Gribov-Pomeranchuk Poles in Scattering Amplitudes*

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It is shown that the argument of Gribov and Pomeranchuk for the existence of fixed poles in the J plane at "nonsense" values of J goes through in the presence of cuts, even though their argument for an essential singularity then fails. Such poles have no effect on the asymptotic behavior but, in cases where the contribution of the third double-spectral function is large, they will invalidate both the Schwarz superconvergence relations and the presence of dips in the asymptotic region. A Regge trajectory will not choose sense or nonsense at a point where it passes through an integer of the wrong signature.

1. INTRODUCTION

In a well-known paper, Gribov and Pomeranchuk[1] pointed out that a partial-wave amplitude necessarily has singularities in the J plane at "nonsense" values of J, i.e., at all negative integral values of J, and at positive integral values satisfying the inequality $J<\max(\lambda,\lambda')$, where λ and λ' are the incoming and outgoing total helicities. They examined the left-hand cut of the partial-wave amplitude in the s plane, and showed that it had a pole as a function of J, when J assumed a nonsense value. A sense-nonsense matrix element would have a one-over-square-root singularity.[2] It might therefore be expected that the whole amplitude would have a pole in the J plane. However, a unitary amplitude is bounded for real s and J, and it cannot have a fixed pole in the J plane. Gribov and Pomeranchuk showed that the amplitude therefore has an accumulation of poles about nonsense values of J, or, in other words, an essential singularity. Such singularities only occur at integral values of J of the wrong signature, at odd integral values of J for even-signature partial waves and at even integral values of J for odd-signature partial waves.

It was subsequently shown by Mandelstam[3] that the arguments of Gribov and Pomeranchuk must be modified if cuts are present in the J plane, and that the essential singularities do not occur on the physical sheet of the J plane.

Renewed interest in singularities at nonsense integers has recently arisen as a result of the Schwarz[4] superconvergence relations, which we shall mention below. Jones and Teplitz[5] have suggested that a pole is present even when there are cuts in the J plane. They gave arguments which made their suggestion very plausible.

In this paper we wish to point out that the arguments of Gribov and Pomeranchuk for poles (or one-over-square-root singularities) at nonsense integers go through even in the presence of cuts. We thus confirm the suggestion of Jones and Teplitz, but we believe our arguments to be simpler than theirs. We shall contrast our argument for the pole with the subsequent arguments for the essential singularity, which fails in the presence of cuts.

We begin by reminding the reader that a fixed pole in the partial-wave amplitude at a nonsense value of J with the wrong signature has no effect on the asymptotic behavior of the amplitude. We shall not restate the reasons for this fact, which have been given several times before. In essence the full amplitude acquires zeros both from the factor associated with the nonsense value of J and from the signature factor, these zeros cancel the pole in the amplitude and in the factor $1/\sin\pi J$. The infinite accumulation of poles *around* nonsense values of J with the wrong signature does contribute to the asymptotic behavior.

The absence of poles in the scattering amplitude at the values of J under consideration would place a restriction on the scattering amplitude, as was pointed out by Schwarz. He showed that the superconvergence relations in the crossed channel would then be valid for the left- and right-hand cut considered separately. He attempted to fit such relations by truncating at low values of the energy and found that the relations were not satisfied, even in cases where the complete superconvergence relations held.

Another restriction imposed by the absence of poles at nonsense values of J with the wrong signature is the existence of "dips" in the asymptotic behavior. Arbab and Chiu[6] have shown that a Regge trajectory does not contribute to the asymptotic behavior of the scattering amplitude at a value of s where it passes through an integer of the wrong signature and chooses nonsense. They were then able to explain several striking minima in the high-energy scattering data. We shall show that the residue associated with a Regge trajectory has a pole as a function of s at the point where

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[1] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Letters **2**, 232 (1962).

[2] Throughout this paper we shall regard an amplitude as free from singularities at an integer $J=n$ if the sense-sense and nonsense-nonsense elements are finite while the sense-nonsense elements behave like $(J-n)^{1/2}$. A singular amplitude will be an amplitude where the nonsense-nonsense amplitudes behave like $(J-n)^{-1}$ and the sense-nonsense amplitudes like $(J-n)^{-1/2}$. Though we shall not mention the distinction each time, a behavior of the latter type is always implied when we speak of poles at $J=n$.

[3] S. Mandelstam, Nuovo Cimento **30**, 1148 (1963).

[4] J. H. Schwarz, Phys. Rev. **159**, 1269, (1967).

[5] C. E. Jones and V. L. Teplitz, Phys. Rev. **159**, 1271 (1967). Jones and Teplitz (private communication) have shown that their arguments can in fact be used to prove their result.

[6] F. Arbab and C. B. Chiu, Phys. Rev. **147**, 1045 (1966).

the trajectory passes through an integral value of J of the wrong signature. As Jones and Teplitz pointed out, it then follows that the contribution to the asymptotic behavior is not strictly zero. We may still have a minimum if effects due to the third double-spectral function are small.

In Sec. 2 we show that a scattering amplitude with a third double-spectral function has a pole but not an essential singularity on the physical sheet of the J plane at the integers in question. This section really contains nothing new, and its reasoning is implicit in previous papers on essential singularities and on cuts in the J plane. Nevertheless, we feel it worthwhile to go through the reasoning, with emphasis on the points under consideration, in the interests of clarity. In Sec. 3 we examine the residues associated with a Regge trajectory at a point where the trajectory passes through an integer of the wrong signature at which nonsense states are present. We show that the nonsense-nonsense elements have poles in s, and the sense-nonsense elements have one-over-square-root singularities in s, at such values. Residues associated with all trajectories have a similar behavior, and the distinction between those which choose sense and those which choose nonsense no longer exists at an integer of the wrong signature. In Sec. 4 we add a few concluding remarks, with special reference to the significance of "dips."

2. THE GRIGOV-POMERANCHUK POLE

We now show that the discontinuity of a partial-wave amplitude across the left-hand cut in the s plane has a pole in J at nonsense values of the wrong signature. What we shall give is nothing more than a restatement of the argument of Gribov and Pomeranchuk, but we shall emphasize that it is true even in the presence of cuts in the J plane. We shall contrast this argument with the argument for an infinite accumulation of poles on the physical sheet of the J plane, which is invalid in the presence of moving cuts.

The discontinuity across the left-hand cut in the s plane can be expressed as the sum of two terms. One is an entire function of J, the other is given by the formula

$$[a_{\lambda\lambda'}(s,J)]_L = -\frac{1}{2q^2\pi}\int dt\, A_{tu}(s,t) \times\left\{e_{\lambda\lambda'}{}^J\left(1+\frac{t}{2q^2}\right) \pm (-1)^\lambda e_{\lambda,-\lambda'}{}^J\left(-1-\frac{t}{2q^2}\right)\right\}, \quad (2.1)$$

where A_{tu} is the third double-spectral function, q the center-of-mass momentum, and the $e_{\lambda\lambda'}{}^J$'s bear the same relation to the Wigner $d_{\lambda\lambda'}{}^J$'s as the Legendre Q^J's do to the P^J's. The $\pm$ sign is positive for even signature, negative for odd signature. The function $e_{\lambda\lambda'}{}^J$ has a pole at a nonsense value of J (or a square root pole for values of J satisfying $|\lambda'| \leqq J < |\lambda|$ or vice versa). Thus $a_{\lambda\lambda'}$ has a pole at a nonsense value of J. The two terms in the curly bracket of (2.1) add when J has the wrong signature and cancel when J has the right signature. It follows that the partial-wave amplitude has a pole at a nonsense value of J with the wrong signature. For values of s sufficiently small in magnitude the third double-spectral function is known exactly, and the t integral in (1) does not vanish. If the left-hand cut of an amplitude has a pole in J for a range of values of s, it can be shown that the complete amplitude has such a pole. The remainder of the left-hand cut and the right-hand cut cannot give a cancelling contribution except possibly at isolated values of s.

It is now easy to see that the argument goes through even in the presence of cuts. We begin at a value of J sufficiently far to the right, where cuts play no part. Equation (2.1) is then valid. We can now continue analytically in J, and the left-hand side will continue to be given by (2.1), which is an analytic function of J. The only way in which such a conclusion could be altered would be for another cut in the s plane to move onto the left-hand cut as J is varied. However, the motion of cuts in the s plane was studied in Ref. 3, and it was found that the moving cut did not overlap the left-hand cut as J was varied from a large real value to the first nonsense integer. (We have in mind processes where the dominating cuts are due to the trajectories associated with the particles being scattered. We feel that it is unlikely that a cancellation takes place in some processes and not in others.)

Let us now contrast this argument with the argument for an infinite accumulation of ploes. Gribov and Pomeranchuk argue that a unitary scattering amplitude is bounded when s and J are real with s above threshold, and that it can therefore not have a pole at a fixed integral value of J. They show that the right-hand discontinuity of the amplitude has an infinite accumulation of poles around the value of J in question. This argument of Gribov and Pomeranchuk, unlike the original argument for the first pole, involves the right-hand cut in the s plane. Now the moving cuts in the s plane do overlap the right-hand cut as the value of J is decreased to the integer in question. The unitarity equation in the form Im $a = ak^*a$ cannot therefore be used if J is real and sufficiently small, and the argument of Gribov and Pomeranchuk breaks down. In Ref. 3 it was shown that the singularity in question is in fact absent from the simplest diagram where it would be expected.

Our conclusion is thus that a scattering amplitude with a third double-spectral function possesses simple poles at nonsense values of J with the wrong signature, but no accumulation of poles.

Having shown that the scattering amplitude has a pole on the first sheet of the J plane, we can easily find its value on the second sheet by unitarity.[7] When

[7] Our argument at this point parallels that of Jones and Teplitz in their N/D formalism.

J is equal to the value under consideration, the fixed unitarity cut and the moving cut in the s plane will both lie along the real axis starting from threshold. The amplitude on the second sheet of the J plane will correspond to the amplitude between these two cuts in the s plane. Now the change of the amplitude across the fixed cut in the J plane is still given by the unitarity condition

$$[1+2ika_1(s,J)][1+2ika_2(s,J)]=1, \qquad (2.2)$$

where the kinematical factor k is defined to be positive just above the fixed right-hand cut on the first sheet, and therefore negative just below the cut on the first sheet. If we first consider a negative value of J, where all states are nonsense states and all matrix elements of a_1 have a pole, we see from (2.2) that

$$a_2(s,J)=-1/2ik, \quad J=n. \qquad (2.3)$$

There is thus no singularity at $J=n$. At a value of J where sense and nonsense states are present, the matrix elements involving the nonsense states will have the behavior (2.3), while those involving only sense states will have an arbitrary finite value (unless a Regge trajectory passes through $J=n$ on the second sheet at the value of s under consideration). There is no Gribov-Pomeranchuk essential singularity on either the first or the second sheet. We thus confirm the suggestions of Jones and Teplitz regarding the behavior of $a(s,J)$ on the second sheet.

3. SINGULARITIES OF THE REGGE RESIDUE

We now show that the residue associated with a Regge trajectory has the behavior

$$\beta_{ss}\approx c_1, \qquad (3.1a)$$

$$\beta_{sn}\approx c_2(s-s_1)^{-1/2}, \qquad (3.1b)$$

$$\beta_{nn}\approx c_3(s-s_1)^{-1}, \qquad (3.1c)$$

where s_1 is the value at which the trajectory goes through an integral value of J of the wrong signature. The subscripts s and n refer to the sense and nonsense channels, respectively.

Our method will be to examine a case in which the third double-spectral function is small, so that terms involving the square of the third double-spectral function may be neglected. The result will then be a direct consequence of those already established. By working with an example with a small third double-spectral function we are able to avoid complications due to cuts in the angular-momentum plane, since diagrams with cuts contain the third double-spectral function at least twice. We can therefore use the unitarity condition for nonintegral J:

$$[a(s,J)]=ka_1(s,J)a_2(s,J), \qquad (3.2)$$

where the subscripts 1 and 2 refer to the first and second sheets in the s plane. If we were working to second order in the third double-spectral function we would not be able to use Eq. (3.2).

To first order in the third double-spectral function, we may write

$$[a(s,J)]^{(1)}=ka_1^{(0)}(s,J)a_2^{(1)}(s,J)+ka_1^{(1)}(s,J)a_2^{(0)}(s,J), \qquad (3.3)$$

where the superscripts refer to the order of smallness in the third double-spectral function. Let us examine the first term of (3.3). The factor $a_1^{(0)}$ will have a Regge pole at $J=\alpha(s)$:

$$a_1^{(0)}(s,J)=\frac{\beta^{(0)}(s)}{J-\alpha(s)}+\text{nonsingular terms}. \qquad (3.4)$$

Since there is no third double-spectral function involved in $\beta^{(0)}$, the elements will have one of the two behaviors

$$\beta_{ss}^{(0)}\sim c_1, \quad \beta_{sn}^{(0)}\sim c_2(s-s_1)^{1/2}, \quad \beta_{nn}^{(0)}\sim c_3(s-s_1), \qquad (3.5a)$$

or

$$\beta_{ss}^{(0)}\sim c_1'(s-s_1), \quad \beta_{sn}^{(0)}\sim c_2'(s-s_1)^{1/2}, \quad \beta_{nn}^{(0)}\sim c_3'. \qquad (3.5b)$$

The quantity $a_2^{(1)}(s,J)$ will have no pole at $J=\alpha(s)$, since we are on the second sheet in the s plane, but it will have the behavior

$$a_{2ss}^{(1)}\sim c_1'', \quad a_{2sn}^{(1)}\sim c_2''(j-n)^{-1/2}, \quad a_{2nn}^{(1)}\sim c_3''(j-n)^{-1}. \qquad (3.6)$$

Thus, combining Eqs. (3.3)–(3.6), we find that

$$[a(s,J)]^{(1a)}=\frac{\beta^{(1a)}(s)}{J-\alpha(s)}, \qquad (3.7)$$

where

$$\beta_{ss}^{(1a)}\sim k_1, \quad \beta_{sn}^{(1a)}\sim k_2(s-s_1)^{-1/2}, \quad \beta_{nn}^{(1a)}\sim k_3, \qquad (3.8a)$$

or

$$\beta_{ss}^{(1a)}\sim k_1', \quad \beta_{sn}^{(1a)}\sim k_2'(s-s_1)^{-1/2}, \quad \beta_{nn}^{(1a)}\sim k_3'(s-s_1)^{-1}. \qquad (3.8b)$$

The superscript a indicates that we are examining the first term of (3.3). Equations (3.8a) and (3.8b) correspond to (3.5a) and (3.5b), respectively. Since the amplitude $a(s,J)$ on the second sheet in the s plane has no pole at $J=\alpha(s)$, we can conclude from (3.7) that the amplitude $a(s,J)$ on the first sheet has a pole at $J=\alpha(s)$ whose residue β has the behavior (3.8).

Finally we can consider the second term of (3.3). The reasoning just given shows that the first factor $a_1^{(1)}(s,J)$ will have a pole at $j=\alpha(s)$, and the resisdue of the pole will behave as indicated in (3.8). The second factor $a_2^{(0)}(s,J)$ will have the behavior

$$a_{2ss}^{(0)}\sim c_1''', \quad a_{2sn}^{(0)}\sim c_2'''(s-s_1)^{1/2}, \quad a_{2nn}^{(0)}\sim c_3'''. \qquad (3.9)$$

We thus find that $\beta^{(1b)}$ behaves like $\beta^{(1a)}$. By use of the factorization theorem, we then see that β_{nn} must have a pole at $s=s_1$ in higher orders. The alternative that β_{ss} has such a pole is excluded by the reasoning of Ref. 3, which shows that there are no fixed powers in the asymptotic behavior. The factorization theorem is valid in the presence of cuts, as may be shown by analytic continuation from high values of J.

We can easily see by *reductio ad absurdum* that the singularities contributed to β by the two terms in (3.3) cannot cancel against one another. For, if we assume a cancellation, we conclude that the β corresponding to the first factor in the second term of (3.3) is finite at $j=n$. The second factor again behaves as in (3.9). Thus the second term of (3.3) gives a contribution to β which is finite at $j=n$ and the singularity of the first term cannot be cancelled.

We have no proof that the singularity of the Regge residue does not cancel through some mechanism as yet unknown. However, in the absence of such an unknown mechanism the β's would be expected to behave as has been indicated above, and we have no reason to believe that a cancellation exists.

It should be emphasized that multiple poles of the scattering amplitude do not occur at the values of J under consideration when higher-order terms in the third double-spectral function are taken into account. (We are assuming that no elementary particles are present.) This follows from the reasoning of Ref. 3, where it is shown that the corresponding terms in the asymptotic behavior are absent. It is important to mention this point, since one familiar mechanism for the cancellation of a pole is the occurrence of multiple poles in higher terms of a perturbation series. In such a case the pole may move from its original position when the series is summed. Such a mechanism does not occur in our example.

We thus conclude that β_{sn} has a one-over-square-root singularity, and β_{nn} a pole, at a value of s where a trajectory passes through an integer of the wrong signature. The residues $\beta^{(1)}$ have such a behavior whether $\beta^{(0)}$ has the behavior (3.5a) or (3.5b), so that there is no precise distinction between trajectories which choose sense and those which choose nonsense at an integer of the wrong signature. There may still be an approximate distinction if effects due to the third double-spectral function are small.

Another point worth mentioning is that the Pomeranchuk trajectory now does contribute to forward Compton scattering. It had been pointed out by Mur[8] and by Abarbanel and Nussinov[9] that only nonsense states contributed to this process, so that the nonsense wrong-signature dip reduced the contribution to zero. According to the reasoning of this section, the contribution is no longer zero, and the difficulties pointed out by Mur no longer exist, even if cuts in the J plane are neglected.

[8] V. D. Mur, Zh. Eksperim. i Teor. Fiz. 44, 2173 (1963); **45**, 1051 (1963) [English transls.: Soviet Phys.—JETP **17**, 1458 (1963); **18**, 727 (1964)].

4. CONCLUDING REMARKS

We first observe that the scattering amplitude does not have any effective singularities at integral values of J of the wrong signature with nonsense channels. We could redefine the scattering amplitude with an extra factor $(J-n)^{-1/2}$ in a sense-nonsense element and a factor $(J-n)^{-1}$ in a nonsense-nonsense element. Such an amplitude would normally be finite at $J=n$. A zero in the amplitude would correspond to a restriction, a pole to an observable term in the asymptotic behavior.

One must now re-examine the significance of the experimental "dips" in the asymptotic behavior of scattering amplitudes at momentum transfers where the Regge trajectory passes through an integer of the wrong signature. In the presence of a third double-spectral function the term in the asymptotic behavior associated with a particular Regge pole will no longer contain a zero at such a point. Nevertheless, if the effects of the third double-spectral function are not too large we might still expect a dip with a minimum not too far from the point in question. If the contribution of the third double-spectral function is large, one would expect the cuts in the J plane to give appreciable contributions, and one would not expect the scattering amplitude to have a Regge asymptotic behavior. One might therefore conclude that dips should still be present in an amplitude which has a Regge asymptotic behavior. One would probably expect dips in some channels and not in others, but, if they occur in a number of cases at the expected values, one would be justified in explaining them in the usual way. The gross failure of the Schwarz superconvergence relation, while it may well be due to truncation at too low a value, should be taken as a warning against a consistent neglect of effects of the third double-spectral function.

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[9] H. D. I. Abarbanel and S. Nussinov, Phys. Rev. **158**, 1462 (1967).