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Estimate of the π-π Total Cross Section and Diffraction Width

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In this note we estimate the total π-π cross section and diffraction width at high energy via a simple approximation procedure for the reduced residue which was recently noted in potential theory. Results consistent with the present known facts on the π-π system are obtained, and it is concluded that this approximation procedure may provide a simple and yet accurate means of estimating total high-energy cross sections and diffraction widths once the crossed-channel trajectories are known.

I. INTRODUCTION

In potential theory, it has been noted that the reduced residue, for the first trajectory, may be approximated by[1]

$$\gamma(\nu)\approx C_1, \quad \nu\approx 0, \tag{1}$$

where C_1 is the "slope" of $\mathrm{Im}\alpha$; namely,

$$\mathrm{Im}\alpha(\nu)\underset{\nu\to 0}{\longrightarrow} C_1\nu^{\mathrm{Re}\alpha(0)+1/2}, \tag{2}$$

where ν is the usual c.m. momentum squared. For example, for a potential of $-1.8e^{-r}/r$ a rough estimate of C_1 yields, in a one-trajectory approximation, $\gamma(0)\approx 0.58$ while $\gamma_{\mathrm{ex}}(0)\approx 0.60$; for the stronger potential $-5e^{-r}/r$ we have $\gamma(0)\approx 1.03$ while $\gamma_{\mathrm{ex}}(0)\approx 0.80$.[2] It was suggested[1] that this might make it possible to make simple and fairly accurate estimates of total high-energy cross sections and diffraction widths.

The purpose of this note is to demonstrate this for the case of π-π scattering. We obtain values of the total π-π cross section and diffraction width which are consistent both with the approximations made and with previous estimates.[3]

II. EQUATIONS AND CALCULATIONS

The invariant amplitude $A^I(s,t)$ for the π-π system may be written[4]

$$A^I(s,t)=\sum_{l=0}^{\infty}(2l+1)f_l{}^I(s)P_l(z_s)[1+(-)^{I+l}], \tag{3}$$

where $I=0$, 1, 2 is the total isospin quantum number for the π-π system, $z_s=\cos\theta_s=1+2t/(s-4)$, and

$$f_l{}^I(s)=\frac{S^I(l,s)-1}{2i\rho(s)}, \tag{4}$$

$$\rho(s)=\left(\frac{s-1}{s}\right)^{1/2}\equiv\left(\frac{\nu}{\nu+1}\right)^{1/2}. \tag{5}$$

The quantity $A^I(s,t)$ satisfies the crossing relations

$$A^I(s,t)=\sum_{I'=0}^{2}\chi^{II'}A^{I'}(t,s), \tag{6}$$

where

$$\chi^{II'}=\begin{bmatrix}1/3 & 1 & 5/3\\ 1/3 & 1/2 & -5/6\\ 1/3 & -1/2 & 1/6\end{bmatrix}. \tag{7}$$

In terms of $A^I(s,t)$, the total cross section is

$$\sigma_{\mathrm{tot}}{}^I(s)=\frac{16\pi}{s}\frac{\mathrm{Im}A^I(s,0)}{\rho(s)}, \tag{8}$$

while the diffraction width Δt is defined by

$$\left(\frac{1}{\Delta t}\right)_I=\left[\frac{(\partial/\partial t)|A^I(s,t)|^2}{|A^I(s,t)|^2}\right]_{t=0}. \tag{9}$$

We now make the usual Watson-Sommerfeld transformation on (3), along with the asymptotic form for the Legendre function

$$P_\nu(z)\underset{z\to\infty}{\longrightarrow}\frac{1}{\sqrt{\pi}}\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)}(2z)^\nu, \tag{10}$$

and letting $\alpha^I(t)$ represent the pole (with the greatest real part) of the partial-wave amplitude $f_l{}^I(t)$ at the pole $l=\alpha^I(t)$, with $\beta^I(t)$ the corresponding residue, we have the asymptotic form

$$A^I(t,s)\underset{\substack{s\to\infty\\ t<0}}{\longrightarrow}-\pi[2\alpha^I(t)+1]\gamma^I(t)\frac{\Gamma(\alpha^I(t)+\frac{1}{2})}{\Gamma(\alpha^I(t)+1)\sqrt{\pi}}\times s^{\alpha^I(t)}\left[\frac{e^{-i\pi\alpha^I(t)}+(-)^I}{\sin\pi\alpha^I(t)}\right], \tag{11}$$

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[1] W. J. Abbe and G. A. Gary, preceding paper, Phys. Rev. **160**, 1510 (1967).

[2] The exact reduced residues are given by A. Ahmadzadeh, Ph.D. thesis, UCRL 11096, University of California, Berkeley, California, 1963 (unpublished).

[3] G. F. Chew and V. L. Teplitz, Phys. Rev. **136**, B1154 (1964).

[4] We use units such that $m_\pi=c=\hbar=1$. Lengths are measured in units of $R_0=\hbar c/(m_\pi c^2)\approx 2^{1/2}$ F and 1 $\mathrm{GeV}^{-2}\approx 0.4$ mb. The variables s and t are, respectively, the total energy and the momentum transfer squared or vice versa depending on whether channel s or channel t is open. If channel s is open we have $s>s_{\mathrm{th}}$, $t<0$; if channel t is open we have $t>t_{\mathrm{th}}$, $s<0$.

where the reduced residue $\gamma^I(t)$ is defined by

$$\gamma^I(t)=\frac{\beta^I(t)}{\nu_t^{\alpha^I(t)}},\quad t=4\nu_t+4. \tag{12}$$

Once the quantities $\alpha^I(t)$ and $\gamma^I(t)$ are given for each $I=0$, 1, 2, the invariant amplitude $A^I(s,t)$ may be calculated via the crossing relations (6), and hence the total cross section and diffraction width Δt from Eqs. (8) and (9) for $s\to\infty$.

Before proceeding with the calculation, a word about the reduced residue is appropriate. In potential theory the reduced residue, defined by

$$b(\nu)=\beta(\nu)/\nu^{\alpha(\nu)}, \tag{13}$$

is a real-analytic function of ν in the ν plane with only a right-hand cut, and $\beta(\nu)$ is the residue of

$$f_l(\nu)=\frac{S(l,\nu)-1}{2i\sqrt{\nu}} \tag{14}$$

at the pole $l=\alpha(\nu)$. This result follows immediately from the Schrödinger equation.[5] In the relativistic problem, the reduced residue defined by

$$\gamma(\nu)=\beta(\nu)/\nu^{\alpha(\nu)} \tag{15}$$

is *conjectured* to be real analytic in the ν plane[6] where $\beta(\nu)$ is now the residue of

$$f_l(\nu)=(\nu+1)^{1/2}\left[\frac{S(l,\nu)-1}{2i\sqrt{\nu}}\right] \tag{16}$$

at the corresponding pole $l=\alpha(\nu)$. Now it is obvious that if the residue of $f_l(\nu)$ at the pole $l=\alpha(\nu)$ is to be real analytic in the ν plane, then there must be singularities in $S(l,\nu)$ to cancel that arising from the factor $(\nu+1)^{1/2}$ at $\nu=-1$. The usual procedure for continuing (16) to $\nu<0$ is to force the statement of its analyticity via Cauchy's theorem. However, as noted above, our purpose here is to test the approximation from potential theory

$$\beta(\nu)\underset{\nu\to 0}{\longrightarrow}\operatorname{Im}\alpha(\nu)/\sqrt{\nu} \tag{17}$$

and, therefore, since $\operatorname{Im}\alpha(\nu)\to C_1\nu^{\operatorname{Re}\alpha(0)+1/2}$, then

$$b(\nu)\underset{\nu\to 0}{\longrightarrow}C_1, \tag{18}$$

Eq. (18) being approximately valid even for $\nu<0$ in potential theory since $b(\nu)$ is roughly constant over quite a wide range.[2] In the relativistic case we shall also approximate $\gamma(\nu)$ by

$$\gamma(\nu)\approx C_1, \tag{19}$$

where again C_1 is defined as in (18). This is tantamount to expanding $(\nu+1)^{1/2}$ about threshold ($\nu=0$) and taking the first term. Although this may seem inconsistent, since we shall be using the result at $\nu=-1$ (zero total energy), we are simply saying that $\gamma(\nu)$ is real analytic to this order in the threshold expansion; indeed, if $\gamma(\nu)$ is in fact a real analytic function of ν, then it must be to each order in ν, and the approximation made here is to retain only the first term; note that it is very simply obtained once the trajectory is given. This procedure has the added advantage (over Cauchy's theorem) that statements about the high-energy behavior of $\beta(\nu)$ are not required.

For the $I=0$ trajectory, we use the phenomenological one which assumes that the f^0(1250-MeV) resonance is indeed the Pomeranchuk particle[7]

$$\operatorname{Im}\alpha^{I=0}(\nu)\underset{\nu\to 0}{\longrightarrow}(1.15\times10^{-3})\nu^{\alpha^{I=0}(\nu)+1/2},\quad t=4\nu+4 \tag{20}$$

$$(d\alpha^{I=0}/dt)_{t=0}\approx\tfrac{1}{3}\ \text{GeV}^{-2}, \tag{21}$$

$$\alpha^{I=0}(t=0)=1, \tag{22}$$

while for the $I=1$ channel we use the trajectory resulting from a recent approximate bootstrap calculation of the ρ (750-MeV) resonance[8]

$$\operatorname{Im}\alpha^{I=1}(\nu)\underset{\nu\to 0}{\longrightarrow}0.065\nu^{\alpha^{I=1}(\nu)+1/2}, \tag{23}$$

$$(d\alpha^{I=1}/dt)_{t=0}\approx 1.06\ \text{GeV}^{-2}, \tag{24}$$

$$\alpha^{I=1}(t=0)\approx\tfrac{1}{2}. \tag{25}$$

The $I=2$ channel will be neglected; since no meson has been (at the present time) observed with $I=2$, the force in this state is presumably repulsive, and hence $\operatorname{Re}\alpha^{I=2}(t)<0$. Consequently the $I=2$ state will contribute a term of order $O(s^{\operatorname{Re}\alpha^{I=2}(t)})$ to the crossing relations (6) which will presumably be negligible. However, as will be seen below, with the trajectories given by (20)–(25), the $I=1$ channel cannot be neglected even at laboratory energies of 20 GeV corresponding to $N=N$ scattering.

Since the functions $\gamma^I(t)$ and $\alpha^I(t)$ are slowly varying functions of t (for $t<0$), we make the further approximation that the entire t dependence of the invariant amplitude $A^I(t,s)$ in (11) resides in the factor $s^{\alpha^I(t)}$, the remaining factors being evaluated at $t=0$. With these approximations, and the trajectories given above, Eq. (11) for $A^I(t,s)$ then becomes

$$A^{I=0}(t,s)\underset{\substack{s\to\infty\\ t<0}}{\longrightarrow}\tfrac{1}{3}\pi i\gamma^{I=0}(0)s^{\alpha^{I=0}(t)}, \tag{26}$$

$$A^{I=1}(t,s)\underset{\substack{s\to\infty\\ t<0}}{\longrightarrow}4(1+i)\gamma^{I=1}(0)s^{\alpha^{I=1}}(t). \tag{27}$$

After a few lines of algebra, using (26), (27), the crossing relations (6), and definitions of the total cross

[5] J. R. Taylor, Phys. Rev. **127**, 2257 (1962).
[6] G. F. Chew, Phys. Rev. **129**, 2363 (1963).
[7] A. Ahmadzadeh and I. A. Sakmar, Phys. Letters **5**, 145 (1963).
[8] W. J. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, Phys. Rev. **154**, 1515 (1967).

section (8) and diffraction width (9), we have

$$\sigma_{\text{tot}}{}^{I}(s) \underset{s\to\infty}{\longrightarrow} 8\pi^2\gamma^{I=0}(0)+\frac{64\pi\gamma^{I=1}(0)\chi^{I1}}{\sqrt{s}}, \qquad (28)$$

$$\left(\frac{1}{\Delta t}\right)_I = (2\ln s)\left[\frac{a_1+a_2s^{1/2}+a_3s}{b_1+2b_2s^{1/2}+b_3s}\right], \qquad (29)$$

where

$$\begin{aligned} b_1&=2[4\gamma^{I=1}(0)\chi^{I1}]^2,\\ b_2&=2\pi\gamma^{I=0}(0)\gamma^{I=1}(0)\chi^{I1},\\ b_3&=\tfrac{1}{4}\pi^2[\gamma^{I=0}(0)]^2,\\ a_1&=b_1(d\alpha^{I=1}/dt)_{t=0},\\ a_2&=b_2[d\alpha^{I=0}/dt+d\alpha^{I=1}/dt]_{t=0},\\ a_3&=b_3(d\alpha^{I=0}/dt)_{t=0}. \end{aligned} \qquad (30)$$

From the trajectories (20)–(25) and stated approximation for the reduced residue $\gamma^I(t)$ discussed above, we have

$$\begin{aligned}\gamma^{I=0}(0)&\approx 1.15\times10^{-3},\\ \gamma^{I=1}(0)&\approx65.0\times10^{-3}.\end{aligned} \qquad (31)$$

For the purpose of comparison with other calculations,[3] (28) and (29) will be evaluated at the $N=N$ lab energy of 20 GeV or a c.m. energy squared of about $s\approx41$ GeV^2 ($\approx$2100 pion mass units). Evaluating (28) and (29) under these conditions then gives for the total cross section:

$$\begin{aligned} &\qquad\quad I'=0 \qquad\quad I'=1\\ \sigma_{\text{tot}}{}^{I=0}&=1.82 \text{ mb}+5.68 \text{ mb}=7.5 \text{ mb}, \end{aligned} \qquad (32)$$

$$\sigma_{\text{tot}}{}^{I=1}=1.82 \text{ mb}+2.84 \text{ mb}=4.7 \text{ mb}; \qquad (33)$$

here the contributions from the respective crossed channels are indicated by I'. It would, of course, not be meaningful to calculate the $I=2$ channel since it has been neglected in the crossed channel. For the diffraction width we find

$$(\Delta t)^{-1}=14.5 \text{ GeV}^{-2} \quad \text{for} \quad I=0 \qquad (34)$$

and

$$(\Delta t)^{-1}=13.0 \text{ GeV}^{-2} \quad \text{for} \quad I=1. \qquad (35)$$

These results compare favorably with a previous estimate of the total π-π cross section and diffraction width,[3] made with the help of the factorization theorem[9]:

$$\sigma_{\pi\pi}=(\sigma_{\pi N})^2/\sigma_{NN}\approx10 \text{ mb}, \qquad (36)$$

$$\left(\frac{1}{\Delta t}\right)_{\pi\pi}=\left(\frac{2}{\Delta t}\right)_{\pi N}-\left(\frac{1}{\Delta t}\right)_{NN}\approx4 \text{ GeV}^{-2}, \qquad (37)$$

where in the estimates (36) and (37), differences in charge states have been neglected. Our results indicate a cross section somewhat lower than that predicted by the factorization theorem (36), while the diffraction width is smaller by a factor of about 3. Unfortunately, there are no direct experimental results for the π-π scattering yet. However, if the reduced residues $\gamma(t)$ behave for $t<t_{\text{th}}$ as in potential theory, then we expect the approximation (28) to underestimate the total cross section, since in potential theory the reduced residue increases as the energy t becomes more negative. The magnitude of the diffraction widths (34) and (35) can be seen to be a consequence of the fairly low ratio $\gamma^{I=0}(0)/\gamma^{I=1}(0)$, and if the factorization theorem is valid, this ratio will hopefully increase when the trajectories are given more accurately.

It appears therefore that the above procedure of approximating the reduced residue $\gamma^I(t)$ is not inconsistent with any known facts for the π-π system, and may provide a fairly accurate and yet simple means of estimating high-energy diffraction widths and total cross sections.[10]

[9] V. N. Gribov and I. Pomeranchuk, Phys. Rev. Letters **8**, 343 (1962); M. Gell-Mann, *ibid.* **8**, 263 (1962).

[10] The approximation $\beta(\nu) \to_{\nu\to0} \text{Im}\alpha(\nu)/\sqrt{\nu}$ also follows from the early unitarity arguments of Gribov and Pomeranchuk {V. N. Gribov and I. Y. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. **43**, 308 (1962) [English transl.: Soviet Phys.—JETP **16**, 220 (1963)]}.