$=\nu^l B(s,l)]$ is

$$A(s,t)=-\frac{C_1\sin\beta\pi}{2}\int^{\alpha_c(s)}\frac{dl\,\nu^l}{\sin\pi l}[s_c'(l)]$$
$$\times[\alpha_c(s)-l]^\beta[P_l(-z)\pm P_l(z)], \quad (17)$$

where $z=1+t/2\nu$, and a given cut, of course, contributes to only one of the signatured amplitudes. The leading behavior of Eq. (11) for large z is

$$A(s,t)\xrightarrow[z\to\infty]{} C_2 z^{\alpha_c(s)}/(\ln z)^{1+\beta}. \quad (18)$$

This differs from the usual formula only by the presence of β in the denominator. Other forms of B we have studied lead to equally mild modifications, typically involving powers of ln$z$, ln ln$z$, etc. An example of asymptotic behavior like that of Eq. (12) is to be found in the Bethe-Salpeter amplitude in $\lambda\phi^4$ theory. In the ladder approximation, and with the masses of exchanged mesons taken to be zero, a cut determines the asymptotic behavior, and $\beta=\frac{1}{2}$.[10,11]

On the other hand, our results do not agree with published expressions for moving Regge cuts.[3,6–8] However, all these expressions are based on sums of perturbation graphs, and while they have elastic cuts, they do not satisfy elastic unitarity. It is therefore not surprising that in such approximations γ turns out to be infinite at the end of the cut.

We point out that our results, that cut discontinuities vanish and are singular, apply to inelastic thresholds such as s_i. Our conclusions agree with results in the literature.[12]

[10] M. K. Banerjee, L. S. Kisslinger, C. A. Levinson, and I. J. Muzinich (unpublished).
[11] B. W. Lee and A. R. Swift, J. Math. Phys. 5, 908 (1964).
[12] A. J. Dragt and R. Karplus, Nuovo Cimento 26, 168 (1962).

PHYSICAL REVIEW VOLUME 160, NUMBER 5 25 AUGUST 1967

Approximate Formulas for Photoelectric Counting Distributions*

GABRIEL BÉDARD, JANE C. CHANG, AND L. MANDEL
Department of Physics and Astronomy, University of Rochester, Rochester, New York
(Received 26 September 1966; revised manuscript received 19 April 1967)

The validity of a simple approximate formula for the photoelectric counting probability in a thermal optical field, which was proposed by one of us (L.M.) in 1959, is investigated. The formula is based on a generalization of the Bose-Einstein distribution and should hold for light of arbitrary spectral density. It is shown by explicit calculation for three different spectral distributions that the formula holds with good accuracy over a very wide range of conditions. It should therefore prove useful when the spectral distribution of the light being studied is not known.

1. INTRODUCTION

THERE has recently been a good deal of interest in measurements of the probability distribution of photoelectric counts, when light falls on a photodetector.[1–9] For a plane, polarized, quasimonochromatic beam having the statistical properties of thermal light, which is incident normally on the detector, the general expression for the probability $p(n;T)$ that n photoelectrons will be registered in a time interval T may be expressed in the form[10–13]

$$p(n;T)=\int\prod_{\mathbf{k}}\left[d^2v_{\mathbf{k}}\frac{1}{\pi w_{\mathbf{k}}}\right.$$
$$\left.\times\exp(-|v_{\mathbf{k}}|^2/w_{\mathbf{k}})\right]\frac{1}{n!}U^n e^{-U}, \quad (1)$$

where

$$U=\alpha cS\int_t^{t+T}|V(\mathbf{x},t')|^2dt', \quad (2)$$

* This research was supported by the U. S. Air Force Cambridge Research Laboratories, Office of Aerospace Research.

[1] F. A. Johnson, T. P. McLean, and E. R. Pike, in *Physics of Quantum Electronics*, edited by P. L. Kelley, B. Lax, and P. E. Tannenwald (McGraw-Hill Book Company, Inc., New York, 1966), 1st ed., p. 706.
[2] F. T. Arecchi, Phys. Rev. Letters 15, 912 (1965).
[3] C. Freed and H. A. Haus, Phys. Rev. Letters 15, 943 (1965).
[4] A. W. Smith and J. A. Armstrong, Phys. Letters 19, 650 (1966); Phys. Rev. Letters 16, 1169 (1966).
[5] F. T. Arecchi, A. Berné, and P. Burlamacchi, Phys. Rev. Letters 16, 32 (1966).
[6] W. Martienssen and E. Spiller, Phys. Rev. 145, 285 (1966).
[7] W. Martienssen and E. Spiller, Phys. Rev. Letters 16, 531 (1966).
[8] F. T. Arecchi, A. Berné, and A. Sona, Phys. Rev. Letters 17, 260 (1966).
[9] S. Fray, F. A. Johnson, R. Jones, T. P. Mclean, and E. R. Pike, Phys. Rev. 153, 357 (1967).
[10] L. Mandel, Proc. Phys. Soc. (London) 72, 1037 (1958).
[11] L. Mandel, E. C. G. Sudarshan, and E. Wolf, Proc. Phys. Soc. (London) 84, 435 (1964).
[12] P. L. Kelley and W. H. Kleiner, Phys. Rev. 136, A316 (1964).
[13] R. J. Glauber, in *Quantum Optics and Electronics*, edited by C. deWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach Science Publishers, Inc., New York, 1965), 1st ed., p. 65.

and

$$V(\mathbf{x},t)=\frac{1}{L^{3/2}}\sum_{\mathbf{k}} v_{\mathbf{k}} \exp i(\mathbf{k}\cdot\mathbf{x}-ckt).$$

Here the $v_{\mathbf{k}}$ are the eigenvalues of the photon annihilation operators $\hat{a}_{\mathbf{k}}$, α is the quantum efficiency of the detector, $\mathbf{x}$ is any point on its sensitive surface of area S, and $w_{\mathbf{k}}$ is the average photon occupation of the mode of the radiation field labeled by the wave vector $\mathbf{k}$. The integral in (1) is usually very difficult to evaluate analytically, and in general no closed form expressions for $p(n;T)$ are known.

An exception occurs for counting time intervals T which are very short compared with the coherence time T_c, or the reciprocal frequency spread $c/\Delta k$, of the light, when $p(n;T\ll T_c)$ has the well-known Bose-Einstein form[10,13,14]

$$p(n;T\ll T_c)=\frac{1}{[1+\langle n\rangle][1+1/\langle n\rangle]^n}, \qquad (3)$$

where $\langle n\rangle$ is the expectation value of the number of counts registered in the time interval T.

This formula is encountered in statistical mechanics, in connection with the fluctuations of boson numbers within one cell of phase space. The following heuristic argument may indicate why it arises here. The frequency spread $\Delta k/c$ of each photon of the beam leads to a fundamental uncertainty in the "localization time" of each photon of order $T_c\sim c/\Delta k$, which defines the "length" of the unit cell of phase space. Since the observations are limited to a time interval T which is much less than T_c, the measurements correspond to counts of photons within one unit cell of phase space.

2. APPROXIMATE FORMULAS

The foregoing considerations suggest that one might attempt to approximate the more general formula (1) for an arbitrary time interval T by a generalization of (3), corresponding to the distribution of n bosons over a number of cells s of phase space. Thus we write[15]

$$p_M(n;T)=\frac{\Gamma(n+s)}{n!\Gamma(s)}\frac{1}{[1+\langle n\rangle/s]^s[1+s/\langle n\rangle]^n}. \qquad (4)$$

The parameter s, which remains to be determined, can be adjusted so that the second moment of n given by (4) agrees with the second moment of n given by the correct formula (1). This leads to a value of s which is nonintegral in general and is given by

$$s=T^2\Big/2\int_0^T(T-\tau)|\gamma(\tau)|^2d\tau, \qquad (5)$$

where $\gamma(\tau)$ is the normalized autocorrelation function of the optical field amplitude, or the Fourier transform of the spectral distribution.[14] As T becomes very large, s tends to T/T_c, with

$$T_c=2\int_0^\infty|\gamma(\tau)|^2d\tau, \qquad (6)$$

which appears here as a natural measure of coherence time.

Equation (4) with s given by (5) was proposed[15] in 1959 as an approximation to $p(n;T)$, partly on the basis of a mathematical conjecture due to Rice.[16] The formula can be shown to hold in the asymptotic limit $T\to\infty$, but, since $s=1$ for $T\ll T_c$ according to (5), it is at the same time valid also for $T\ll T_c$. This suggests that $p_M(n;T)$ is unlikely to depart too far from the correct probability $p(n;T)$ for any T. Moreover, the second moment of n given by Eq. (4) is necessarily correct, and since no particular form of the spectral distribution was assumed, the formula (4) might be thought to be a reasonable first approximation to $p(n;T)$ over a wide range of conditions. The equation has, however, tended to remain virtually unknown and unused in practice. In the following we shall give several examples illustrating its validity.

More recently, Glauber[13,17] has given an asymptotic formula for $p(n;T)$ when T is large, for the special case in which the spectral profile is Lorentzian, when $|\gamma(\tau)|$ has the well-known form $\exp(-|\tau|/T_c)$. His expression may be written

$$p_G(n;T)=\frac{1}{n!}\left(\frac{\langle n\rangle}{CT_c}\right)^n s_n(CT)\exp[-(C-1/T_c)T], \qquad (7)$$

with

$$C=[1/T_c^2+2\langle n\rangle/TT_c]^{1/2},$$

and

$$s_n(x)=e^x(2x/\pi)^{1/2}K_{n-1/2}(x), \qquad (8)$$

where $K_{n-1/2}(x)$ is the modified Hankel function of half-integral order. Some experimental measurements of $p(n;T)$ obtained with a laser operating below threshold have recently[3] been checked against this formula, although it is not clear how well the spectral distribution in the experiment was approximated by a Lorentzian form. Yet another asymptotic expression for $p(n;T)$ has been suggested by McLean and Pike.[18]

Since experimental measurements of photoelectric counting distributions are now becoming more frequent, and checks against a formula are desirable, it is of interest to examine the accuracies of the approximations (4) and (7). Fortunately, Bédard[19] has recently

[14] See, for example, L. Mandel and E. Wolf, Rev. Mod. Phys. **37**, 231 (1965).

[15] L. Mandel, Proc. Phys. Soc. (London) 74, 233 (1959).

[16] S. O. Rice, Bell System Tech. J. **23**, 1 (1944); **23**, 282 (1944); **24**, 46 (1945).

[17] R. J. Glauber, in *Physics of Quantum Electronics*, edited by P. L. Kelley, B. Lax, and P. E. Tannenwald, (McGraw-Hill Book Company, Inc., New York, 1966), 1st ed., p. 788.

[18] T. P. McLean and E. R. Pike, Phys. Letters **15**, 318 (1965). The accuracy of this approximate formula appears to be somewhat worse than that of the other two.

[19] G. Bédard, Phys. Rev. **151**, 1038 (1966).

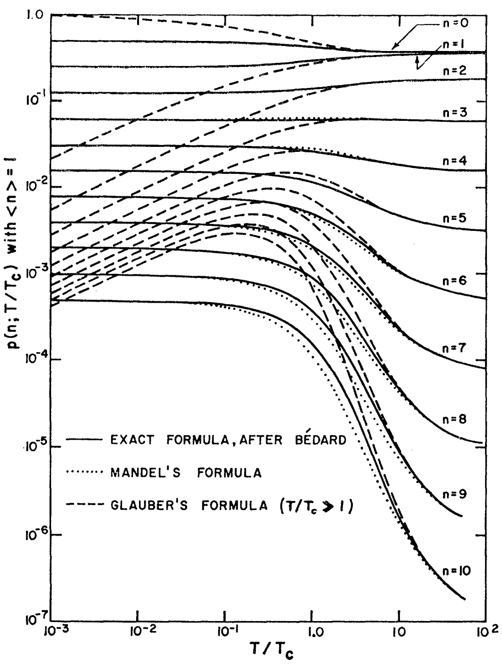

FIG. 1. A comparison of the probability distributions $p_M(n;T)$ and $p_G(n;T)$ with the exact values calculated from Bédard's recurrence relation. The spectral density is Lorentzian. Where the broken curves are not shown they coincide with the full curves.

succeeded in obtaining an exact expression in closed form for the factorial moment generating function of $p(n;T)$, when the spectral distribution of the light is Lorentzian. This led to a recurrence relation for $p(n;T)$ in the form[19]

$$p(n;T)=\sum_{r=0}^{n-1}\frac{(-1)^{n+r+1}}{(n-r)!}D_{n-r}(1)p(r;T)\,, \tag{9}$$

where $D_s(x)$ is given by

$$D_s(x)=\left(\frac{\langle n\rangle T}{zT_c}\right)^s\left[\cosh z+\sinh z\left(\frac{T}{2zT_c}+\frac{zT_c}{2T}\right)\right]^{-1}\times\left[\frac{T}{2T_c}\mathcal{I}_s(z)+z\left(1+\frac{T_c}{2T}\right)\mathcal{I}_{s-1}(z)+\left(\frac{z^2T_c}{2T}\right)\mathcal{I}_{s-2}(z)\right], \tag{10}$$

with

$$z=[2x\langle n\rangle T/T_c+T^2/T_c^2]^{1/2},$$

and $\mathcal{I}_s(x)$ is the modified spherical Bessel function of the first kind of order s.

This formula, which is exact, allows us to examine the accuracies of both the approximate formulas (4) and (7). Such a comparison is illustrated in Fig. 1 for a certain range of values of n and T/T_c with $\langle n\rangle=1$. It will be seen that both $p_M(n;T)$ and $p_G(n;T)$ tend rapidly towards $p(n;T)$ when T/T_c increases beyond 3 or 4. Since the approximate equations were derived for large T/T_c, good agreement for small T/T_c is not of course to be expected. However, it is interesting to note that $p_M(n;T)$ tends to follow the correct curve almost everywhere, as was suggested by the simple argument above.

It is worth noting that, despite the very different structures of the relations (4) and (7), they agree in the asymptotic limit $T\to\infty$. This becomes clear when we examine the factorial moment generating functions defined by

$$Q(x;T)\equiv\langle(1-x)^n\rangle=\sum_{r=0}^{\infty}\frac{\langle n\rangle^{(r)}}{r!}(-x)^r\,, \tag{11}$$

where $\langle n^{(r)}\rangle$ is the rth factorial moment of n. From Eq. (4) we readily find

$$\langle n^{(r)}\rangle_M=\langle n\rangle^r\frac{\Gamma(r+s)}{\Gamma(s)s^r}\,, \tag{12}$$

so that

$$Q_M(x;T)=\sum_{r=0}^{\infty}\frac{s(s+1)\cdots(s+r-1)}{r!}\left(-\frac{x\langle n\rangle}{s}\right)^r=[1+x\langle n\rangle/s]^{-s}. \tag{13}$$

The generating function corresponding to the asymptotic formula (7) has been given by Glauber[13]:

$$Q_G(x;T)=\exp\left[-T\left(\frac{1}{T_c^2}+\frac{2\langle n\rangle x}{TT_c}\right)^{1/2}+\frac{T}{T_c}\right]. \tag{14}$$

If we now approach the limit $T/T_c\to\infty$, while keeping $\langle n\rangle$ fixed, Eq. (14) reduces to

$$Q_G(x;T)=\exp\{-\langle n\rangle x-\tfrac{1}{2}\langle n\rangle^2x^2T_c/T+O[T_c/T]^2\}. \tag{15}$$

On the other hand, from Eq. (5) with

$$|\gamma(\tau)|=\exp(-|\tau|/T_c)\,,$$

it follows that, for sufficiently large T/T_c,

$$s=T/T_c+\tfrac{1}{2}+T_c/4T+O[T_c/T]^2, \tag{16}$$

so that Eq. (13) reduces to

$$Q_M(x;T)=\exp\{-\langle n\rangle x-\tfrac{1}{2}\langle n\rangle^2x^2T_c/T+O[T_c/T]^2\}. \tag{17}$$

We see that the generating functions $Q_M(x;T)$ and $Q_G(x;T)$ coincide to the first order in T_c/T in the exponent.

If the limit $T/T_c\to\infty$ is approached while the mean light intensity is maintained constant, the situation is a little different. Then the parameter $\langle n\rangle T_c/T\equiv\delta$, which is a measure of the average number of counts registered in a time equal to the coherence time, remains constant. From Eqs. (13) and (16) we find, when T/T_c becomes large,

$$Q_M(x;T)=\exp\left\{-\left(\frac{T}{T_c}+\tfrac{1}{2}+\frac{T_c}{4T}+O[T_c/T]^2\right)\times\ln\left(1+\frac{x\delta}{1+T_c/2T+T_c^2/4T^2+O[T_c/T]^3}\right)\right\}$$

$$=\exp\left\{-x\delta\left(\frac{T}{T_c}+O[T_c/T]^2\right)+\tfrac{1}{2}x^2\delta^2\left(\frac{T}{T_c}-\tfrac{1}{2}+O[T_c/T]^2\right)+O[x^3\delta^3]\right\}, \quad (18)$$

while from Eq. (14),

$$Q_G(x;T)=\exp\left\{-x\delta\left(\frac{T}{T_c}\right)+\tfrac{1}{2}x^2\delta^2\left(\frac{T}{T_c}\right)+O[x^3\delta^3]\right\}. \quad (19)$$

The parameter δ is always very small for light from typical thermal sources, and can be shown[14] to have an upper bound given by $(e^{E/kT}-1)^{-1}$, where E is the mean photon energy and T the temperature. Under the usual conditions δ is much less than 10^{-3}, so that any difference between (18) and (19) is unimportant. However, if δ should become larger, then (18) is the slightly better approximation to the asymptotic form of the true generating function given by Bédard,[19] which may be shown to be

$$Q(x;T)=\exp\left\{-x\delta\left(\frac{T}{T_c}\right)+\tfrac{1}{2}x^2\delta^2\left(\frac{T}{T_c}-\tfrac{1}{2}\right)+O[x^3\delta^3]\right\}. \quad (20)$$

3. FACTORIAL MOMENTS

Although the results shown in Fig. 1 are encouraging, and suggest that $p_M(n;T)$ is a good approximation to $p(n;T)$ over a wide range of T/T_c, they refer to only one value of $\langle n\rangle$. Rather than repeat the calculations over a wide range of values of $\langle n\rangle$, we shall consider a simple test which is independent of $\langle n\rangle$. Such a test is provided by the ratio $\langle n^{(r)}\rangle/\langle n\rangle^r$, where $\langle n^{(r)}\rangle$ is the rth factorial moment of n.

We have already found from Eq. (4) that

$$\frac{\langle n^{(r)}\rangle_M}{\langle n\rangle^r}=\frac{\Gamma(r+s)}{\Gamma(s)s^r}, \quad (21)$$

whereas, from the approximate formula (7), it follows that[13]

$$\frac{\langle n^{(r)}\rangle_G}{\langle n\rangle^r}=s_r\left(\frac{T}{T_c}\right). \quad (22)$$

Both ratios are independent of $\langle n\rangle$. The factorial

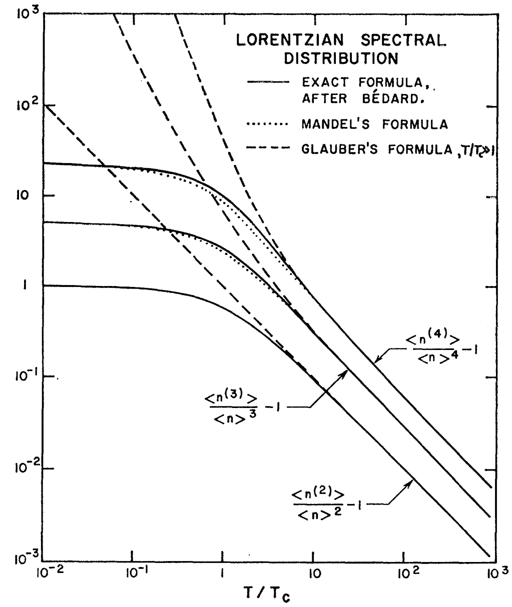

FIG. 2. A comparison of the factorial moments calculated from the approximate probabilities $p_M(n;T)$ and $p_G(n;T)$ with the exact values calculated from Bédard's recurrence relation. The spectral density is Lorentzian. Where the broken curves are not shown they coincide with the full curves.

moments can be compared with the values calculated from Bédard's exact recurrence formula

$$\langle n^{(r)}\rangle=\sum_{s=0}^{r-1}(-1)^{r+s+1}\binom{r}{s}D_{r-s}(0)\langle n^{(s)}\rangle. \quad (23)$$

The results of this comparison are shown in normalized form in Fig. 2. Once again it will be seen that the moments calculated from the formula (4) tend to agree very closely with the correct values. Good agreement is therefore not confined to any particular value of $\langle n\rangle$.

4. OTHER SPECTRAL DISTRIBUTIONS

All the foregoing tests refer to thermal light having a Lorentzian spectral distribution, whereas the arguments leading to Eq. (4) suggest that the formula should hold for an arbitrary spectral distribution. Accordingly we shall now examine the validity of Eq. (21) giving the normalized factorial moments for two other spectral distributions, for which the parameter s takes on values determined by Eq. (5).

In practice the Gaussian spectral density is of particular importance. For a Gaussian spectral density the normalized correlation function is itself Gaussian, and we have

$$\gamma(\tau)=\exp[-\pi^2\tau^2/2T_c^2-2\pi i\nu_0\tau], \quad (24)$$

where ν_0 is the midfrequency of the light. The other spectral density we shall consider is the rectangular

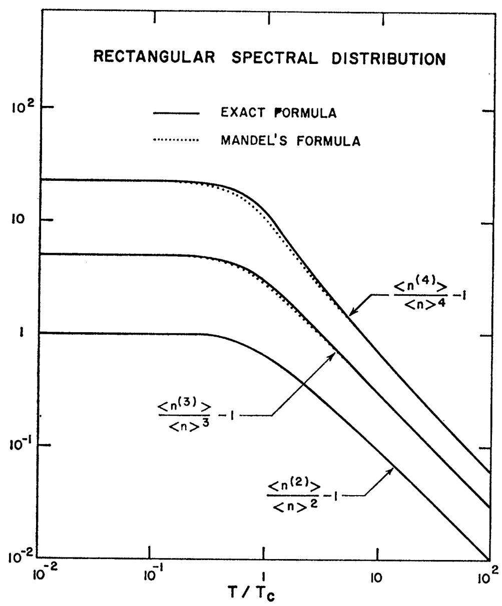

FIG. 3. A comparison of the factorial moments calculated from the approximate probability $p_M(n;T)$ with the exact values calculated from the cumulants. The spectral density is Gaussian. Where the broken curves are not shown they coincide with the full curves.

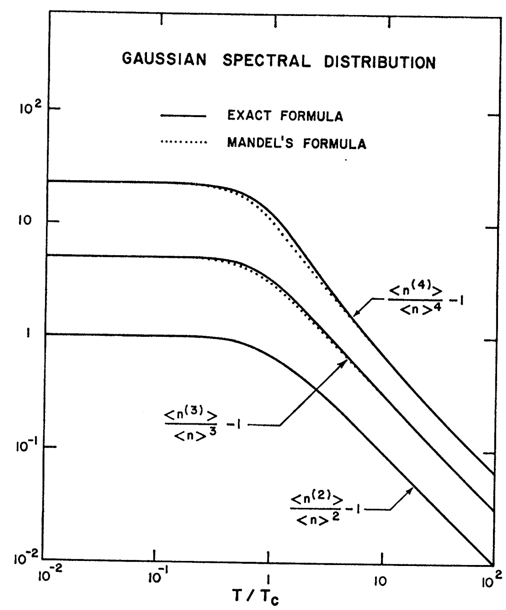

FIG. 4. A comparison of the factorial moments calculated from the approximate probability $p_M(n;T)$ with the exact values calculated from the cumulants. The spectral density is rectangular. Where the broken curves are not shown they coincide with the full curves.

distribution, not because it has any practical importance, but because it represents a rather extreme departure from the Lorentzian form. For a rectangular distribution we find, on taking the Fourier transform,

$$\gamma(\tau)=\frac{\sin(\pi\tau/T_c)}{(\pi\tau/T_c)}\exp(-2\pi i\nu_0\tau). \qquad (25)$$

These equations can now be used to determine s from Eq. (5). The parameters have been adjusted so that both Eqs. (24) and (25) automatically satisfy Eq. (6).

There remains the problem of calculating the factorial moments exactly, for comparison with Eq. (21). To do this we shall make use of some results of Slepian[20] for the cumulants of the random variable U given by Eq. (2), together with some expansions[15,21] which relate the cumulants of U with those of n. It can be shown[14] that, for any spectral distribution,

$$\frac{\langle n^{(2)}\rangle}{\langle n\rangle^2}=1+\kappa_2, \qquad (26)$$

$$\frac{\langle n^{(3)}\rangle}{\langle n\rangle^3}=1+3\kappa_2+\kappa_3, \qquad (27)$$

$$\frac{\langle n^{(4)}\rangle}{\langle n\rangle^4}=1+6\kappa_2+3\kappa_2^2+4\kappa_3+\kappa_4, \quad \text{etc.}, \qquad (28)$$

where κ_r is closely related to the rth cumulant of U and is given by[20]

$$\kappa_r=\frac{1}{T^r}\int_{-(1/2)T}^{(1/2)T}\cdots\int\gamma(t_1-t_2)\gamma(t_2-t_3)\cdots \times\gamma(t_r-t_1)dt_1dt_2\cdots dt_r. \qquad (29)$$

These relations allow us to compute the normalized factorial moments directly for the two spectral distributions in question, and to compare them with the corresponding moments derived from the approximate formula (4).

The results of such a computation are shown in Figs. 3 and 4. Once again it will be seen that there is good agreement between the results calculated from the exact and from the approximate formula, for both Gaussian and rectangular spectral distributions. These conclusions again hold for all values of $\langle n\rangle$.

Our conjecture that the formula (4) holds generally as a good approximation to the counting probability is therefore supported for three different spectral distributions. Moreover, since the expression for $p_M(n;T)$ contains only one parameter s that is not directly measurable (approximated by T/T_c for long time intervals), the formula may prove to be particularly useful in connection with photoelectric counting experiments when the spectral distribution is not known precisely.

ACKNOWLEDGMENT

We would like to acknowledge assistance from E. Martin with some of the computations.

[20] D. Slepian, Bell System Tech. J. **37**, 163 (1958).
[21] L. Mandel, in *Progress in Optics*, edited by E. Wolf (North-Holland Publishing Company, Amsterdam, 1963), Vol. II p. 181.