$. We write it

$$\lim_{|\mathbf{q}|,|\mathbf{q}'|\to q(s)} T_l(s,|\mathbf{q}|,|\mathbf{q}'|)\equiv T_l(s,q(s),q'(s))=T_l(s)\,, \quad (54)$$

and we can see that (54) is true by taking the on-shell limits in (52) or (51).

One can look at an example of the Yukawa potential and see explicitly that (50), (51), (52), and (54) give familiar results,[13] while (53) does not.

Considering the homogeneous term of Eq. (14), let us start from the expression for $\mathrm{Disc}_{s''}(IG_0T)\equiv F(s'',\cos(\mathbf{q},\mathbf{q}'))$, defined by (12). In this expression we have to make the substitution for $\cos(\mathbf{q},\mathbf{q}')$, and then make the partial-wave decomposition of such a continued function. This procedure can be written as follows:

$$F_l(s'',|\mathbf{q}|,|\mathbf{q}'|)=\frac{1}{2}\int_{-1}^{1} d\cos(\mathbf{q},\mathbf{q}')P_l(\cos(\mathbf{q},\mathbf{q}'))\times\left[\int\int d\Omega''T(s'',\cos(\mathbf{q},\mathbf{q}''))T^\dagger(s'',\cos(\mathbf{q}'',\mathbf{q}'))\right]\Bigg|_{\cos(\mathbf{q},\mathbf{q}')\to 1+[-|\mathbf{q}|^2-|\mathbf{q}'|^2+2|\mathbf{q}||\mathbf{q}'|\cos(\mathbf{q},\mathbf{q}')]/2q''^2(s'')}\,. \quad (55)$$

From (55) and (14) we get the following partial-wave equation:

$$T_l(s,|\mathbf{q}|,|\mathbf{q}'|)=I_l(s,|\mathbf{q}|,|\mathbf{q}'|)-\frac{1}{4(2\pi)^3}\int_{-\infty}^{0} d\hat{q}''^2\frac{(-\hat{q}''^2s'')^{1/2}F_l(s'',|\mathbf{q}|,|\mathbf{q}'|)}{(-\hat{q}''^2+m_1^2)^{1/2}(-\hat{q}''^2+m_2^2)^{1/2}(s''-s)}\,. \quad (56)$$

Comparing (56) with its analog (49) in the second approach, we see that on the right-hand side of (56) we have all partial-wave amplitudes, because of the definition (55). To calculate any F_l we have to consider the whole T, i.e., all its partial waves. If in (55) one first decomposes $T(s'',\cos(\mathbf{q},\mathbf{q}''))$ and $T^\dagger(s'',\cos(\mathbf{q}'',\mathbf{q}'))$ into partial waves, then one should sum the whole series before making the substitution for $\cos(\mathbf{q},\mathbf{q}')$. To show this explicitly we write the partial-wave decomposition of the on-shell T and $T^\dagger$ in the form (51). We have

$$T(s'',\cos(\mathbf{q},\mathbf{q}''))\equiv T(s'',|\mathbf{q}(s'')|,|\mathbf{q}''(s'')|,\cos(\mathbf{q},\mathbf{q}''))=\sum_{l'=0}^{\infty}(2l'+1)P_{l'}(\cos(\mathbf{q},\mathbf{q}''))T_{l'}(s'',|\mathbf{q}(s'')|,|\mathbf{q}''(s'')|)\,,$$

and a similar expression for $T^\dagger(s'',\cos(\mathbf{q}'',\mathbf{q}'))$. Using these formulas we can easily perform the integration over Ω'' in (55) and obtain

$$\int d\Omega''T(s'',\cos(\mathbf{q},\mathbf{q}''))T^\dagger(s'',\cos(\mathbf{q}'',\mathbf{q}'))=4\pi\sum_{l'=0}^{\infty}(2l'+1)P_{l'}(\cos(\mathbf{q}.\mathbf{q}'))\times T_{l'}(s'',|\mathbf{q}(s'')|,|\mathbf{q}''(s'')|)\times T_{l'}^\dagger(s'',|\mathbf{q}''(s'')|,|\mathbf{q}'(s'')|)\,. \quad (57)$$

Then we make the substitution for $\cos(\mathbf{q},\mathbf{q}')$, however not in the argument of $P_{l'}(\cos(\mathbf{q},\mathbf{q}'))$, but in the expression which results *after* summation on the right-hand side of (57).

Finally, after substituting for $\cos(\mathbf{q}.\mathbf{q}')$, we make the partial-wave projection as indicated in (55), and get the following set of equations, written shortly as (56):

$$T_l(s,|\mathbf{q}|,|\mathbf{q}'|)=I_l(s,|\mathbf{q}|,|\mathbf{q}'|)-\frac{1}{(4\pi)^2}\int_{-\infty}^{0} d\hat{q}''^2\frac{(-\hat{q}''^2s'')^{1/2}(s''-s)^{-1}}{(-\hat{q}''^2+m_1^2)^{1/2}(-\hat{q}''^2+m_2^2)^{1/2}}\times\int_{-1}^{1} d\cos(\mathbf{q},\mathbf{q}')P_l(\cos(\mathbf{q},\mathbf{q}'))\left[\sum_{l'=0}^{\infty}(2l'+1)P_{l'}(\cos(\mathbf{q},\mathbf{q}'))T_{l'}(s'',|\mathbf{q}(s'')|,|\mathbf{q}''(s'')|)\times T_{l'}^\dagger(s'',|\mathbf{q}''(s'')|,|\mathbf{q}'(s'')|)\right]\Bigg|_{\cos(\mathbf{q},\mathbf{q}')\to 1+[-|\mathbf{q}|^2-|\mathbf{q}'|^2+2|\mathbf{q}||\mathbf{q}'|\cos(\mathbf{q},\mathbf{q}')]/2q''^2(s'')}\,. \quad (58)$$

We see that (58) represents a set of infinitely many coupled partial-wave equations.

One of the immediate applications of Eq. (58) is that it can be used for obtaining the off-energy-shell continuation of the on-shell partial-wave amplitudes, which can be expressed in terms of the phase-shifts by

$$T_{l'}(s'',|\mathbf{q}(s''),|\mathbf{q}''(s'')|)T_{l'}^\dagger(s'',|\mathbf{q}''(s'')|,|\mathbf{q}'(s'')|)\equiv|T_{l'}(s'')|^2=\rho^2(s'')\sin^2\delta_{l'}(s'')\,,$$

[13] B. W. Lee and R. Sawyer, Phys. Rev. **127**, 2266 (1962).

where

$$\rho(s'')=\frac{[s''(q''^2+m_1{}^2)(q''^2+m_2{}^2)]^{1/2}}{q''[q''^2+\frac{1}{2}(m_1{}^2+m_2{}^2)]}.$$

7. METHODS OF SOLVING REDUCED EQUATIONS

We shall consider several methods of solving the reduced equations (9), (11), and (14), according to the three approaches.

In the first approach the basic equation is the equation for the completely on-shell amplitude (9d) and we can solve it by referring to the partial-wave Eq. (46). This equation can be linearized by the usual N/D method and solved in a standard way. Having the solution of (9d) we can solve the *linear* equations (9b) and (9c) which have the on-shell amplitude as a part of the kernel. We can do it for the partial-wave amplitudes by writing equations of the form similar to (49). Finally, using solutions of (9b) and (9c), we put them in (9a) and get the fully off-shell amplitude.

In the second approach, to solve Eq. (11) we use its partial-wave projection given by (49). We can start by considering the half-on-shell case, by putting either $\hat{q}^2=\hat{q}^2(s)$ or $\hat{q}'^2=\hat{q}^2(s)$, and use a symmetry property of the following form

$$T_l(s,\hat{q}^2(s),\hat{q}'^2)=T_l(s,\hat{q}^2,\hat{q}'^2(s)),\quad \text{if}\quad \hat{q}'^2=\hat{q}^2.$$

The half-on-shell amplitude $T_l(s,\hat{q}^2(s),\hat{q}'^2)$ is a function of two continuous variables: energy s and the magnitude of one of the relative momenta q'. If we want to solve numerically Eq. (49), we get an equation for a matrix where rows correspond to the discrete values of energy and columns to the discrete values of the relative momenta. We would then have to solve a matrix equation. Having the solution of (49), we use (49) once more to get the completely off-shell amplitude from the half-on-shell amplitudes. Marchesini and Wong[2] have recently discussed a similar method of solving Eq. (49), by rewriting it as an integral equation in *two* variables for the completely off-shell amplitude, with the help of one δ function. They have used the matrix N/D method and shown several numerical examples.

In the third approach we have to solve Eq. (14). Let us discuss three topics related with this problem:

(1) Iteration in (14).

(2) An ansatz for a solution, with a given angular dependence.

(3) Truncated partial-wave series.

For a *given* inhomogeneous term I, either as a function of s and $\cos(\mathbf{q},\mathbf{q}')$, or already given as a function of the off-shell variables, we can calculate any of the terms in the iteration series. We have explained in Sec. 4 how to obtain the inhomogeneous term in the off-shell variables using the substitution for $\cos(\mathbf{p},\mathbf{p}')$. Of course to obtain the on-shell limit of it does not produce any problem, and the next-order term in the iteration series can be calculated by explicitly evaluating the angular integrals in (14) and afterwards making the substitution for $\cos(\mathbf{q},\mathbf{q}')$. This substitution can be simplified by utilizing the frame of reference where $\cos\theta=1$ and $\cos(\mathbf{q},\mathbf{q}')=\cos\theta'$. This was done in Sec. 4, where we have explained the example of a box graph. Thus, the iteration series in (14) can be well defined term by term. The convergence of such a series is an open question and may be positively answered only for some class of the inhomogeneous terms.

Let us now consider an arbitrary ansatz for a solution of (14) which has a *given* angular dependence. The dependence on angles can be assumed in any form, not necessarily in a separable one. We can be guided by the completely on-shell amplitude and check whether the on-shell limit of our ansatz corresponds to the experimentally observed situation. Having a justified ansatz, with an *explicit* angular dependence, we can perform the angular integration in (14) and be left with only a one-dimensional integral relation. This relation can be used either for adjusting some parameters in the assumed ansatz, or as an equation for a function of energy and magnitudes of the relative momenta which was put in the ansatz.

As a particular case of an ansatz with a given angular dependence, we can consider a truncated partial-wave series. We truncate the series (51), which is a convergent series, or its on-shell limit which is also convergent. The reason for truncating the series (51) is *not* to avoid a divergence, but to simplify the partial-wave equations (56) or (58), by making them a finite set of equations. By dealing with a finite number of partial waves we make it feasible to find a solution of (56) or (58). The resummation of partial waves which is needed in (55) is trivial if we have only a finite number of partial waves, and usually it is a small number. The substitution for $\cos(\mathbf{q},\mathbf{q}')$ can be made term by term.

The assumption of a truncated partial-wave series is reasonable for lower energies and rather dubious for higher. However, in the higher-energy region we have made serious approximations anyway be excluding all higher unitarity cuts except the 2-particle cut. Our formalism is an approximate one in the first place.

The number of partial waves which will be left in the series (51) should be chosen in such a way that they correspond to the most dominant partial waves found from the phase-shift analysis. This should be so, because the on-shell limit of (51) exactly coincides with the partial-wave decomposition of the on-shell amplitude.

We present an example where only the S and P waves are kept. To simplify notation we consider equal masses and denote the relative momentum by p. We take the following ansatz for a solution of Eq. (14) (for the case

$p^0=p^{0\prime}=0$)

$$T(s,p,p',\cos(\mathbf{p},\mathbf{p}'))\approx T_0(s,p,p')+3T_1(s,p,p')\cos(\mathbf{p},\mathbf{p}'). \quad (59)$$

Substituting (59) in (55), performing trivial angular integrations, and then making substitution for $\cos(p,p')$, we get[14]

$$F_l(s'',p,p')=2\pi\int_{-1}^{1}d\cos(\mathbf{p},\mathbf{p}')P_l(\cos(\mathbf{p},\mathbf{p}'))\times\{T_0^2(s'',p'',p'')+3T_1^2(s'',p'',p'')\times\{1+[-p^2-p'^2+2pp'\cos(\mathbf{p},\mathbf{p}')]/2p^2(s'')\}\}. \quad (60)$$

Finally, we make the partial-wave projection in (60), and get the following set of partly coupled equations:

$$T_1(s,p,p')=I_1(s,p,p')+\frac{pp'}{2\pi^2}\times\int_0^\infty dp''\frac{T_1^2(s'',p'',p'')}{(p''^2+1)^{1/2}[4(p''^2+1)-s]}, \quad (61)$$

$$T_0(s,p,p')=\left\{I_0(s,p,p')+\frac{3}{2\pi^2}\int_0^\infty dp''\times\frac{p''^2T_1^2(s'',p'',p'')}{(p''^2+1)^{1/2}[4(p''^2+1)-s]}\left(1-\frac{p^2+p'^2}{2p''^2}\right)\right\}+\frac{1}{2\pi^2}\int_0^\infty dp''\frac{p''^2T_0^2(s'',p'',p'')}{(p''^2+1)^{1/2}[4(p''^2+1)-s]}. \quad (62)$$

To solve the set of equations (61) and (62) we start from (61) and, having its solution, we substitute it into (62) to get the full inhomogeneous term of (62). We then solve (62).

If we include more than the S and P waves, then we get a more complicated set of equations. Instead of solving them we can use these equations as an off-shell continuation of the on-shell amplitudes. On the right-hand side of these equations we have the radial parts of the completely on-shell amplitudes which can be expressed in terms of the experimentally known phase-shifts. For the equal-mass case we have

$$|T_l(s'',p'',p'')|^2\equiv|T_l(s'')|^2=16\pi\left(\frac{s''}{s''-4}\right)^{1/2}\sin^2\delta_l(s'').$$

8. SUMMARY AND FINAL REMARKS

We have discussed three approaches of formulating an approximate scheme for calculating the 2-body off-shell and on-shell T matrix. All these approaches led to approximate schemes and they took into account only the 2-particle right-hand cut and some left-hand cuts. These approaches can provide a first step in an approximate calculation of the off-shell T matrix.

The three approaches differ in several details, but they have the following common features: (1) 2-particle unitarity; (2) manifestly covariant formulation; (3) relativistic, kinematical weighting factors. The first approach ignores the left-hand cut except for the one which could be included in the inhomogeneous term. That approach is the most primitive one, and it shows the necessity of including both left and right hand cuts. The second approach is the simplest one in its form, and it takes into account the left-hand cuts. It develops, however, some additional incorrect left-hand cuts, which are removed in the third approach. The last approach has built in the correct left-hand-cut structure, but at the expense of some more complicated formulas. Thus, in the third approach one can consider only a truncated partial-wave series for any practical calculation.

In further investigations one can follow several paths:

(1) Three-particle theory.

(2) Comparison of the numerical results obtained from the full Bethe-Salpeter equation by Schwartz and Zemach[15] with the results from the reduced equations.

(3) Investigation of different type of perturbed solutions, to see how the off-shell amplitudes behave under small perturbations of the completely on-shell amplitude.

(4) Study of the variety of solutions due to the Castillejo-Dalitz-Dyson-poles ambiguity.

(5) Bootstrap-type calculations for a different inhomogeneous term as a driving force.

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APPENDIX

To define the accessible region in the space of different invariants we shall first repeat Taylor's[9] considerations for invariants which he has introduced, namely,

$$s,s_5,s_6,s_{15},s_{35}, \quad (A1)$$

[14] It should be noticed that if we have a truncated series then we can substitute for $\cos(\mathbf{p},\mathbf{p}')$ term by term. Such a procedure is incorrect only in the case of an infinite series.

[15] C. Schwartz and C. Zemach, Phys. Rev. 141, 1454 (1966).

defined in Sec. 5 by (27a). Following Taylor's[9] paper, we write

$$\frac{\partial(s_5,s_6,s_{15},s_{35})}{\partial(k_{5x},k_{5y},k_{5z},k_5{}^0)}=8(-\det Z)^{1/2}, \tag{A2}$$

where Z is given by

$$Z=\begin{pmatrix} \hat{k}_1{}^2 & \frac{1}{2}(s-\hat{k}_1{}^2-\hat{k}_2{}^2) & \frac{1}{2}(s_{13}-\hat{k}_1{}^2-\hat{k}_3{}^2) & \frac{1}{2}(\hat{k}_1{}^2+s_5-s_{15}) \\ \frac{1}{2}(s-\hat{k}_1{}^2-\hat{k}_2{}^2) & \hat{k}_2{}^2 & \frac{1}{2}(\hat{k}_1{}^2+\hat{k}_4{}^2-s-s_{13}) & \frac{1}{2}(s+s_{15}-\hat{k}_1{}^2-s_6) \\ \frac{1}{2}(s_{13}-\hat{k}_1{}^2-\hat{k}_3{}^2) & \frac{1}{2}(\hat{k}_1{}^2+\hat{k}_4{}^2-s-s_{13}) & \hat{k}_3{}^2 & \frac{1}{2}(s_{35}-\hat{k}_3{}^2-s_5) \\ \frac{1}{2}(\hat{k}_1{}^2+s_5-s_{15}) & \frac{1}{2}(s+s_{15}-\hat{k}_1{}^2-s_6) & \frac{1}{2}(s_{35}-\hat{k}_3{}^2-s_5) & s_5 \end{pmatrix}. \tag{A3}$$

The region of integration in the Bethe-Salpeter equation, written in the invariant variables s, s_5, s_6, s_{15}, s_{35}, is the intersection of the region $\det Z<0$ and the accessibility region D. The region D is given by the intersection of the regions D_{12} and D_{34}, where D_{34} is obtained from D_{12} by replacing the subscripts 1 and 2 by 3 and 4 everywhere in the condition for D_{12}. Finally, D_{12} is the union of the regions $D_{12}{}^+$ and $D_{12}{}^-$, with $D_{12}{}^+$ defined by: $\det D_{12}>0$, and at least one of $\hat{k}_1{}^2$, or $\hat{k}_2{}^2$, or s_5 negative, or one of $\Delta(\hat{k}_1{}^2,\hat{k}_2{}^2,s)$, or $\Delta(\hat{k}_1{}^2,s_5,d_{15})$, or $\Delta'(\hat{k}_1{}^2,\hat{k}_2{}^2,\ s_5,s_6,s,s_{15})$ positive. $D_{12}{}^-$ is defined by $\det D_{12}<0$, $\hat{k}_1{}^2$, $\hat{k}_2{}^2$, $s_5,\Delta(\hat{k}_1{}^2,\hat{k}_2{}^2,s)$, $\Delta(\hat{k}_1{}^2,s_5,s_{15})$, and $\Delta'(\hat{k}_1{}^2,\hat{k}_2{}^2,s_5,s_6,s,s_{15})$ all negative. The notation is explained by

$$\Delta(s_1,s_2,s)=s_1{}^2+s_2{}^2+s^2-2s_1s_2-2s_1s-2s_2s\,,$$
$$\Delta'(s_1,s_2,s_5,s_6,s,s_{15})=\tfrac{1}{4}(s+s_{15}-s_1-s_6)^2-s_2s_5\,,$$
$$D_{12}=\det\begin{pmatrix} \hat{k}_1{}^2 & \frac{1}{2}(s-\hat{k}_1{}^2-\hat{k}_2{}^2) & \frac{1}{2}(\hat{k}_1{}^2+s_5-s_{15}) \\ \frac{1}{2}(s-\hat{k}_1{}^2-\hat{k}_2{}^2) & \hat{k}_2{}^2 & \frac{1}{2}(s+s_{15}-\hat{k}_1{}^2-s_6) \\ \frac{1}{2}(\hat{k}_1{}^2+s_5-s_{15}) & \frac{1}{2}(s+s_{15}-\hat{k}_1{}^2-s_6) & s_5 \end{pmatrix}.$$

Although these formulas look lengthy, they follow, however, from straightforward considerations, presented in Appendix II of Taylor's[9] paper. The proof of the above formulas is based on the theorem that a symmetric matrix is positive definite if and only if every principal minor is positive.

Having the framework of Taylor's[9] arguments, we can immediately extend it to the set of invariants which are built up from the Wightman-Gårding relative momenta $\hat{q}$, $\hat{q}'$, $\hat{q}''$, and the total momentum $\hat{K}$:

$$\hat{K}^2,\hat{q}''\cdot\hat{K},\hat{q}''^2,\hat{q}''\cdot\hat{q},\hat{q}''\cdot\hat{q}'. \tag{A4}$$

We write

$$\frac{\partial(\hat{q}''^2,\hat{q}''\cdot\hat{K},\hat{q}''\cdot\hat{q},\hat{q}''\cdot\hat{q}')}{\partial(\hat{q}_x'',q_y'',q_z'',q^{0\prime\prime})}=8(-\det\tilde{Z})^{1/2}, \tag{A5}$$

where $\tilde{Z}$ is given by

$$\tilde{Z}=\begin{pmatrix} \hat{q}''^2 & \hat{q}''\cdot\hat{K} & \hat{q}''\cdot\hat{q} & \hat{q}''\cdot\hat{q}' \\ \hat{q}''\cdot\hat{K} & \hat{K}^2 & \hat{q}\cdot\hat{K} & \hat{q}'\cdot\hat{K} \\ \hat{q}''\cdot\hat{q} & \hat{q}\cdot\hat{K} & \hat{q}^2 & \hat{q}\cdot\hat{q}' \\ \hat{q}''\cdot\hat{q}' & \hat{q}'\cdot\hat{K} & \hat{q}\cdot\hat{q}' & \hat{q}'^2 \end{pmatrix}. \tag{A6}$$

Comparing (A6) with (A3), we can already see some notational simplification, if working in terms of invariants (A4) instead of (A1). One gets an *essential* simplification in (A6) if the reduction procedure is applied which, in the covariant notation, means

$$\hat{q}''\cdot\hat{K}=0. \tag{A7}$$

Also the different limits, obtained by putting

$$\hat{q}\cdot\hat{K}=0 \quad\text{and/or}\quad \hat{q}'\cdot\hat{K}=0, \tag{A8}$$

significantly simplify the symmetric matrix (A6).

The other determinants, which correspond to D_{12} and D_{34}, are formed from the matrix $\tilde{Z}$ and therefore they simplify together with $\tilde{Z}$.

The upper limits in the integrals over the variables $\hat{q}''\cdot\hat{q}$ and $\hat{q}''\cdot\hat{q}'$, which play the role of angles in the covariant formulation of the Bethe-Salpeter equation (34), are given by equating to zero the determinants of of submatrices corresponding to D_{12} and D_{34}. As an example we shall write one of the limits

$$[f_1(\hat{q}''\cdot\hat{K}=0,\ \hat{q}\cdot\hat{K},\ \hat{q}^2,\ \hat{K}^2)]^2 =\frac{\hat{q}''^2}{\hat{K}^2}[\hat{K}^2\hat{q}^2-(\hat{q}\cdot\hat{K})^2]. \tag{A9}$$

It should be noticed that in (A9) we have written the simplified form for the limit f_1, assuming the reduction condition $\hat{q}''\cdot\hat{K}=0$.

The above remarks and Taylor's[9] arguments should be sufficient to explain the accessible region in the full 4-dimensional space of different invariants.

Finally, let us list three Jacobians which were used in our calculations:

$$\frac{\partial(|k_5|,k_5^0)}{\partial(s_5,s_6)}=-\frac{1}{4(\sqrt{s})|\mathbf{k}_5|}=-\frac{1}{4(\sqrt{s})|\mathbf{p}''|}=-\frac{1}{4(\sqrt{s})|\mathbf{q}''|}\text{ in the c.m.s.;}$$

$$\frac{\partial(s_5,s_6)}{\partial(\hat{q}''^2,\hat{q}''\cdot\hat{K})}=-2;\quad \frac{\partial(s_5,s_6)}{\partial(s'',\hat{q}''\cdot\hat{K})}=-\frac{2}{s''}(-\hat{q}''^2+m_1^2)^{1/2}(-\hat{q}''^2+m_2^2)^{1/2}+2\hat{q}''\cdot\hat{K}\frac{m_1^2-m_2^2}{s''^2}.$$

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Particle-Nucleus Interactions and the Regge Model*

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We propose that the Regge-pole model may be applied to the high-energy scattering of particles (π,K,N) by nuclei. The deuteron total cross sections are shown to be consistent with this hypothesis.

We suggest here that the Regge-pole model may be applicable to the scattering of high-energy particles (π,K,N) by nuclei. To examine this possibility further it is proposed that the energy dependence of nuclear total cross sections at high energies be investigated experimentally. The present deuteron data are shown to be consistent with the Regge model.

Total cross-section measurements[1] provide a good test of the Regge-pole model[2] since the total cross section is related to the imaginary part of the forward elastic-scattering amplitude by the optical theorem. The total cross sections usually considered are $\pi^\pm$, $K^\pm$, $\bar{p}$, and p on protons and neutrons.[2] The neutron cross sections are obtained from the deuteron total cross sections using the method of Glauber,[3] in which corrections due to the mutual screening of the two nucleons in the deuteron are taken into account. The Glauber correction is only an approximation and, since its contribution to the neutron cross section can be as large as 20% [e.g., $\sigma(\bar{p}n)$ at 6 GeV], it is a potential source of a large systematic error. This error can be extremely crucial when one is considering the differences of cross sections, as one often does in trying to isolate the contribution of a single trajectory.

In an attempt to avoid the difficulties of the Glauber correction, we shall treat the deuteron as a system which interacts at high energies by the exchange of Regge pole in much the same way as an elementary particle. It is our hope that the structure of the deuteron will affect only the t dependence of the residue function, leaving the characteristic energy dependence of Regge exchange unaffected.

The role of the deuteron in our study is similar to the one it plays in the quark-model analysis of the deuteron total cross sections recently performed by Levinson, Wall, and Lipkin.[4] Their sum rules are obtained by performing a quark decomposition of the incident particles (π, K, and N), but *not* of the deuteron which is regarded as a "black box."

We consider interactions of the form indicated schematically in Fig. 1. Since the deuteron has $I=0$, we are restricted to isoscalar trajectories. We therefore consider the contribution of two $C=+1$ trajectories, the P and P', and one $C=-1$ trajectory, the ω.[5]

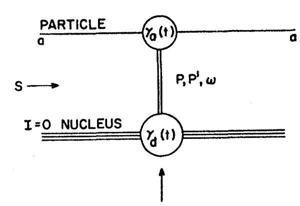

FIG. 1. This diagram is a schematic representation of the type of elastic-scattering process considered here.

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[1] W. Galbraith *et al.*, Phys. Rev. **138**, B913 (1965).

[2] V. Barger and M. Olsson, Phys. Rev. **146**, 1080 (1966); Phys. Rev. Letters **16**, 545 (1966); *ibid.* **15**, 930 (1965).

[3] R. J. Glauber, Phys. Rev. **100**, 242 (1955); B. Udgaonkar and M. Gell-Mann, Phys. Rev. Letters **8**, 346 (1962); D. Harrington, Phys. Rev. **135**, B358 (1964).

[4] C. A. Levinson, N. S. Wall, and H. J. Lipkin, Phys. Rev. Letters **17**, 1122 (1966).

[5] We take the point of view that there are two $C=+1$ trajectories contributing here, the Pomeranchuk trajectory P and a P'. A good fit may also be obtained with a single trajectory having these quantum numbers and an intercept of 0.88. No additional light is shed on how many vacuum trajectories are necessary, however. The φ trajectory has not been included here because of the rapidly accumulating evidence that φ does not couple to nonstrange hadrons. A residual φ contribution would, of course, affect our fits to the Δ's.