Contour Distortions in Relativistic Three-Particle Scattering Equations*

J. NUTTALL
Texas A & M University, College Station, Texas
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It is shown how to relate the physical three-baryon scattering amplitude as given by a generalized Bethe-Salpeter equation to the solution of an integral equation with an L^2 kernel. This is done by demonstrating that the contours for the integration variables in a perturbation contribution to the amplitude may be distorted so that no propagator vanishes. The method is an extension of techniques used earlier by Tiktopoulos for the two-particle Bethe-Salpeter equation. The results hold for center-of-mass energies less than half way to the threshold for production of a meson.

1. INTRODUCTION

AMONG the difficulties encountered in connection with the integral equations of relativistic, off-mass-shell scattering theory is that caused by the vanishing of the propagators. This complicates a numerical solution and stops the kernels from having useful formal properties, such as being L^2 or compact. This paper begins an investigation of the problem when three-particle states are involved. For simplicity, we assume that the three-particle channel is the one of lowest mass (we call the particles "baryons") and that the next threshold is due to three baryons and a meson. We hope to consider coupled two- and three-particle channels later. Under certain assumptions, discussed below, we show that the physical three-particle elastic scattering amplitude may be written in terms of the solution of an integral equation with an L^2 kernel.

We study generalized Bethe-Salpeter equations such as those discussed by Taylor[1] where three-particle intermediate states are exposed. The complete three-body scattering amplitude is given in terms of the irreducible three-body amplitude and the two-body scattering amplitude, and our approach is to assume that the last two amplitudes are known. We make use of analytic properties that they would have in perturbation theory.

The technique employed is analogous to that used in the nonrelativistic problem (Rubin, Sugar, and Tiktopoulos,[2] Nuttall[3]), where we investigate a contribution to the amplitude obtained by iterating the integral equations. We must show that the contours of integration for the variables appearing explicitly may by distorted, without altering the integral, to positions where the three-particle propagators do not vanish. Moreover, the distortions must be such that the resulting terms may be resummed by an integral equation.

We are able to find satisfactory displaced contours so long as the center-of-mass (c.m.) energy does not lie beyond half-way to the first inelastic threshold. If both the initial and final states correspond to points lying within a region Δ of the Dalitz plot, the contour, and corresponding integral equation, is simpler. We found an analogous result in the nonrelativistic three-body problem.[3]

For the two-particle Bethe-Salpeter equation, Tiktopoulos[4] found that a satisfactory contour displacement existed for c.m. energies below the first inelastic threshold. The method that we describe here for the three-body problem fails to work for a similar range mainly because of the meson pole contribution to the crossed channel of the two-body scattering amplitude. To accommodate this, a more complicated contour distortion will be needed and we hope to return to this question later.

Throughout we disregard all problems concerned with large momentum behavior and renormalization. We omit self-energy contributions to the propagators in the exposed states—their inclusion would not affect the substance of the argument. We assume that the necessary amplitudes decrease fast enough to lead to L^2 kernels.

It has already been demonstrated by Schwartz and Zemach[5] that a numerical solution of the two-body Bethe-Salpeter equation is feasible if the equation is first transformed into one with an L^2 kernel. Their coordinate-space method of achieving this is actually equivalent to the contour-distortion method of Tiktopoulos.[4] In the same way we hope that the present work may be useful in connection with the numerical solution of three-particle off-mass-shell equations.

We begin by studying under what conditions it is possible to carry out the Wick rotation[6] of the internal energy variables in a general Feynman integral. The results are used later to demonstrate the necessary analyticity of the "known" two-particle and three-particle irreducible amplitudes. In Sec. 3 we obtain a satisfactory distorted contour for a special choice of the external variables, and the results are extended in Sec. 4 to include all physical external variables. There is a brief discussion of integral equations in Sec. 5.

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[1] J. G. Taylor, Phys. Rev. **150**, 1321 (1966).
[2] M. Rubin, R. Sugar, and G. Tiktopoulos, Phys. Rev. **146**, 1130 (1966).
[3] J. Nuttall (unpublished).
[4] G. Tiktopoulos, Phys. Rev. **136**, B275 (1964).
[5] C. Schwartz and C. Zemach, Phys. Rev. **141**, 1454 (1966).
[6] G. C. Wick, Phys. Rev. **96**, 1124 (1954).

2. WICK ROTATIONS

Before analyzing the three-body scattering problem we need a result concerning the Wick rotation[6] of all the internal energy variables of the Feynman integral derived from a connected diagram which may have any number of external lines. Let the real external momenta be denoted by $\hat{k}^{(i)}$, $i=1, \cdots, n$, $(\sum \hat{k}^{(i)}=0)$, the loop momenta to be integrated over $\hat{p}^{(i)}$, $i=1, \cdots, l$ and the momenta of the internal lines $\hat{q}^{(i)}$, $i=1, \cdots, m$. These momenta are related by equations of the form

$$\hat{q}^{(i)}=\hat{K}^{(i)}+\textstyle\sum_j \epsilon_{ij}\hat{p}^{(j)}, \tag{1}$$

where $\hat{K}^{(i)}$ is a linear combination of the external momenta and $\epsilon_{ij}=-1$, 0 or 1. Suppose that the mass of the particle corresponding to the ith internal line is μ_i.

The result specifies a region $\mathfrak{D}$ of the space of the external energy variables $k_0^{(i)}$ for which the Wick rotation of the internal energy variables is possible. To determine $\mathfrak{D}$ we group the external momenta into two sets $\{\hat{k}^{(i)}:i\in I\}$ and $\{\hat{k}^{(i)}:i\notin I\}$, I a set of integers, defining a channel, and find the mass m_I of the lowest mass intermediate state in channel I. The region $\mathfrak{D}$ is the set of all $k_0^{(i)}$ satisfying

$$\Big|\sum_{i\in I} k_0^{(i)}\Big|<m_I \quad \text{for all } I. \tag{2}$$

We demonstrate that a given set $(k_0^{(i)})$ belongs to $\mathfrak{D}$ by showing that it is possible to find a set of loop energies $p_0^{(i)}=\bar{p}_i$ and corresponding internal line energies $q_0^{(i)}=\bar{q}_i$ such that

$$|\bar{q}_i|<\mu_i, \quad i=1, \cdots, m. \tag{3}$$

We transform $p_0^{(i)}$ to π_i by

$$p_0^{(i)}=\bar{p}_i+e^{i\theta}\pi_i \tag{4}$$

and using (1) we have

$$q_0^{(i)}=\bar{q}_i+e^{i\theta}\textstyle\sum_j \epsilon_{ij}\pi_j. \tag{5}$$

Initially $\theta=0$ and each π_i runs from $-\infty$ to ∞.

The denominator associated with the ith line is $[\hat{q}^{(i)}]^2-\mu_i^2+i\epsilon$ which vanishes at

$$q_0^{(i)}=\pm[\mu_i^2+\mathbf{q}^{(i)2}-i\epsilon]^{1/2}. \tag{6}$$

If each $\mathbf{q}^{(i)}$ remains real and θ varies from 0 to $\frac{1}{2}\pi$, it is easy to see that no denominator vanishes so long as (3) is obeyed. Thus the Wick rotation is possible and results in an integration contour

$$\operatorname{Re} p_0^{(i)}=\bar{p}_i, \quad \mathbf{p}^{(i)} \text{ real}, \tag{7}$$

with the corresponding relation for $q^{(i)}$

$$\operatorname{Re} q_0^{(i)}=\bar{q}_i, \quad \mathbf{q}^{(i)} \text{ real}. \tag{8}$$

We now must find the shape of the region $\mathfrak{D}$ which may be defined as the set of $k_0^{(i)}$ for which it is possible to find $q_0^{(i)}=\bar{q}_i$ satisfying (3). $\mathfrak{D}$ is not empty, for it includes the origin, $k_0^{(i)}=0$ with $\bar{q}_i=0$. On the other hand, $\mathfrak{D}$ cannot extend beyond the limit laid down by (2), for if Q is the sum of the internal energies of an intermediate state for channel I, then

$$Q=\sum_{i\in I} k_0^{(i)}, \tag{9}$$

and (3) shows that $|Q|$ must be less than the sum of the masses of the particles in the intermediate state. Another useful property of $\mathfrak{D}$ is that if a set $(k_0^{(i)})\in\mathfrak{D}$ then all sets of the form $(tk_0^{(i)})$ with $0\leq t\leq 1$ also belong to $\mathfrak{D}$. To prove this we replace the $(\bar{p}_i)$ which correspond to $(k_0^{(i)})$ by $(t\bar{p}_i)$, and (1) shows that all $\bar{q}_i$ are multiplied by t, so that (3) are still satisfied.

To determine whether or not a given set $(k_0^{(i)})\in\mathfrak{D}$ we consider moving along a straight line from the origin, $k_0^{(i)}=0$ to the point $(k_0^{(i)})$ in the space $\mathcal{K}$ of all $k_0^{(i)}$. For each set $(k_0^{(i)})$ near enough to the origin there is a region $S(\hat{k})$ in the space $\mathcal{P}$ of all $p_0^{(i)}$ in which all the restrictions (3) are obeyed. In other words $S(\hat{k})$ contains all allowed $(\bar{p}_i)$.

When $k_0^{(i)}=0$ we know $S(\hat{k})$ is not empty and we take any $(p_0^{(i)})$ inside $S(\hat{k})$ which will satisfy our conditions until $(k_0^{(i)})$ moves in such a way as to make $(p_0^{(i)})$ lie on one of the edges of $S(\hat{k})$. This implies that for this set $(k_0^{(i)})$, for a certain set J of internal lines

$$q_0^{(i)}=x_i\mu_i, \quad i\in J, \qquad x_i=\pm 1, \tag{10}$$

and

$$|q_0^{(i)}|<\mu_i, \quad i\notin J. \tag{11}$$

The region $S(\hat{k})$ will still not be empty if we can change $p_0^{(i)}$ by $\delta p_0^{(i)}$ in such a way as to change $q_0^{(i)}$ so that

$$x_i\delta q_0^{(i)}<0 \quad \text{for all} \quad i\in J. \tag{12}$$

Using (1) we deduce that

$$q_0^{(i)}\delta q_0^{(i)}=q_0^{(i)}\textstyle\sum_j \epsilon_{ij}\delta p_0^{(j)}<0 \quad \text{for all} \quad i\in J. \tag{13}$$

From this point the argument follows that of Stapp[7] to the point where we see that it is possible to satisfy (12) unless there exists a set of positive α_i satisfying the Landau equations[8]

$$\sum_{\text{closed loops}} \alpha_i q_0^{(i)}=0$$

and

$$q_0^{(i)}=\pm\mu_i \quad \text{for all} \quad i\in J. \tag{14}$$

If these equations have no solution $S(\hat{k})$ is still nonempty and will remain so until $(k_0^{(i)})$ are chosen to satisfy them.

The equations (14) that we have obtained are merely those which give the location of singularities of Feynman integrals in one-dimensional space-time. The only singular surfaces in $\mathcal{K}$ space with positive α_i are the

[7] H. P. Stapp, Phys. Rev. **125**, 2139 (1962).

[8] See, for instance, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Company, Inc., New York, 1965).

normal thresholds that we have already stated are the boundaries of $\mathfrak{D}$.

Assuming as always the convergence of all integrals at infinity, our result shows that the Feynman integral is an analytic function of its external variables in $\mathfrak{D}$. This domain of analyticity may easily be extended in two ways. First we may allow $k_0^{(i)}$ to be complex so long as the real part of $k_0^{(i)}$ satisfy the requirements (2). For after the Wick rotation we have Re $p_0^{(i)}=\bar{p}_i$ and our previous argument shows that the real part of (6) cannot hold.

It is also possible to allow some of the $\mathbf{k}^{(i)}$ to become complex under the transformation

$$\mathbf{k}^{(i)} \to e^{i\phi}\mathbf{k}^{(i)}, \tag{15}$$

with ϕ small, without making $\mathfrak{D}$ much smaller, so long as the $\mathbf{k}^{(i)}$ that remain real astisfy

$$|\mathbf{k}^{(i)}|<L, \tag{16}$$

where L is some fixed limit. To show this, we rotate at the same time all the internal $\mathbf{p}^{(i)}$ by the same transformation as (15). Any $\mathbf{q}^{(i)}$ will now have the form

$$\mathbf{q}^{(i)}=\mathbf{K}+e^{i\phi}\mathbf{P}, \tag{17}$$

where $\mathbf{K}$ and $\mathbf{P}$ are real and $\mathbf{K}$ is bounded. It is easy to see that

$$\mathrm{Re}[\mathbf{q}^{(i)}]^2 \geq -K^2 \sin^2\phi/\cos^2\phi, \tag{18}$$

which may be made as small as we please by choosing small enough ϕ. Consequently $\mathrm{Re}[\mu_i^2+\mathbf{q}^{(i)2}]^{1/2}$ is not much less than μ_i. The new $\mathfrak{D}$ is therefore specified by (2) with each μ_i replaced by $\mu_i-\epsilon_i$, where ϵ_i is some small quantity.

3. CONTOUR DISTORTIONS

In this section we begin the program we have outlined in the Introduction by studying contour deformations for diagrams contributing to the physical three-baryon elastic scattering amplitude [Fig. 1(a)] with the external variables fixed at special values. In addition we impose the condition maintained throughout this paper, that the total c.m. energy is not beyond half-way to the first inelastic threshold.

More explicitly, we investigate the structure of the three-baryon intermediate states in a perturbation-theory contribution to the amplitude. An arbitrary term must consist of a chain of two-body scattering diagrams and three-body irreducible diagrams linked together in any order, except that the same pair of particles must not interact twice in succession. An example is shown in Fig. 2. In this figure we have used the symbol of Fig. 1(b) to represent the sum of all connected contributions to the two-baryon scattering amplitude while that of Fig. 1(c) denotes the three-baryon irreducible part of the three-baryon scattering amplitude. This is defined, following Taylor,[1] as the sum of all connected contributions to the three-body amplitude which do not contain three-particle intermediate states.

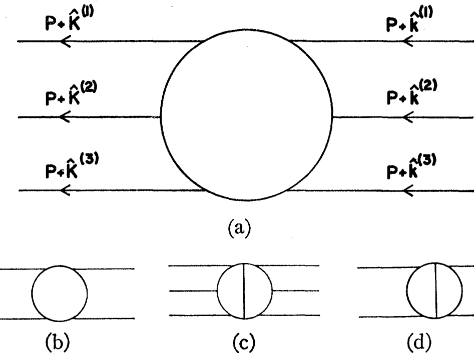


FIG. 1. (a) Three-baryon scattering amplitude; (b) Two-baryon scattering amplitude; (c) Irreducible three-baryon scattering amplitude; (d) Irreducible two-baryon scattering amplitude.

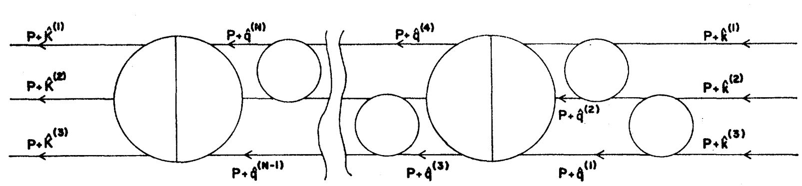
We shall need to make use of certain analyticity properties as functions of their external momentum variables possessed by the amplitudes of Figs. 1(b) and 1(c). These properties will be derived on the assumption that Fig. 1(c) is a finite sum of perturbation contributions. For the two-body scattering amplitude we shall make use of the iteration of the ordinary Bethe-Salpeter equation, which states that the amplitude is the sum of chains of any number of two-baryon irreducible amplitudes, Fig. 1(d). Again we shall make use of analyticity properties of Fig. 1(d) deduced from perturbation theory.

Each of the internal lines in Fig. 2 corresponds to a propagator which we shall take to be $(\hat{q}^2-m^2+i\epsilon)^{-1}$, with m the baryon mass, disregarding self-energy effects. The inclusion of these terms, apart from renormalization problems, would not lead to any serious modifications of our argument.

Our approach is to assume that the irreducible amplitudes of Figs. 1(b) and 1(d) are known and study the problem of finding the complete three-body scattering amplitude in terms of them. Our scheme will not calculate some terms of the type of Fig. 2 with a small number of links in them, and these must be evaluated separately.

We shall work in the c.m. frame where the total

FIG. 2. Typical contribution to three-baryon scattering amplitude with three-baryon states exposed.



energy is $3P$ and we label the external momenta as shown in Fig. 1(a). Thus the following 4-vector equations hold:

$$\hat{k}^{(1)}+\hat{k}^{(2)}+\hat{k}^{(3)}=\hat{K}^{(1)}+\hat{K}^{(2)}+\hat{K}^{(3)}=0. \quad (19)$$

For simplicity in this section let us only consider a special set of values of the external variables with

$$k_0^{(i)}=K_0^{(i)}=0, \quad i=1,2,3$$

and (20)

$$|\mathbf{k}^{(i)}|=|\mathbf{K}^{(i)}|=k \quad \text{say, } i=1,2,3,$$

where k is fixed.

For physical scattering we need $P^2=k^2+m^2$, but we first assume that P is real and less than m. In this case the external energies are clearly within the region $\mathfrak{D}$ of Sec. 2 for any connected three-body diagram and we may therefore perform the Wick rotation on all the internal energies in each of the parts of Fig. 2. After the rotation each $q_0^{(i)}$ has a fixed real part, and if $P<m$, these may all be taken to be zero. With this choice the requirements of (3) are met for all lines shown explicitly in Fig. 2. For the other lines we may apply the result of Sec. 2 to the different amplitudes making up the chain of Fig. 2, with the particles in the exposed three-baryon states playing the role of the external lines.

After the Wick rotation none of the denominators for the three particles states vanish, and the amplitudes of Figs. 1(b) and 1(c) are analytic functions of their external variables where they are required. Moreover, all the $q^{(i)}$ are integrated over the same contour, $\mathrm{Re}q_0^{(i)}=0$ and $\mathbf{q}^{(i)}$ real, so that it is straightforward to sum all diagrams of the type of Fig. 2 by means of an integral equation. The equation will be a generalization of the Watson-Faddeev equations[9] to include the effects of the three-body irreducible amplitude, but the variables will lie on the contour mentioned above.

We must investigate whether this situation still holds for values of $P>m$. Of course we know that, as long as the three-body amplitude Fig. 1(a) is not singular, it must be possible to distort the contours over which the internal variables are integrated to make the denominators nonvanishing. However, there is no certainty that the distorted contour will be a topological product of the same contour for each of the $\hat{q}^{(i)}$ variables, independent of the order of the term considered. We shall show that this is indeed the case for a range of values of $P>m$, providing that $\hat{k}^{(i)}$, $\hat{K}^{(i)}$ are also restricted. It is then quite easy to write the required integral equations. In the next section we show that a more complicated contour distortion is necessary to deal with the values of the external momenta, but that if $3P$ is less than half-way to the energy of the next threshold, an integral equation may still be derived.

We shall prove that all the terms of Fig. 2 are continued analytically to values of $P>m$ by distorting each $q_0^{(i)}$ contour from the imaginary axis to the one

[9] See, for instance, K. M. Watson and J. Nuttall, *Topics in Several Particle Dynamics* (Holden-Day, San Francisco, California, 1967).

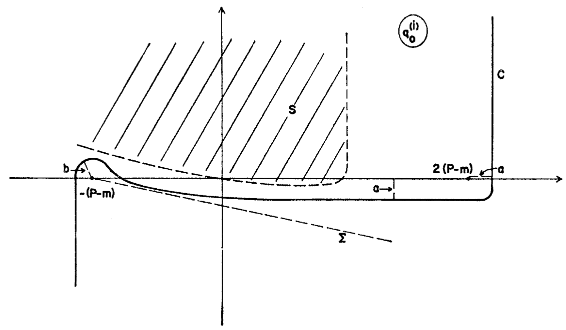


FIG. 3. Standard distorted $q_0^{(i)}$ contour C.

shown in Fig. 3, which we call contour C. In this figure, a and b are small, with $a>b$, and the exact shape of C near real, negative $q_0^{(i)}$ will be discussed later. If the ϵ in each denominator is allowed to approach zero, we shall also have to distort the $\mathbf{q}^{(i)}$ contours slightly, and we shall use a rotation

$$\mathbf{q}^{(i)}\rightarrow e^{-i\phi}\mathbf{q}^{(i)}, \quad (21)$$

with ϕ a small positive angle.

Let us first verify that none of the denominators corresponding to particles in the exposed three-baryon states of Fig. 2 vanishes. The types of denominator involved are

$$(P+\hat{q})^2-m^2+i\epsilon, \quad (22a)$$

$$(P-\hat{q}-\hat{p})^2-m^2+i\epsilon, \quad (22b)$$

$$(P-\hat{q}-\hat{k})^2-m^2+i\epsilon, \quad (22c)$$

where q_0, p_0 lie on C, $\mathbf{q}$, $\mathbf{p}$ obey (21), and $\hat{k}$ is one of the 4-vectors associated with an external particle, which means that $\mathbf{k}$ is real and for the moment $k_0=0$. The denominators vanish when the following equations are satisfied:

$$q_0=-(P\pm[\mathbf{q}^2+m^2-i\epsilon]^{1/2}), \quad (23a)$$

$$q_0+p_0=P\pm[(\mathbf{q}+\mathbf{p})^2+m^2-i\epsilon]^{1/2}, \quad (23b)$$

$$q_0=P-k_0\pm[(\mathbf{q}+\mathbf{k})^2+m^2-i\epsilon]^{1/2}. \quad (23c)$$

The small parameters ϕ, a, and b must be chosen so that none of these equations is satisfied. Suppose we first take the positive sign in (23a) and (23b). With q_0 and p_0 on C, it is clear that the real parts of these equations cannot hold if

$$4(P-m)<P+m,$$

i.e.,

$$3P<5m. \quad (24)$$

We shall henceforth assume this requirement, which means that the c.m. energy is below that needed for production of a baryon-antibaryon pair.

If we take the negative signs in (23a) and (23b), we must study the imaginary parts of q_0 and q_0+p_0 given by these equations. From (23a) we find, using (21) with ϕ small,

$$\mathrm{Im}q_0\approx-\phi q^2/(q^2+m^2)^{1/2}, \quad (25)$$

so that q_0 lies on the curve Σ of Fig. 3. Having fixed ϕ, we choose a and the shape of C near real, negative q_0 to make C lie just above Σ in this region, which means that

$$a=\phi(P^2-m^2)/P-\delta, \qquad (26)$$

where δ is small and positive. With this choice (23a) will never hold.

With q_0 and p_0 on C we observe that q_0+p_0 must lie in a region to the right of the contour shown in Fig. 4. As above, we find that $\text{Im}(q_0+p_0)$ determined from (23b) will be large enough to make q_0+p_0 of (23b) avoid this region if

$$b<2P(P-m)\phi/(2P-m). \qquad (27)$$

Now we analyze Eq. (23c) in the same way, remembering that we are working with $k_0=0$ and $k^2=P^2-m^2$. Again the positive sign leads to no trouble, but with the negative sign we find that q_0 has an imaginary part given by

$$\text{Im}q_0=\phi(q^2+\mathbf{q}\cdot\mathbf{k})/[(\mathbf{q}+\mathbf{k})^2+m^2]^{1/2}, \qquad (28)$$

where $\mathbf{q}$ is now real. The values of q_0 given by (23c) with the negative sign lie in the region S of Fig. 3. We note that $\text{Im}q_0$ may be negative for $\text{Re}q_0$ between 0 and $P-m$ corresponding to q between $[P^2-m^2]^{1/2}$ and 0, but we may show that δ may be chosen small enough to ensure $|\text{Im}q_0|<a$. Thus to show that no q_0 given by (23c) lies on C we need only investigate values of $\text{Re}q_0$ near $-(P-m)$ and make sure that here $\text{Im}q_0>b$. This will be so if (27) is replaced by a stronger condition

$$b<x(x-k)\phi/(2P-m), \qquad (29)$$

where $x^2=4P(P-m)$. Clearly (26), (29), and the condition $a>b$ are easy to satisfy simultaneously.

So far we have proved that, with our particular choice of external momenta, a distorted contour for $\hat{q}^{(i)}$ may be found on which none of the denominators (22) vanishes, even if $\epsilon=0$. To show that the amplitude of Fig. 2 is analytically continued by this process we must also demonstrate that the iterated amplitudes that appear are analytic for the necessary values of their external variables.

Take first the case of the three-baryon irreducible amplitude of Fig. 1(c). The six external momenta needed must be chosen from the following types

$$P+\hat{q},\quad P+\hat{k},\quad P-\hat{q}^{(1)}-\hat{q}^{(2)},\quad P-\hat{k}-\hat{q}^{(1)}, \qquad (30)$$

depending on whether the amplitude appears at the end or in the middle of Fig. 2. We showed at the end of

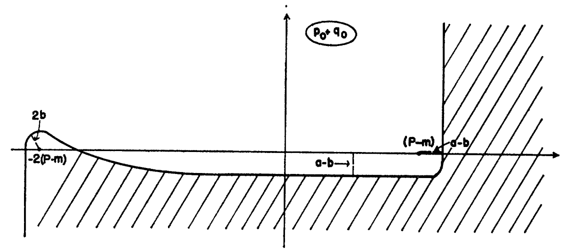


FIG. 4. Range of values taken on by (p_0+q_0), shown shaded.

Sec. 2 that a small rotation of some of the external three-momenta did not affect the essential conclusions of the analysis. Consequently we may deduce that the irreducible amplitude is analytic in the region $\mathfrak{D}$ depending on the variables of the type

$$\text{Re}(P+q_0),\quad P+k_0,\quad \text{Re}(P-q_0^{(1)}-q_0^{(2)}),\quad \text{Re}(P-k_0-q_0). \qquad (31)$$

We suppose the next threshold is due to a state consisting of three baryons and a meson of mass μ. In the crossed channels the irreducible three-baryon amplitude will have lowest-mass intermediate states of mass μ and $\mu+m$. Using the restrictions (2) we find that, wherever they appear in Fig. 2, the irreducible amplitudes wil be analytic so long as

$$6(P-m)<\mu,$$

i.e.,

$$3P<3m+\tfrac{1}{2}\mu. \qquad (32)$$

The critical restriction (32) comes for instance in the crossed channel with momenta (into diagram) of $P-\hat{q}^{(1)}-\hat{q}^{(2)}$ and $-P-\hat{q}^{(3)}$.

Thus with our present contour we are able to continue only half-way to the next threshold at $3P=3m+\mu$ (we assume $\mu<2m$). The problem of how to modify the contour to enable us to continue all the way to the next threshold is deferred to another paper.

It remains to show that the two-particle scattering amplitudes [Fig. 1(b)] entering into Fig. 2 are also analytic on the distorted contours. The result of Sec. 2 does not immediately suffice, and we are forced to analyze the two-baryon intermediate state structure of Fig. 1(b), which may be written as a sum of terms involving iterations of the irreducible two-body amplitude Fig. 1(d). An example is shown in Fig. 5. We cannot use the results of Tiktopoulos[4] on the Bethe-Salpeter equation, for the external vectors are not real, but we may apply the same technique that he used and that we have used earlier in this section.

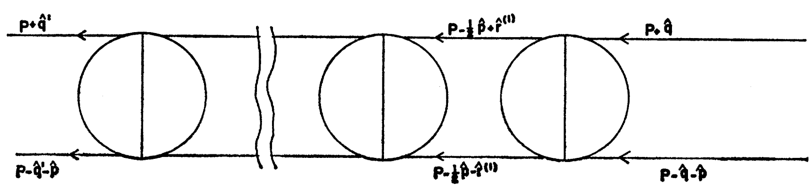


FIG. 5. A contribution to the two-baryon scattering amplitude, formed by iterating several irreducible two-baryon amplitudes.

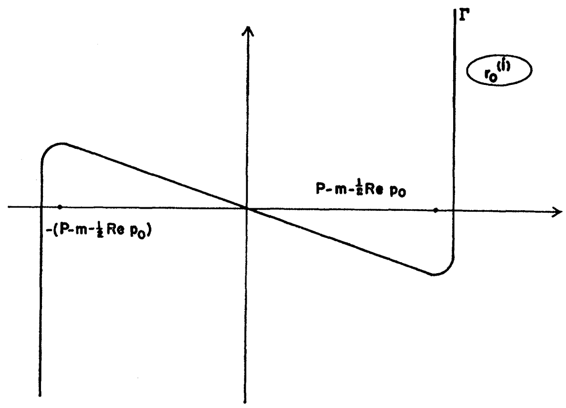


FIG. 6. Distorted contour Γ for $r_0^{(i)}$.

The external momenta for the two-body scattering amplitude are shown in Fig. 5. Here $\hat{p}$, $\hat{q}$, and $\hat{q}'$ are either on the distorted contour or are one of $\hat{k}^{(i)}$, $\hat{K}^{(i)}$. Suppose first that they all lie on the distorted contour. We specify a distorted contour for the internal momenta $\hat{r}^{(i)}$ of Fig. 5 and show that the necessary denominators do not vanish. We let each $r_0^{(i)}$ lie on the contour Γ of Fig. 6, whose shape depends on the value of p_0. As before, we rotate each $\mathbf{r}^{(i)} \to e^{-i\phi}\mathbf{r}^{(i)}$. An analysis analogous to the previous discussion shows that none of the internal denominators of Fig. 5, $[P\pm\hat{r}^{(i)}]^2-m^2+i\epsilon$, vanishes with this choice of contours. We now apply the result of Sec. 2 to demonstrate the necessary analyticity of the two-baryon irreducible amplitudes occurring in Fig. 5. The various types of total energy appearing in the crossed channels are $r_0^{(1)}\pm r_0^{(2)}$ and $q_0+\frac{1}{2}p_0\pm r_0$. With our choice of $r_0^{(i)}$ contour the requirements of (2) are satisfied if

$$3P<3m+\mu\,, \tag{33}$$

and the same result follows from the direct channel energy $2P-p_0$. Thus this continuation works all the way up to the next threshold, so long as the sum of all terms of the Fig. 5 has the same analyticity as an individual term. However, the restriction (32) again appears when we study the contribution of just one irreducible two-body amplitude. If any of $\hat{p}$, $\hat{q}$, $\hat{q}'$ are $\hat{k}^{(i)}$, $\hat{K}^{(i)}$ the above considerations certainly still hold when $k_0^{(i)}=0$.

This completes the discussion of the contour distortions for the special case when the external variables satisfy the conditions (20) and (32). In Sec. 5 we discuss how to use our results to obtain integral equations leading to the three-body amplitude but first we investigate the problem of extending the values of the external variables for which our method works.

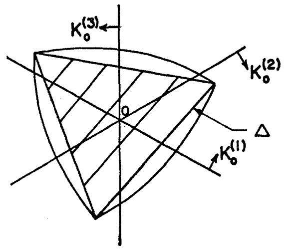


FIG. 7. The Dalitz plot showing the physical region and the region Δ.

4. EXTERNAL OTHER MOMENTA

We now consider whether our previous analysis may be extended when we allow the $k_0^{(i)}$, $K_0^{(i)}$ to vary from zero, but maintain the condition $3P<3m+\frac{1}{2}\mu$. It is convenient to use a Dalitz plot[10] to describe the values of $k_0^{(i)}$ and a similar one for the $K_0^{(i)}$. So far we have been working with both $(k_0^{(1)})$ and $(K_0^{(i)})$ at O, the center of the physical region in the plot of Fig. 7. The boundary of the physical region is given by the equation

$$(k^{(1)})^4+(k^{(2)})^4+(k^{(3)})^4-2[(k^{(1)}k^{(2)})^2+(k^{(2)}k^{(3)})^2+(k^{(3)}k^{(1)})^2]=0 \tag{34}$$

and the extreme allowed values of any particular k_0 are

$$k_0=-(P-m)\,,\quad (P-m)(P+m)/2P\,. \tag{35}$$

If k_0 varies from zero, with k^2 given by

$$k^2+m^2=(P+k_0)^2\,, \tag{36}$$

then the values of q_0 given by (23c) are changed, and in particular the shape of the region S of Fig. 3 alters. From (28) we may deduce that Imq_0 from (23c) may be negative for values of q_0 in the range

$$-2k_0\leq q_0\leq P-m-k_0\,. \tag{37}$$

We must ensure that S never intersects C, which from (37) certainly means that k_0 must lie within range

$$-(P-m)+\delta'\leq k_0\leq \tfrac{1}{2}(P-m)-\delta'\,, \tag{38}$$

with δ' small. In fact a little algebra shows us that if k_0 satisfies (28), δ and b may be chosen small enough to ensure that S actually does not intersect C.

The result is that, if (38) holds for all $k_0^{(i)}$, $K_0^{(i)}$, the denominators of the exposed three-particle states do not vanish on the distorted contour, but we must still verify that the component amplitudes of Fig. 2 are analytic. Our previous analysis applies except where an external momentum variable is involved. This case is

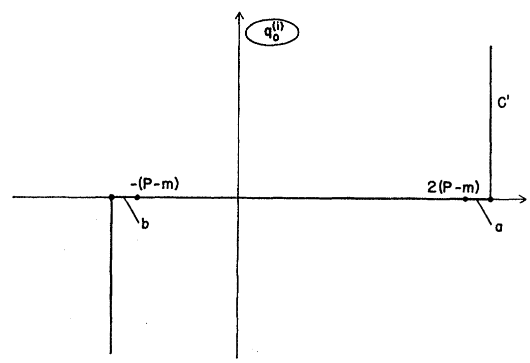


FIG. 8. Contour C', useful for nonzero ϵ.

[10] See, for instance, G. Källén, *Elementary Particle Physics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1964).

easily treated by the method of the previous section and again leads to the condition $3P<3m+\frac{1}{2}\mu$.

Consequently, as long as both $(k_0^{(i)})$ and $(K_0^{(i)})$ lie inside a region contained in the triangular part Δ of the physical region in Fig. 7, all terms of the Fig. 2 type may be written as integrals over the distorted contours we have described, on which no denominator vanishes.

We have obtained an analogous result in the nonrelativistic case,[3] and, to find a satisfactory contour distortion for $k_0^{(i)}$, $K_0^{(i)}$ lying outside Δ, we shall follow the procedure that worked before. It is not now possible to distort all the integration contours in the same way. Instead we continue to displace in the way we have described all $\hat{q}^{(i)}$ except the two nearest the ends of the diagram of Fig. 2, $\hat{q}^{(1)}$ and $\hat{q}^{(N)}$. We shall show that for a given $\hat{k}^{(i)}$, $\hat{K}^{(i)}$, other displacements of the $\hat{q}^{(1)}$, $\hat{q}^{(N)}$ contours may be found which provide the required continuation.

To start this task we consider a term such as Fig. 2 before the ϵ in each Feynman propagator has been allowed to tend to zero. A study of Eqs. (23) shows that no denominator will vanish (with nonzero ϵ) if each $q_0^{(i)}$ lies on the contour C' of Fig. 8, while the $\mathbf{q}^{(i)}$ are all real. Let us assume that the initial part of the term we are considering consists of two-particle scattering amplitudes as shown in Fig. 9. As $\epsilon\to 0$ we displace the $q_0^{(2)}$ contour to C and $\mathbf{q}^{(2)}\to e^{-i\phi}\mathbf{q}^{(2)}$ and for each $\hat{q}^{(2)}$ consider whether a satisfactory distortion of $\hat{q}^{(1)}$ exists.

Let us use the notation of Eqs. (23) and set

$$\hat{q}^{(1)}=\hat{q},\quad \hat{q}^{(2)}=\hat{p},\quad \hat{k}^{(3)}=\hat{k},\tag{39}$$

so that the three denominators we must study are those of (22). If $\mathrm{Im}p_0$ is not near zero, then (23b) cannot be satisfied and we need only find a distortion that avoids (23a) and (23c). Of course, any distortion need only be small and we may use the familiar argument leading to the Landau equations (see Stapp[7]) to show that a $\hat{q}$ displacement exists unless there is a pinch between (23a) and (23c). This can only happen at the normal threshold

$$(2P-\hat{k})^2=4m^2$$

or

$$k_0=(P^2-m^2)/2P\tag{40}$$

which lies on the edge of the physical region and must be avoided.

If $\mathrm{Im}p_0$ is near zero we must also consider the possibility of a pinch involving all three denominators. This

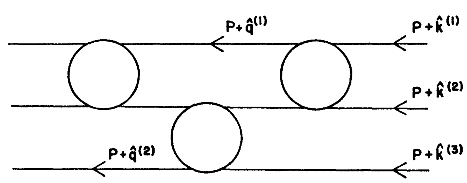


FIG. 9. Beginning of a particular contribution to the three-baryon amplitude.

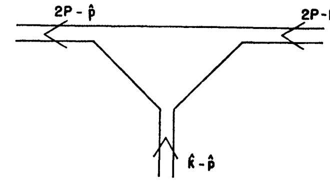


FIG. 10. Equivalent triangle diagram.

is a definite danger, for we know[2] that there exist real $\hat{p}$, $\hat{k}$ for which the triangle graph of Fig. 10 (wihch is equivalent to what we are studying) is singular on the physical sheet—i.e. with all the Feynman ϵ's approaching zero through positive values. We must study particularly those values of $\hat{p}$ with real parts near this singularity and show that a small displacement to the $\hat{p}$ on our contour does not lie in the complex part of the singular surface. Since the equation of the singular surface has real coefficients this may be done by studying the directions of tangents to its real section, as described by Eden.[11]

Since the displacement $\mathbf{p}\to e^{-i\phi}\mathbf{p}$ may be thought of as one where the direction of $\mathbf{p}$ remains unchanged but $p=|\mathbf{p}|$ develops an imaginary part, we assume that the angle between $\mathbf{p}$ and $\mathbf{k}$ is fixed and study the shape of the singular curve in (p_0,p) space with fixed $\hat{k}$. In terms of the invariants α and β given by

$$\alpha=Q^2-p^2,\quad \beta=Qz-pkc,\tag{41}$$

$$Q=2P-p_0,\quad z=2P-k_0,\quad c=\mathbf{p}\cdot\mathbf{k}/pk,\tag{42}$$

the singular surface is given by

$$K^2\alpha^2-2K^2\alpha\beta+(K^2-4m^2)K^2\alpha+4m^2\beta^2=0,\tag{43}$$

with $K^2=z^2-k^2$.

In α, β space this is the hyperbola shown in Fig. 11.

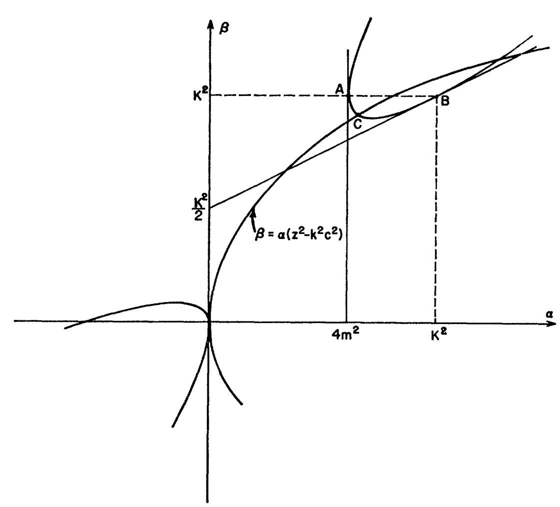


FIG. 11. The singular hyperbola in (α,β) space, with AB singular on the physical sheet. The region to the left of the parabola $\beta^2=\alpha(z^2-k^2c^2)$ corresponds to real (Q,p).

[11] R. J. Eden, *Brandeis University Summer Institute, Lecture Notes 1, 1961* (W. A. Benjamin, Inc., New York, 1962).

It is well known, as described by Polkinghorne,[12] that only the part of the curve lying between A and B corresponds to a physical sheet singularity. Not all of this curve will transform into a real curve in (p_0,p) [or (Q,p)] space. For a given α, β we may solve (41) for Q to give

$$Q=(z^2-k^2c^2)^{-1}\{z\beta\pm kc[\beta^2-\alpha(z^2-k^2c^2)]^{1/2}\}, \quad (44)$$

which will be real only if

$$\beta^2\geq\alpha(z^2-k^2c^2). \quad (45)$$

Thus (α,β) must lie to the left of the parabola shown in Fig. 11.

Each point of the interesting part of the singular curve transforms into two points in (Q,p) space, and for values of k_0 lying in the range not dealt with by our earlier method,

$$\tfrac{1}{2}(P-m)\leq k_0\leq(P^2-m^2)/2P, \quad (46)$$

we have the situation shown in Fig. 12. The branch of the singular curve of interest lies within the hyperbola $\alpha=4m^2$, touching it on two points A' and A'', both corresponding to A. Only the part between A' and A'' is singular on the physical sheet, and this may not exist at all if c is too small. If c is negative, $A'A''$ lies below the Q axis and corresponds to negative p, of no interest. It may be shown that A' and A'' have Q's lying between $2m$ and $3P-m$, which means that

$$-(P-m)\leq p_0\leq 2(P-m), \quad (47)$$

i.e., within the range of $\text{Re}p_0$ on the contour C.

Now suppose that the singular curve is given by the equation $f(p_0,p)=0$, and that (p_0,p) is a real point on

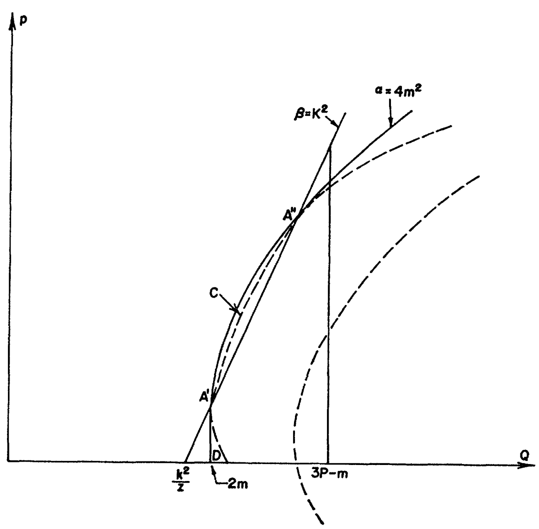


FIG. 12. The singular curves in (Q,p) space. The part between $A'A''$ is singular on the physical sheet.

[12] J. C. Polkinghorne, *Brandeis University Summer Institute, Lecture Notes 1, 1961* (W. A. Benjamin, Inc., New York, 1962).

this curve. A nearby point $(p_0+i\delta p_0,\ p+i\delta p)$ will also be on the curve if

$$i[\delta p_0(\partial f/\partial p^0)+\delta p(\partial f/\partial p)]=0. \quad (48)$$

Except when p_0 is near to $-(P-m)$ or $2(P-m)$, points on our contour C have real negative δp_0 and δp, or correspondingly positive δQ and negative δp. They clearly do not satisfy (48), for Fig. 12 shows that tangent vectors to the singular curve have both δQ and δp positive.

The special case of $k_0=\frac{1}{2}(P-m)$ must be considered separately, for then $A'=D$ and if $c=1$, $A''=E$, and δp_0 may be positive. Near D, the singular curve has the approximate form

$$p_0=2(P-m)-\lambda p^4 \quad (49)$$

and with $p\rightarrow e^{-i\phi}p$, the values of p_0 given by (49) do not intersect C (see Fig. 13). In the same way, for small enough b, we find that the singular values of p_0 near $p_0=-(P-m)$ lie above C.

We therefore have shown that for each (p_0,p) on our distorted contour, a displacement of the $\hat{q}$ contour exists which makes the denominators of the graph Fig. 9 nonzero (the above argument must also be applied to normal threshold singularities to complete the discussion). It is necessary to point out that for large q, we may take q_0 on C and $\mathbf{q}\rightarrow e^{-i\phi}\mathbf{q}$, for (23a) and (23c) cannot vanish while we have already shown (23b) does not vanish in this case. Only for smaller q do we need a different displacement. This result is important when we consider the analyticity of the two-body scattering amplitudes in Fig. 9 on our distorted contour. The arguments of Sec. 2 may be modified slightly to show again that the two-body amplitudes are indeed analytic where required so long as $3P<3m+\frac{1}{2}\mu$.

If an irreducible three-body amplitude appears near the ends of Fig. 2, the denominators involved will be just a subset of those for a diagram containing a succession of two-body scatterings, so that the distortion we have described above will work for all cases. It is only necessary to distort $\hat{q}^{(1)}$ and $\hat{q}^{(N)}$ in a nonstandard manner.

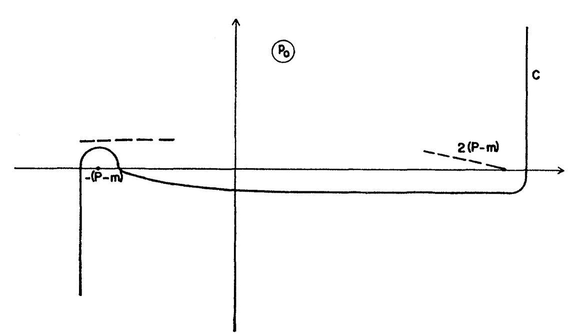


FIG. 13. The parts of the singular curve shown by dashed lines near $-(P-m)$ and $2(P-m)$, for the special case $k_0=\frac{1}{2}(P-m)$, $c=1$. Note that they do not intersect contour C.

5. INTEGRAL EQUATIONS

It is a relatively straightforward matter to formally sum all but a few contributions to the scattering amplitude by means of an integral equation with variables on our distorted contour. The kernel will be analytic and, with satisfactory behavior for large momenta, will be L^2. We assume the baryons are not identical (the equations could easily be symmetrized) and use an obvious extension of the Watson-Faddeev[9] technique.

We define a set of amplitudes $T_{ij}^{(\alpha\beta)}(\hat{p},\hat{q};\hat{p}',\hat{q}')$ to represent various contributions to the full three-baryon scattering amplitude. Here $\alpha(\beta)=0$, 1, 2, 3 describe the nature of the last (first) part of Fig. 2, with $\alpha=0$ meaning a three-body irreducible amplitude and other values indicating a two-body amplitude with particle α not being scattered. The ith particle ($i=1$, 2, 3) in state α has momentum $P-\hat{p}-\hat{q}$, and the other two have $P+\hat{p}$, $P+\hat{q}$, and similarly for j, β and $\hat{p}'$, $\hat{q}'$. If $\alpha=1$, 2, 3 then i will not equal α and we take particle α to be the one with momentum $P+\hat{p}$. If $\alpha=0$, p and q are assigned in cyclic fashion. All the momenta $\hat{p}$, $\hat{q}$, $\hat{p}'$, $\hat{q}'$ lie on the distorted contour $p_0\in C$ and $\mathbf{p}\rightarrow e^{-i\phi}\mathbf{p}$. In addition we shall need the amplitudes $t_{ij}^{(\alpha)}(\hat{p},\hat{q};\hat{p}',\hat{q}')$ which describe two-body scattering ($\alpha=1$, 2, 3) and the three-body irreducible amplitude ($\alpha=0$). For $\alpha\neq 0$, $t^{(\alpha)}$ will contain a factor $\delta^4(\hat{p}-\hat{p}')$.

The equations satisfied by $T_{ij}^{(\alpha\beta)}$ are

$$T_{ij}^{(\alpha\beta)}(\hat{p},\hat{q};\hat{p}',\hat{q}')=\delta_{\alpha\beta}t_{ij}^{(\alpha)}(\hat{p},\hat{q};\hat{p}',\hat{q}')+\sum_{\gamma/\alpha}\int d^4\hat{p}''d^4\hat{q}''t_{ik}^{(\alpha)}(\hat{p},\hat{q};\hat{p}'',\hat{q}'')G_\alpha(\hat{p}'',\hat{q}'')T_{kj}^{(\gamma\beta)}(\hat{p}'',\hat{q}'';\hat{p}',\hat{q}'),\quad(50)$$

where

$$G_\alpha(\hat{p},\hat{q})=\{[(P+\hat{p})^2-m^2][(P+\hat{q})^2-m^2]\times[(P-\hat{p}-\hat{q})^2-m^2]\}^{-1},\quad\text{for}\quad\alpha=0$$

$$G_\alpha(\hat{p},\hat{q})=\{[(P+\hat{q})^2-m^2][(P-\hat{p}-\hat{q})^2-m^2]\}^{-1},\quad\text{for}\quad\alpha\neq 0.\quad(51)$$

The symbol under the sum over γ means that we include all values if $\alpha=0$ but omit α if $\alpha\neq 0$. We choose k to be not equal to either α or γ. If $\alpha\neq 0$ and $\gamma=0$ we take $k=3$, 1, 2 for $\alpha=1$, 2, 3.

The kernel of this equation consists of analytic functions and its square will be L^2 just as in the nonrelativistic case so long as the necessary integrals converge at infinity. We must now relate the physical scattering amplitude to the quantities $T_{ij}^{(\alpha\beta)}$. It is simplest to do this when both the initial and final external variables lie inside the region Δ of the Dalitz plot. We showed in the previous section that a satisfactory continuation of a term was provided by distorting all internal momenta in standard fashion. Consequently, the sum of all contributions to the three-body scattering amplitude except those containing up to four terms may be written

$$T=\sum_\alpha\sum_\beta\sum_{\gamma/\alpha}\sum_{\delta/\gamma}\sum_{\gamma/\alpha}\sum_{\nu/\delta}\sum_{\delta/\beta}t^{(\epsilon)}G_\epsilon t^{(\gamma)}G_\gamma\times T^{(\alpha\beta)}G_\delta t^{(\delta)}G_\nu t^{(\nu)}.\quad(52)$$

For ease of writing we have not filled in the other indices or the momentum variables, which are allotted in a straightforward manner. The internal variables are integrated over the distorted contour, while the external variables are pairs of $\hat{k}^{(i)}$ and $\hat{K}^{(i)}$.

A similar procedure works when the external variables do not both lie within the region Δ. It is necessary to place three $t^{(\alpha)}$ on either side of the $T^{(\alpha\beta)}$ in an expression analogous to (52) and the integrations involved will be over contours different from the standard one.

By defining the operators in a Banach space rather than a Hilbert space as above, we could include more of the first few terms.

6. CONCLUSIONS

Our work leaves many questions unanswered. We would like to extend the allowed energy range to $3P<3m+\mu$, but meson pole terms in crossed channels of the two-body and three-body irreducible amplitudes prevent this. Above $3P=3m+\frac{1}{2}\mu$ these terms need a different, more complicated distortion which is not a topological product of distorted contours for the different internal variables. This makes it harder to derive integral equations. For the parts not containing the meson pole, our present contour is probably the "best" possible.

If this problem is overcome, the next step might be to study systems where both two- and three-particle intermediate states are exposed. In the present example this would mean exposing the (two-baryon +meson) state in the baryon-baryon scattering amplitude. There is a possibility that the two-particle results may be generalized to the case of n particles. It may be that by exposing all intermediate states with numbers of particles up to and including n, that the method will provide well-behaved equations for energies up to the $(n+1)$ particle threshold. An understanding of physical region singularities will be important for a study of any such generalization.

ACKNOWLEDGMENT

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