Elastic Unitarity and Regge-Cut Discontinuities*

J. B. BRONZAN AND C. EDWARD JONES
Laboratory for Nuclear Science and Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts

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We examine the constraints imposed on Regge-cut discontinuities by elastic unitarity. We find that discontinuities must be singular at their endpoints, and, contrary to published examples, must vanish there. We give particular attention to "wrong-signature" negative integer angular momenta in the spinless problem. There, one Regge cut must exactly mask the elastic unitarity cut; its discontinuity contains a pole in the angular momentum. Our results modify the usual expression for the contribution of a cut to high-energy scattering in crossed channels.

RECENTLY there has been interest in the use of Regge cuts as well as poles for the fitting of high-energy data.[1] Cuts were originally proposed by Mandelstam for the purpose of removing the Gribov-Pomeranchuk essential singularity from the physical sheet of the complex angular-momentum plane, and Mandlestam found such cuts in a certain class of diagrams with three-particle intermediate states.[2,3] At present, not very much is known about the discontinuity across Regge cuts; for instance, only recently has it been realized that the discontinuity need not contain the Gribov-Pomeranchuk singularity.[3-5]

In this note we establish two endpoint properties of the discontinuity which follow from elastic unitarity. These are that the discontinuity is singular at the end of the cut, and that it vanishes there. We point out here—and later discuss—that an endpoint behavior of this kind is *not* found in expressions given in the literature for discontinuities.[3,6-8] We give particular attention to Regge-cut discontinuities at "wrong-signature" negative integers. At these points, where Regge cuts serve to remove the Gribov-Pomeranchuk essential singularity, we find that exactly one Regge cut on the physical sheet of the energy plane has a fixed pole in its discontinuity, and we examine the behavior of the residue of the fixed pole at threshold. In what follows, we consider spinless, equal-mass particles, but the form of the arguments is quite general.

Regge cuts in the angular-momentum plane manifest themselves in partial-wave amplitudes as cuts in the s plane whose branch points are functions of l. In particular, as l is reduced from large, real values, a branch point $s_c(l)$ emerges from the first inelastic threshold $s=s_i$ of the partial-wave amplitude and moves toward the elastic threshold $s=s_e$. In Fig. 1, we display this cut and the sheets entered by passing its branch point. We denote the discontinuity across the Regge cut by

$$B(s_1)-B(s_1')=2i\gamma(s,l). \tag{1}$$

Here the amplitude

$$B(s,l)=(\sqrt{s}/\nu^{l+1/2})e^{i\delta_l(s)}\sin\delta_l(s) \quad (4\nu=s-s_e),$$

is a real analytic function of s because of ν^{-l}. We suppress superfluous variables, including signature indices, although of course we are dealing with signatured amplitudes.

Elastic unitarity, continued in s for real l, is

$$[1/B(s_1')]-[1/B(s_2)]=-2i\nu^{l+1/2}/\sqrt{s}. \tag{2}$$

Combining Eqs. (1) and (2), we obtain

$$B(s_1)-B(s_2)-(2i\nu^{l+1/2}/\sqrt{s})B(s_1)B(s_2) =2i\gamma[1-(2i\nu^{l+1/2}/\sqrt{s})B(s_2)]. \tag{3}$$

It is interesting to note that $\gamma[1-(2i\nu^{l+1/2}/\sqrt{s})B(s_2)]$ must be real, thus linking the phases of B and γ. Suppose now that $\gamma(s,l)$ are analytic in s at $s=s_c(l)$. Then we find

$$B(s)=-(1/\pi)\gamma(s)\ln(s_c-s)+b(s), \tag{4}$$

where $b(s)$ is analytic at s_c. However, these amplitudes cannot possibly satisfy Eq. (3) as $s\rightarrow s_c$, because the left side of the equation would be quadratic in $\ln(s_c-s)$, and the right side only linear. We conclude that γ is singular at s_c.

We can be more specific about the behavior of γ at s_c if we adopt the representation

$$B(s,l)=[W(s,l)+Y(s,l)]^{-1}, \quad W(s,l)=-ie^{-i\pi l}\nu^{l+1/2}/[(\cos\pi l)\sqrt{s}]. \tag{5}$$

Here Y is a real, analytic function of s for real l. It

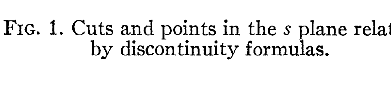

FIG. 1. Cuts and points in the s plane related by discontinuity formulas.

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bears all the dynamical singularities of B other than the elastic branch point, and it has a kinematic cut beginning at $s=0$ in order to cancel the kinematic cut of the function W. Among the singularities of Y is the branch point $s_c(l)$; for $0<s<s_c$, Y is real. In terms of Y,

$$\gamma=-\mathrm{Im}Y/\{[\mathrm{Re}Y+W(s_1)]^2+[\mathrm{Im}Y]^2\}. \quad (6)$$

If $\mathrm{Im}Y$ vanishes at s_c, so does γ. To see this, one need only note that $W(s_1)$ has a nonzero imaginary part. If $\mathrm{Im}Y$ is finite at s_c, $\mathrm{Re}Y$ has a logarithmic singularity there, and γ again vanishes at s_c. Finally, if $\mathrm{Im}Y$ diverges at s_c, it is convenient to write Eq. (6) in the form

$$\gamma=-1/[1+(\mathrm{Re}Y/\mathrm{Im}Y)^2+(W(s)/\mathrm{Im}Y)^2 +2W(s_1)\mathrm{Re}Y/(\mathrm{Im}Y)^2]\mathrm{Im}Y. \quad (7)$$

The bracket in the denominator cannot vanish at s_c because the first two terms, which are positive, dominate the last two terms in magnitude. The discontinuity, γ, therefore, vanishes at s_c, independent of the behavior of $\mathrm{Im}Y$ there. Similar methods yield the same conclusions about γ when $s_c<s_e$.

A special discussion must be given for angular-momentum l_0, where $s_c(l_0)=s_e$. Calculations[3,6–8] of $s_c(l)$ for spinless particles invariably determine l_0 to be a negative integer. This fits in neatly with the requirement that the elastic unitarity cut be masked at wrong-signature negative-integer angular momentum, thereby permitting the Gribov-Pomeranchuk essential singularity to be absent from $B(s,l)$. To analyze the behavior of $\gamma(s,l)$ in the neighborhood of l_0, consider the example of the even-signature amplitude, $B^+(s,l)$. We focus attention on the cut in the energy plane at $s_c(l)$ that just masks the elastic unitarity cut at the first wrong-signature negative integer $l=-1$. Since the thresholds of W and Y are coincident at $l=-1$, it is possible for these functions to cancel. In fact, such a cancellation must occur, because $B^+(s,l)$ is known to have a fixed simple pole at $l=-1$.[4,5] (The fixed pole is all that remains of the Gribov-Pomeranchuk essential singularity when the Regge cut is present.) Without loss of generality, we may write

$$Y^+(s_1,l)=2i\{s[s-s_c(l)]\}^{-1/2}+F^+(s_1,l), \quad (8)$$

and then we find

$$B^+(s_1,l)=[(-ie^{-i\pi l}\nu^{l+1/2}/(\cos\pi l)\sqrt{s}) +2i\{s[s-s_c(l)]\}^{-1/2}+F^+(s_1,l)]^{-1}, \quad (9)$$

where the argument s_1 defines the sheet of the functions as in Fig. 1. For $s\neq s_e$, we find

$$B^+(s_1,l)\xrightarrow[l\to-1]{}(-[i(l+1)/\sqrt{(s\nu)}] \times\{\ln\nu-i\pi-[s_c'(-1)/8\nu]\} +F^+(s_1,l)+O[(l+1)^2])^{-1}. \quad (10)$$

If $B^+(s,l)$ is to have a simple fixed pole at $l=-1$, we require that

$$F^+(s_1,l)=(l+1)y^+(s_1,l). \quad (11)$$

The residue of the fixed pole in $B^+(s_1,l)$ is

$$b(s)=\lim_{l\to-1}(l+1)B^+(s_1,l)=(y^+(s_1,-1)-(i/\sqrt{s\nu}) \times\{\ln\nu-i\pi-[s_c'(-1)/8\nu]\})^{-1}. \quad (12)$$

By means of Eqs. (6), (8), and (11), we find that $\gamma^+(s,l)$ also has a fixed pole at $l=-1$ with residue

$$\lim_{l\to-1}(l+1)\gamma^+(s,l)=b(s)/2i. \quad (13)$$

From these equations we verify that the fixed pole is missing from $B^+(s_1',l)=B^+(s_1,l)-2i\gamma^+(s,l)$, which is a result obtained previously.[4,5] It is important to observe that $y^+(s_1,-1)$ has a branch point at $s=s_e$. Without knowledge of dynamics it is therefore impossible to state the nature of the singularity of $b(s)$ at threshold. In particular there is no reason to believe that this singularity has a square-root character, as has been asserted.[9] It should be noted that unless $y^+(s_1,-1)$ just cancels the second term in Eq. (12) at threshold, $b(s_e)=0$.

If Regge cuts other than $s_c(l)$ are on the physical sheet when $l=-1$, they are contained in $(l+1)y^+(s,l)$, and their discontinuities also contain fixed poles. If $B^+(s,l)$ is to have a fixed pole at $l=-3$, a second Regge cut, $\tilde{s}_c(l)$, must be present with $\tilde{s}_c(-3)=s_e$ to cancel W. This result makes contact with previous work,[4,5] since the Gribov-Pomeranchuk essential singularity is removed from the amplitude only if the Regge cut cannot be deformed away from the elastic unitarity cut. This requires a coincidence of thresholds, and results in the fixed pole in $B^\pm(s,l)$.

So far we have dealt with the singularity of $\gamma(s,l)$ at $s=s_c(l)$ for fixed l. If we write the cut of $B(s,l)$ in the angular-momentum plane as

$$B(s,l)=-\frac{1}{\pi}\int^{\alpha_c(s)}\frac{dl'\bar{\gamma}(s,l')}{l'-l} \qquad (\alpha_c=s_c^{-1}), \quad (14)$$

then $\bar{\gamma}(s,l)=\gamma(s,l)$. Following from the conclusions in the s plane, it is easy to see that the discontinuity is singular and vanishes at the endpoint $l=\alpha_c(s)$ of the cut in the angular-momentum plane.

It is interesting to obtain the consequences of our findings for the asymptotic behavior in the crossed channel. For this purpose, suppose that

$$B(s,l)=C_1[s_c(l)-s]^\beta+f(s), \quad (15)$$

where $\beta>0$ and $f(s)$ is analytic at s_c. Then

$$\gamma(s,l)=-C_1\sin\pi\beta[s-s_c(l)]^\beta\approx-C_1\sin\pi\beta \times[s_c'(l)][\alpha_c(s)-l]^\beta. \quad (16)$$

After a Sommerfeld-Watson transformation, the contribution of the cut to the full amplitude $A(s,t)$ [$A(s,l)$

[9] John H. Schwartz (unpublished).

$=\nu^l B(s,l)]$ is

$$A(s,t)=-\frac{C_1\sin\beta\pi}{2}\int^{\alpha_c(s)}\frac{dl\,\nu^l}{\sin\pi l}[s_c'(l)]$$
$$\times[\alpha_c(s)-l]^\beta[P_l(-z)\pm P_l(z)], \quad (17)$$

where $z=1+t/2\nu$, and a given cut, of course, contributes to only one of the signatured amplitudes. The leading behavior of Eq. (11) for large z is

$$A(s,t)\underset{z\to\infty}{\longrightarrow} C_2 z^{\alpha_c(s)}/(\ln z)^{1+\beta}. \quad (18)$$

This differs from the usual formula only by the presence of β in the denominator. Other forms of B we have studied lead to equally mild modifications, typically involving powers of $\ln z$, $\ln\ln z$, etc. An example of asymptotic behavior like that of Eq. (12) is to be found in the Bethe-Salpeter amplitude in $\lambda\phi^4$ theory. In the ladder approximation, and with the masses of exchanged mesons taken to be zero, a cut determines the asymptotic behavior, and $\beta=\frac{1}{2}$.[10,11]

On the other hand, our results do not agree with published expressions for moving Regge cuts.[3,6–8] However, all these expressions are based on sums of perturbation graphs, and while they have elastic cuts, they do not satisfy elastic unitarity. It is therefore not surprising that in such approximations γ turns out to be infinite at the end of the cut.

We point out that our results, that cut discontinuities vanish and are singular, apply to inelastic thresholds such as s_i. Our conclusions agree with results in the literature.[12]

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Approximate Formulas for Photoelectric Counting Distributions*

GABRIEL BÉDARD, JANE C. CHANG, AND L. MANDEL
Department of Physics and Astronomy, University of Rochester, Rochester, New York
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The validity of a simple approximate formula for the photoelectric counting probability in a thermal optical field, which was proposed by one of us (L.M.) in 1959, is investigated. The formula is based on a generalization of the Bose-Einstein distribution and should hold for light of arbitrary spectral density. It is shown by explicit calculation for three different spectral distributions that the formula holds with good accuracy over a very wide range of conditions. It should therefore prove useful when the spectral distribution of the light being studied is not known.

1. INTRODUCTION

THERE has recently been a good deal of interest in measurements of the probability distribution of photoelectric counts, when light falls on a photodetector.[1–9] For a plane, polarized, quasimonochromatic beam having the statistical properties of thermal light, which is incident normally on the detector, the general expression for the probability $p(n;T)$ that n photoelectrons will be registered in a time interval T may be expressed in the form[10–13]

$$p(n;T)=\int\prod_{\mathbf{k}}\left[d^2v_{\mathbf{k}}\frac{1}{\pi w_{\mathbf{k}}}\right.$$
$$\left.\times\exp(-|v_{\mathbf{k}}|^2/w_{\mathbf{k}})\right]\frac{1}{n!}U^n e^{-U}, \quad (1)$$

where

$$U=\alpha cS\int_t^{t+T}|V(\mathbf{x},t')|^2dt', \quad (2)$$

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