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Fixed-u Sum Rules in πN Scattering*

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It is observed that the πN invariant amplitudes A and B, at fixed $u=0$ and $I_u=\frac{1}{2}$, give rise to sum rules of the type now becoming familiar from superconvergence relations, even though the amplitudes are not superconvergent. With the s-channel ($\pi N\to\pi N$) spectrum approximated by the nucleon and the $\Delta(1238)$, and the t-channel ($\pi\pi\to N\bar{N}$) spectrum approximated by the ρ, it is found that the sum rule obtained from the B amplitude is satisfied very well, while that from the A amplitude is not. It is speculated that this discrepancy might be due to the $\pi\pi$ s wave, which is rigorously absent from the B sum rule.

I. DERIVATION OF SUM RULES

It has been discovered recently[1] that sum rules can be obtained from strong-interaction scattering amplitudes which satisfy dispersion relations and have sufficiently good asymptotic behavior. The usual criterion for obtaining a sum rule is that the amplitude be "superconvergent," that is, that at high energy it decreases faster than s^{-1}. In this paper we would like to observe that sum rules may be obtained even from amplitudes that are not superconvergent, and to discuss such a sum rule relating baryon to meson resonances in πN scattering.

With s, t, and u the usual Mandelstam variables, and the πN amplitudes A and B defined by $T=-A+\gamma\cdot(q_1+q_2)B/2$, it is known[2] that at fixed u, both A and B behave asymptotically as $s^{\alpha(u)-1/2}$ where $\alpha(u)$ is the leading trajectory in the u channel. Thus if for $u=0$ all trajectories were below $J=-\frac{1}{2}$, these amplitudes would be superconvergent, and we could write, for example, for the amplitude B

$$\int_0^\infty ds[\mathrm{Im}B(s,u=0)+\mathrm{Im}B(-s,u=0)]=0\,, \quad (1)$$

where $\mathrm{Im}B(\pm s)$ is evaluated above the cut in the s plane, and for the moment we are ignoring the complications due to the unequal π and N masses. If we consider the amplitude that is pure isospin $\frac{1}{2}$ in the u channel, we would expect that only the nucleon trajectory would be above $J=-\frac{1}{2}$; the trajectory of the $N^*(1518)$ is estimated to have an intercept of $J=-0.9$ or lower.[3] If we write $R(s)$ for the leading term of the nucleon exchange contribution, we have

$$R(s)\equiv\gamma(u=0)[(s)^{\alpha(0)-1/2}+(-s)^{\alpha(0)-1/2}]\,,\qquad -\tfrac{1}{2}<\alpha<\tfrac{1}{2}\,, \quad (2)$$

which takes the form

$$R(s)\equiv\gamma(0)(1+e^{-i\pi(\alpha-1/2)})\,|s^{\alpha-1/2}| \quad (2a)$$

for s above the right-hand cut; we then expect that $F(s)\equiv B(s,u=0)-R(s)$ does decrease faster than s^{-1}, and so we conclude

$$\int_0^\infty ds[\mathrm{Im}F(s)+\mathrm{Im}F(-s)]=0\,. \quad (3)$$

However, we can see that $\mathrm{Im}R(s)+\mathrm{Im}R(-s)=0$, and so Eq. (3) leads us back to Eq. (1). An identical relation holds for the amplitude A. We cannot break up the left-hand side of Eq. (1) into the sum of two integrals, since each separately would be divergent; nevertheless, the imaginary parts cancel in just such a way as to make (1) true. Of course this kind of cancellation is not restricted to backward πN scattering; it is a simple consequence of the nucleon's having even signature, as expressed in Eq. (2).

Since we do not know how to calculate the t-channel ($\pi\pi\to N\bar{N}$) $I=0$ contribution to our sum rule, we could not compare it with experiment unless it should turn out that the $I=0$ contribution is small. We notice that the amplitude we are using is mainly $I=1$ in the t channel:

$$B^{I_u=1/2}=B^{I_t=1}+1/\sqrt{6}B^{I_t=0}\,. \quad (4)$$

Furthermore, there are the following indications that the $I=0$ amplitude should contribute primarily to A rather than to B: (1) The $\pi\pi$ s wave does not contribute to B at all. (2) Therefore, the lowest $I=0$ resonant contribution to B is from the f^0, which is already at a fairly high energy. Although there are no energy denominators in our sum rule, the above considerations show that the high-energy contributions of the s and t channels, while not separately negligible, must tend to cancel. (3) The absence of structure in the near-forward πp differential cross sections indicates that the P (and, to a lesser extent, the P') trajectory is coupled predominantly to A near $t=0$.[4] To the extent that we can

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[1] L. D. Solov'ev, Yadernaya Fiz. **3**, 188 (1966) [English transl.: Soviet J. Nucl. Phys. **3**, 131 (1966)]; V. de Alfaro, S. Fubini, G. Furlan, and G. Rossetti, Phys. Letters **21**, 516 (1966).
[2] S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2204 (1962).
[3] Charles Chiu, Lawrence Radiation Laboratory, Berkeley (private communication).
[4] K. J. Foley *et al.*, Phys. Rev. Letters **14**, 862 (1965); C. B. Chiu, S.-Y. Chu, and L.-L. Wang, Phys. Rev. (to be published).

extrapolate along the trajectory, we would expect that the f^0 is coupled mainly to A.[4a]

Therefore we would hope to get a reasonable sum rule from the B amplitude if we ignore the $I=0$ part; for the A amplitude this is less reasonable, especially if the $\pi\pi$ s wave is important.

One final word on evaluating the sum rule is appropriate at this point. A tractable approximation which we employ is to take the discontinuities to arise from narrow resonances. In this case, we have

$$\begin{aligned}\mathrm{Im}B(s,u=0)&=\sum \text{(residues of } s\text{-channel poles) } \delta(s-M_s^2)\\ &\quad-\sum \text{(residues of } t\text{-channel poles)}\\ &\qquad\times\delta(s+M_t^2-2M^2-2\mu^2)\,. \end{aligned}\tag{5a}$$

The sum rule therefore takes the form

$$\sum (s\text{-channel residues})=\sum (t\text{-channel residues})\,. \tag{5b}$$

To begin, we shall include only the nucleon, $\Delta(1238)$, and ρ pole contributions.

Since the π and N do have different masses, we must replace $\mathrm{Im}B(s)$ and $\mathrm{Im}B(-s)$ in Eq. (1) by the discontinuities B_s and B_t, respectively, and extend the integration down to the lowest t threshold; Eq. (5) is unchanged. Since the Legendre series for the discontinuities (not the full amplitude) converge everywhere along the path of integration, the approximation of keeping only the lowest partial waves can still be justified.

II. CALCULATIONS

In this section we present the relevant pole terms for the s- and t-channel states. We need an amplitude with pure $I=\frac{1}{2}$ in the u channel; this corresponds to the combination $[\frac{4}{3}(I=\frac{3}{2})-\frac{1}{3}(I=\frac{1}{2})]$ in the s channel. Using the notation

$$\begin{pmatrix}x\\y\end{pmatrix}\equiv\begin{pmatrix}I_s=\frac{3}{2}\\I_s=\frac{1}{2}\end{pmatrix},$$

we find for the ρ contribution

$$-A_\rho=\begin{pmatrix}1\\-2\end{pmatrix}2g_1g_2\frac{\mu_\rho M(1-M_\rho^2/2M^2)}{t-M_\rho^2}$$

and

$$B_\rho=\begin{pmatrix}1\\-2\end{pmatrix}2g_1g_2\frac{1+2\mu_\rho}{t-M_\rho^2}\,. \tag{6}$$

Here we have written the ρNN coupling for the ρ of charge i as

$$g_2\left[\gamma_\mu+\frac{2\mu_\rho}{2M}\sigma_{\mu\nu}q_\nu\right]\times\tau_i, \tag{7}$$

with the normalization $\langle\text{proton}|\tau_3|\text{proton}\rangle=1$. The $\rho\pi\pi$ coupling is defined to be $g_1(q_1-q_2)_\mu\times\tau_i$; experimentally, we find $g_1^2/4\pi\approx2.5$ (corresponding to a ρ width of 120 MeV).

Universality[5] of the ρ-meson coupling to the isospin current would imply $g_2=\frac{1}{2}g_1$. To relate g_2 and μ_ρ to measurable quantities, we can look at a dispersion calculation of the nucleon form factors, in which we obtain

$$F^{I=1}\begin{pmatrix}\text{electric}\\\text{magnetic}\end{pmatrix}=\frac{f_{\rho\gamma}g_2M_\rho^2}{M_\rho^2-t}\begin{pmatrix}1\\2\mu_\rho\end{pmatrix}+(\text{nonpole terms})\,. \tag{8}$$

From $\rho\to e^+e^-$ (or e^+e^- colliding-beam) experiments,[6] we can in principle determine $f_{\rho\gamma}$, so that the form-factor data can determine g_2. At present, it appears that $f_{\rho\gamma}=2/g_1$ (as would be expected from a dispersion calculation of the pion form factor), so that the ρ residue in Eq. (8) can be written as

$$M_\rho^2r\begin{pmatrix}1\\2\mu_\rho\end{pmatrix},$$

with $r\equiv2g_2/g_1$. Spearman's fit to the form-factor data[7] implies $r=1.6$ and $\mu_\rho=\mu_V(=1.85)$. Therefore the ρ-meson contributions to our sum rule from Eqs. (6) and (8) are

$$-\mathrm{Res}A_\rho(I_u=\tfrac{1}{2})=2rg_1^2\mu_VM(1-M_\rho^2/2M^2)=4\pi\times9.6\text{ GeV},$$

and

$$\mathrm{Res}B_\rho(I_u=\tfrac{1}{2})=2rg_1^2(1+2\mu_V)=4\pi\times37.5\,. \tag{9}$$

Next, we exhibit the baryon-pole terms. The nucleon pole is

$$A_N=0$$

and

$$B_N=-\begin{pmatrix}0\\3\end{pmatrix}\frac{g^2}{s-M^2}, \tag{10}$$

where $g^2/4\pi=14.5$. Hence the nucleon contributions to the sum rules are

$$\mathrm{Res}A_N(I_u=\tfrac{1}{2})=0$$

and

$$\mathrm{Res}B_N(I_u=\tfrac{1}{2})=g^2=4\pi\times14.5\,. \tag{11}$$

The Δ-pole terms can be evaluated by expressing A and B in terms of the partial-wave amplitudes.[8] Defining

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[4a] *Note added in proof.* J. Schwarz (to be published) has noted that $B^{I_t=0}$ vanishes when $s=u$, which at $u=0$ is $t=(1340\text{ GeV})^2$, thus providing a further suppression of the f^0 contribution to the B sum rule. Schwarz points out that $B^{I_t=0}/(s-u)$ is superconvergent; using this relation one can evaluate the f^0 coupling to B, and verify that its contribution to the sum rule is indeed negligible.

[5] J. J. Sakurai, Ann. Phys. (N. Y.) **11**, 1 (1960).

[6] A. H. Rosenfeld, *et al.*, University of California Lawrence Radiation Laboratory Report No. UCRL-8030, 1967 (unpublished).

[7] T. D. Spearman, Phys. Rev. **129**, 1847 (1963).

[8] See S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

the residue of the (3,3) partial-wave amplitude by

$$\frac{2W}{\Delta+W}\frac{\exp(i\delta_{33})\sin\delta_{33}}{(E-M)q}\approx\frac{\gamma_{33}}{\Delta^2-W^2}, \quad (12)$$

we see that

$$[q(E-M)/2\Delta]\gamma_{33}=\Gamma/2. \quad (13)$$

Experimentally, we have $\Gamma=120$ MeV and $\Delta=\frac{4}{3}M$, and so $\gamma_{33}=22.5\approx\frac{3}{2}(g^2/4\pi)$. This is the same value predicted either by SU_6 or by the static bootstrap model.

We can now write the Δ-pole terms in A and B as

$$A_\Delta(s,u=0)=\begin{pmatrix}-1\\0\end{pmatrix}\frac{4\pi\gamma_{33}}{s-\Delta^2}\left\{\frac{3(W+M)}{2(E+M)^2}\times\left[\frac{(M^2-\mu^2)^2}{2s}-\frac{s}{2}+M^2+\mu^2\right]+(W-M)\right\}_{W=\Delta},$$

and

$$B_\Delta(s,u=0)=\begin{pmatrix}-1\\0\end{pmatrix}\frac{4\pi\gamma_{33}}{s-\Delta^2}\left\{\frac{3}{2(E+M)^2}\times\left[\frac{(M^2-\mu^2)^2}{2s}-\frac{s}{2}+M^2+\mu^2\right]-1\right\}_{W=\Delta}. \quad (14)$$

The contributions to the sum rules are

$$-\mathrm{Res}A_\Delta(I_u=\tfrac{1}{2})=11.3M\gamma_{33}=4\pi\times18.8\ \mathrm{GeV},$$

and

$$\mathrm{Res}B_\Delta(I_u=\tfrac{1}{2})=14\gamma_{33}=4\pi\times25. \quad (15)$$

The masses, widths, and elasticities of many of the higher baryon resonances are known, and so we can compute their contributions to the sum rules. However, it turns out that all of these contributions together are less than 10% of the N and Δ contributions, and so we shall continue to neglect them.

We are now in a position to evaluate our sum rules. If we rewrite Eq. (5b) as

$$\frac{1}{4\pi}(s\text{-channel contribution})=\frac{1}{4\pi}(t\text{-channel contribution}), \quad (16)$$

Eqs. (9), (11), and (15) give

A sum rule: 18.8 (from baryons) versus 9.6 (from ρ) in GeV (17a)

and

B sum rule: 39.5 (from baryons) versus 37.5 (from ρ). (17b)

III. DISCUSSION

We note first that the B sum rule is well satisfied with the experimental values of the ρNN couplings (as interpreted from form-factor analysis). Universality (which implies $r=1.0$ instead of 1.6) would not have given such good agreement. Even within the resonance approximation, we have of necessity neglected the contributions of higher resonances, such as the f^0, any possible second ρ,[9] and the resonances seen in the missing-mass spectrometer work at CERN.[10] According to the arguments presented in Sec. I, these higher. energy contributions to the sum rules should be small-

If the f^0 is indeed coupled to A, it could perhaps account for some of the discrepancy in the A sum rule. In addition, the $\pi\pi$ s wave, which does not contribute to B at all, makes a contribution to A which we do not know how to calculate. We do not even know the sign of this contribution, but can perhaps estimate the magnitude of the low-energy s wave by approximating the low-energy spectrum by a σ meson. From a boson-exchange model of nucleon-nucleon scattering, Ball, Scotti, and Wong[11] estimate the σNN coupling to be $g_{\sigma NN}^2/4\pi\approx5$. If we arbitrarily assign the σ mass of 500 MeV and a width of 200 MeV, the σ would contribute about 2 GeV to the A sum rule, Eq. (17a), and so could account for about $\frac{1}{4}$ of the discrepancy. In addition, there is some evidence for an $I=l=0$ $\pi\pi$ resonance lying underneath the ρ,[12] and we have no way of estimating what its coupling to NN might be.

Because of these ambiguities, it is difficult to draw any conclusions from the A sum rule; we would certainly not like to use this sum rule to estimate the magnitude of the $\pi\pi\rightarrow N\bar{N}$ amplitude! The B sum rule seems to be well satisfied by keeping only the states ρ, N, and Δ.

[9] E. B. Hughes, T. A. Griffy, M. R. Yearian, and R. Hofstadter, Phys. Rev. **139**, B458 (1965); H. Högaasen and W. Fischer, Phys. Letters **22**, 516 (1966) discuss the evidence for a particle with the quantum numbers of the ρ and a mass of 1.2 to 1.5 GeV, from form-factor and Regge analysis, respectively.

[10] G. Chikovani *et al.*, Phys. Letters **22**, 233 (1966).

[11] J. Ball, A. Scotti, and D. Wong, Phys. Rev. **142**, 1000 (1966).

[12] See L. Durand, III and Y. T. Chiu, Phys. Rev. Letters **14**, 329 (1965).