Spin Dependence of High-Energy Scattering Amplitudes. II. Zero-Mass Particles*

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The discussion of spin dependence at high energy is extended to the case of zero-mass particles. Crossing relations for zero-mass particles are derived, showing that the helicity of massless particles is simply reversed under crossing. These are used by way of a Pomeranchuk-Martin–type theorem to obtain restrictions on the high-energy spin dependence. The problem of fixed poles in the angular-momentum plane and the coupling of the Pomeranchuk trajectory to photons is discussed. It is shown that if there are no fixed poles at positive integers for any Compton amplitude, one obtains the (presumably ridiculous) result that the asymptotic total cross section for photons on any particle is proportional to the square of its charge. An approximate dynamical calculation is given which relates the coupling of photons to the Pomeranchuk trajectory with the derivative of the trajectory at $t=0$. This yields a prediction for the high-energy total cross section of photons on protons which is consistent with present data. The convergence of the Drell-Hearn sum rule is discussed; it is argued that even with cuts in the angular momentum, the sum rule will converge. Certain sum rules of Bég and of Pagels and Harari are discussed briefly.

I. INTRODUCTION

THIS paper, second in a series[1] dealing with spin dependence of high-energy scattering, is concerned with the special problems when one of the particles is massless. In particular, we are concerned with photon processes.

In Sec. II, it is shown how the derivation of crossing relations for helicity amplitudes must be modified to cover the zero-mass case. The result is simple: The helicity of massless particles is flipped on crossing. Section III gives an easy application of these relations to obtain restrictions on the asymptotic spin dependence by Pomeranchuk-Martin–type arguments.

Section IV is devoted to the question of fixed poles in the angular-momentum plane for photon processes, and the relation of these to high-energy scattering is obtained. It is shown that, contrary to the belief of some authors, the Pomeranchuk trajectory very naturally couples to photons and contributes to their total cross sections. An approximate dynamical calculation is given which relates the coupling of photons to the Pomeranchuk trajectory to the derivative of the trajectory function $\alpha(t)$ at $t=0$. This yields a prediction for the high-energy total cross section of photons on nucleons which is consistent with present data.

Section V is concerned with the convergence of the Drell-Hearn sum rule. It is argued that even though the Pomeranchuk trajectory can couple to photons, along with its associated cuts, the Drell-Hearn sum rule will converge. The rate of convergence depends on the strength of the cuts. Certain sum rules of Bég and of Pagels and Harari are discussed briefly.

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[1] A. H. Mueller and T. L. Trueman, Phys. Rev. **160**, 1296 (1967) (preceding paper, hereinafter referred to as I).

II. CROSSING RELATIONS FOR ZERO-MASS PARTICLES

The derivation of the crossing relation for helicity amplitudes[2] does not apply directly to zero-mass particles. The reason for this is the path of continuation chosen there. Recall that in continuing $\mathbf{p}\rightarrow-\mathbf{P}$ the path was chosen so that $+(p_0^2-m^2)^{1/2}\rightarrow+(P_0^2-m^2)^{1/2}$ and $\theta,\ \varphi\rightarrow\pi-\Theta,\ \pi+\Phi$. ($\theta$, φ are polar angles of $\mathbf{p}$, Θ, Φ of $\mathbf{P}$.) The absence of a branch point in the relation between $|\mathbf{p}|$ and p_0 in the zero-mass case means there is no such path here. Thus the continuation must result in $|\mathbf{p}|\rightarrow-|\mathbf{P}|,\ \theta,\ \varphi\rightarrow\Theta,\ \Phi$. {One could just as well have chosen this path for massive particles. The resulting relation between general helicity amplitudes [Eq. (31) of Ref. 2] would then be different. However, one must keep in mind that the equation relating the general helicity amplitude to the center-of-mass helicity amplitudes [Eq. (20) of Ref. 2] must also be continued along this path. The resulting equation relating the two sets of center-of-mass helicity amplitudes is exactly the same as obtained in Ref. 2 [Eqs. (39) and (41)], as it must for consistency.}

We outline below the derivation as modified for the zero-mass case. The result can be obtained from the formulas of Ref. 2 by setting the mass equal to zero for $s\neq0$, $t\neq0$ (as must be the case since there should be no significant difference between $m=0$ and $m\neq0$ but $m\ll|s|,\ |t|$).

Introduce the basic helicity state for a zero-mass particle which has four-momentum b:

$$b=\begin{pmatrix}1\\0\\0\\1\end{pmatrix}=\begin{pmatrix}b_0\\b_x\\b_y\\b_z\end{pmatrix},\qquad(2.1)$$

[2] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964).

and define $Z_0(p)$ by

$$Z_0(p)\begin{bmatrix}1\\0\\0\\1\end{bmatrix}=\begin{bmatrix}p\\0\\0\\p\end{bmatrix}. \tag{2.2}$$

Following Jacob and Wick,[3] we define the helicity states for general p by

$$|p,\lambda\rangle=H_0(p)|b,\lambda\rangle, \tag{2.3a}$$

$$H_0(p)=R_z(\varphi)R_y(\theta)R_z(-\varphi)Z_0(p), \tag{2.3b}$$

where $R_i(\alpha)$ denotes a rotation through the angle α about the i axis. Under the arbitrary Lorentz transformation L the helicity state undergoes a rotation $r(L,p)$ given by[4]

$$r(L,p)=H_0^{-1}(Lp)LH_0(p). \tag{2.4}$$

Since

$$Z_0(-P)=\begin{bmatrix}-\frac{1}{2}[P+(1/P)] & 0 & 0 & -\frac{1}{2}[P-(1/P)]\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\frac{1}{2}[P-(1/P)] & 0 & 0 & -\frac{1}{2}[P+(1/P)]\end{bmatrix}$$
$$=-Z_0(P)R_z(\pi), \tag{2.5}$$

and since $\theta,\ \varphi\rightarrow\Theta,\ \Phi$ we have

$$r(L;-P)=R_z^{-1}(\pi)r(L;P)R_z(\pi). \tag{2.6}$$

For simplicity, consider the crossed processes shown in Fig. 1 which involve three spinless particles and one massless particle with spin. The spinning particle is crossed in going from one process to the other. From invariance under Lorentz transformations and Eq. (6) we have

$$F_\lambda(p_2,k;p_1,q)=\sum_\mu \mathfrak{D}_{\mu\lambda}{}^*[r(L;k)] \times F_\mu(Lp_2,Lk;Lp_1,Lq), \tag{2.7}$$

$$G_\lambda(p_2,-p_1;q,-k)=\sum_\mu \mathfrak{D}_{-\mu,-\lambda}{}^*[r(L;k)] \times G_\mu(Lp_2,-Lp_2;Lq,-Lk). \tag{2.8}$$

We are thus led to identify

$$F_\lambda(p_2,k;p_1,q)=G_{-\lambda}(p_2,-p_1;q,-k) \tag{2.9}$$

up to a phase. [Equations (2.6)–(2.9) hold also in the massive case when this alternative path of continuation is used.] Since the helicity of the zero-mass particle is invariant under Lorentz transformations, the same relation must hold between center-of-mass amplitudes:

$$F_\lambda(s,t)=\eta_\lambda G_{-\lambda}(s,t). \tag{2.10}$$

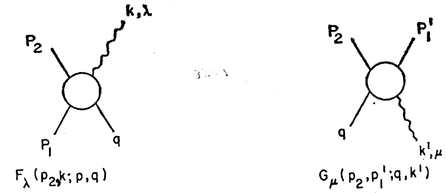

FIG. 1. The processes described by the amplitudes of Eqs. (2.7) and (2.8).

For the same reason, it is not possible to conclude that η_λ is independent of λ. However, if parity is conserved, then $\eta_\lambda=\eta_{-\lambda}$ and (2.10) is independent of helicity. Thus the crossing property for zero-mass particles is very simple: the helicity is reversed. This agrees with the well-known property of photons.

III. POMERANCHUK THEOREM

The crossing relations plus a few standard assumptions can be used to obtain restrictions on the spin dependence when the massless particle is its own antiparticle. Since this is surely academic except for photons, we shall call the particles photons, although the derivation obviously does not depend on the spin of the particle. Thus, if we cross the two photons, whose helicity we denote by α_1 and α_2, we obtain

$$G_{\lambda_2\alpha_2;\lambda_1\alpha_1}(s,t)=\pm G_{\lambda_2,-\alpha_1;\lambda_1,-\alpha_2}(u,t). \tag{3.1}$$

The λ's denote the helicities of the two uncrossed particles which are not changed in the crossing.

By following Martin,[5] we may easily construct a generalization of the Pomeranchuk theorem. One assumes the diagonal amplitude $G_{\lambda_1\alpha_1,\lambda_1\alpha_1}(s,0)\equiv G_{\lambda_1\alpha_1}(s,0)$ is analytic in s except along the real axis and over some finite region like the Bros, Epstein, and Glaser region.[6] With the additional assumptions:

(1) $G_{\lambda_1\alpha_1}(s,0)<C(\epsilon)e^{\epsilon|s|}$,

(2) $\lim_{s\to\pm\infty}\dfrac{|G_{\lambda_1\alpha_1}(s,0)|}{s\ln s}=0$,

(3) $\lim_{s\to\infty}\mathrm{Im}(1/s)[G_{\lambda_1\alpha_1}(s,0)-G_{\lambda_1,\alpha_1}(-s,0)]$ exists,

it follows that

$$\lim_{s\to\infty}(1/s)\ \mathrm{Im}[G_{\lambda_1\alpha_1}(s,0)-G_{\lambda_1\alpha_1}(-s,0)]=0. \tag{3.2}$$

The first is the familiar temperedness condition; (2) is stronger than the Froissart bound, but is necessary for the result (3.2); (3) is needed to avoid cancellation through oscillation as $s\rightarrow\infty$.

[3] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

[4] G. C. Wick, Ann. Phys. (N. Y.) **18**, 65 (1962). Strictly speaking $r(L;p)$ is not a rotation but an element of the little group of the vector b. On the helicity state $|b,\lambda\rangle$ only the rotation about the z axis is effective; the other generators of the little group annihilate $|b,\lambda\rangle$. See, for example, S. S. Schweber, *Relativistic Quantum Field Theory* (Row, Peterson and Company, Evanston, Illinois, 1961), p. 52. We wish to thank Dr. J. H. Lowenstein for a useful communication regarding this and for pointing out an error in the matrix $Z_0(-P)$ in Eq. (2.5) in an early draft of this paper.

[5] A. Martin, Nuovo Cimento **39**, 704 (1965).

[6] S. Bros, H. Epstein, and V. Glaser, Commun. Math. Phys. **1**, 240 (1965). Note that the proof of crossing given here does not apply to zero-mass particles and so must be considered as an additional assumption.

Now for large s, $u \propto (-s)$ and so we have

$$\lim_{s\to\infty} \frac{1}{s} \operatorname{Im}[G_{\lambda_1\alpha_1}(s,0)-G_{\lambda_1,\alpha_1}(u,0)] = \lim_{s\to\infty} \frac{1}{s} \operatorname{Im}[G_{\lambda_1\alpha_1}(s,0)\mp G_{\lambda_1-\alpha_1}(s,0)] = 0.$$

The $\pm$ in (3.1) is determined to be $+$ since otherwise both cross sections would be required to vanish. Hence.

$$\lim_{s\to\infty}[\sigma_{\lambda_1\alpha_1}(s)-\sigma_{\lambda_1,-\alpha_1}(s)]=0. \tag{3.3}$$

The result (3.3) is of interest with regard to the convergence of the Drell-Hearn sum rule.[7] It is, however, not quite sufficient to guarantee that the sum rule converge.[8] See Sec. V for further discussion of this question.

IV. FIXED POLES AND THE COUPLING OF PHOTONS TO THE POMERANCHUK TRAJECTORY

From the crossing relation of Sec. II, it follows that the total cross section for a photon on any target involves helicity flip of two units in the crossed channel; i.e.,

$$G_{\lambda_1+,\lambda_1+}(s,0)=\sum_{\mu} d_{\mu_1\lambda_1}{}^s(\pi/2)d_{\mu_2\lambda_1}{}^s(\pi/2)F_{\mu_2\mu_1;-+}(s,0).$$

Because the leading terms of $d_{2,\mu}{}^J(\theta_t)$ vanish at $J=1$ [see Eqs. (3.9) and (3.10) of I] it appears as if the Pomeranchuk trajectory, with $\alpha(0)=1$, cannot contribute to the total photon cross sections. This situation and the difficulties with it were first noticed by Mur.[9] He realized that a fixed singularity of the partial-wave amplitudes at $J=1$ would allow the Pomeranchuk trajectory to contribute to the total cross section, but assumed that unitarity ruled out such singularities. As we now know, this is not the case[1,10] and we must expect the residue of the Pomeranchuk trajectory to be singular at $J=1$.

In this section, the relationship between this singular residue and a fixed pole at $J=1$ is discussed, and an approximate calculation of the numerical value of high-energy Compton scattering is given.

The amplitude for $\pi(p_1)+\gamma(k_1)\to\pi(p_2)+\gamma(k_2)$ with isotopic spin 0 in the t channel is written in terms of invariants as[11]

$$S_{\mu\nu}=A(s,t)P_\mu P_\nu+\cdots,$$

where

$$P=\tfrac{1}{2}(p_1+p_2),\quad s=-(p_1+k_1)^2,\quad t=-(p_2-p_1)^2,$$
$$p_1{}^2=p_2{}^2=-\mu^2,\quad k_1{}^2=k_2{}^2=0.$$

{A is normalized such that if only $I=0$ in the t channel contributes, then

$$\sigma(s)=\lim_{t\to0}\left[\frac{(s-\mu^2)\operatorname{Im}A(s,t)}{2t}\right](\tfrac{1}{3})^{1/2} \tag{4.1}$$

is the total cross section for γ's on pions. The $(\frac{1}{3})^{1/2}$ is an isotopic spin factor.} The amplitude, $F_{+-}(s,t)$, for the scattering of two photons, one of helicity $+1$ the other of helicity -1, into two pions in an isotopic-spin-zero state is given by

$$F_{+-}(s,t)=-(\tfrac{1}{4}t-\mu^2)\tfrac{1}{2}(1-z^2)A(s,t),$$

where

$$z=\frac{s-\mu^2+\frac{1}{2}t}{2[\frac{1}{4}t(\frac{1}{4}t-\mu^2)]^{1/2}}.$$

From Eq. (4.1) it is clear that $\operatorname{Im}A(s,t)$ goes to zero at $t=0$ while $\operatorname{Im}F_{+-}(s,t)$ remains finite at $t=0$.

The partial-wave amplitude is given by

$$F_{+-}{}^J(t)=\frac{1}{2}\int_{-1}^{1}dz\, d_{20}{}^J(z)F_{+-}(t,z).$$

Using Eq. (3.7) of I we obtain

$$F_{+-}{}^J(t)=\frac{2}{\pi}\sum_{p=0}^{4} C_{J+p-2}(0,2,J)\times\int_{z_0}^{\infty}dz' Q_{J-2+p}(z')\operatorname{Im}\tilde{F}_{+-}(t,z')+F_{+-}{}^{BJ}(t),$$

where

$$\tilde{F}_{+-}=\frac{1}{(1-z^2)}F_{+-}=-\frac{(\frac{1}{4}t-\mu^2)}{2}A$$

and F^B is the Born term. Defining $\hat{F}_{+-}{}^J$ by

$$\hat{F}_{+-}{}^J(t)=[\tfrac{1}{4}t-\mu^2]^{-1}[\tfrac{1}{4}t(\tfrac{1}{4}t-\mu^2)]^{-(J-2)/2}F_{+-}{}^J,$$

we find the explicit form

$$\hat{F}_{+-}{}^J(t)=\frac{2e^2}{[\frac{1}{4}t(\frac{1}{4}t-\mu^2)]^{(J-1)/2}}\sum_{p=0}^{4}C_{J-2+p}(0,2,J)\times\left\{-Q_{J-2+p}\left(\frac{\frac{1}{4}t}{[\frac{1}{4}t(\frac{1}{4}t-\mu^2)]^{1/2}}\right)\frac{2}{\sqrt{3}} -\frac{1}{4\pi e^2}\int_{s_0}^{\infty}ds\, Q_{J-2+p}\left(\frac{s-\mu^2+\frac{1}{2}t}{2[\frac{1}{4}t(\frac{1}{4}t-\mu^2)]^{1/2}}\right)\times\operatorname{Im}A(t,s+i\epsilon)\right\}. \tag{4.2}$$

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[7] S. D. Drell and A. C. Hearn, Phys. Rev. Letters 16, 908 (1966).

[8] The question of the convergence of the Drell-Hearn sum rule was originally called to our attention by Professor F. E. Low.

[9] V. D. Mur, Zh. Eksperim. i Teor. Fiz. 44, 2173 (1963); 45, 1051 (1963) [English transls.: Soviet Phys.—JETP 17, 1458 (1963); 18, 727 (1964)]. See also C. H. Chan and T. K. Kuo (unpublished); H. K. Shepard, Phys. Rev. 159, 1331 (1967); H. Abarbanel and S. Nussinov, *ibid.* 158, 1462 (1967).

[10] S. Mandelstam and L.-L. Wang, Phys. Rev. 160, 1490 (1967).

[11] S. Fubini, Nuovo Cimento 43, 476 (1966).

[The expression $\sum_{p=0}^{4} C_{J-2+p}(0,2,J)P_{J-2+p}(z)$ is equal to the $C_{20}{}^{J+}(z)$ of Ref. 12.]

There are two sources of fixed poles in $\hat{F}_{+-}{}^{J}$ at $J=1$. Due to the presence of a third double spectral function $\hat{F}_{+-}{}^{J}$ must have a fixed pole at $J=1$ for the same reason as the strong-interaction amplitude discussed in I and in Ref. 10. Also, since the photons are treated to lowest order in e^2, there will, in general, be fixed poles even in the absence of a third double spectral function.

Near $J=1$ we find from (4.2)

$$\frac{1}{(J-1)^{1/2}}\hat{F}_{+-}{}^{J}(t)\approx\frac{2e^2}{J-1}(\tfrac{2}{3})^{1/2}\left\{-2(\tfrac{1}{3})^{1/2}-\frac{1}{4\pi e^2}\int_{s_0}^{\infty}ds\,\mathrm{Im}A(t,s+i\epsilon)\right\}. \quad (4.3)$$

For $t<0$ the integral in (4.3) will converge. If $\alpha(t)$ is the trajectory function for the Pomeranchuk trajectory and if we write

$$\mathrm{Im}A(t,s+i\epsilon)\underset{s\to\infty}{\longrightarrow}(6)^{1/2}(\alpha-1)\beta(t)s^{\alpha-2}\pi,$$

then

$$\frac{1}{(J-1)^{1/2}}\hat{F}_{+-}{}^{J}(t)\approx-\frac{2e^2}{J-1}(\tfrac{2}{3})^{1/2}\left\{(6)^{1/2}\frac{\beta(0)}{4e^2}+\frac{2}{(3)^{1/2}}\right\} \quad (4.4)$$

near $J=1$ and near $t=0$. [We have chosen to write $\mathrm{Im}A$ in this way in order to agree with the definition of β in (4.5).]

In the absence of the fixed pole, $\beta(0)=0$. However, as we have argued in I, $\beta(0)\neq0$ violates no general principles and, furthermore, appears to follow in dynamical models from the fact that the *left-hand cut contribution* to $\hat{F}_{+-}{}^{J}$ has a fixed pole at $J=1$. With this in mind we write

$$\frac{1}{(J-1)^{1/2}}\hat{F}_{+-}{}^{J}(t)=\frac{1}{J-1}\frac{\gamma(t)}{J-\alpha(t)}+\cdots\approx\frac{\beta(t)}{J-\alpha(t)} \quad (4.5)$$

near $J=\alpha(t)$, where $\beta(t)$ is expected to be nonzero at $t=0$. Using the inversion formula (4.5) of I we find that a pole of the type given above contributes to F_{+-} as

$$(1-z^2)F_{+-}(t,s)=-\pi(2\alpha+1)\beta C_{\alpha+2}(0,2,\alpha)(\alpha-1)^{1/2}\left(\frac{1+e^{-i\pi\alpha}}{2\sin\pi\alpha}\right)\times(\tfrac{1}{4}t-\mu^2)[\tfrac{1}{4}t(\tfrac{1}{4}t-\mu^2)]^{(\alpha-2)/2}P_{\alpha+2}(z)+\cdots$$

or

$$F_{+-}\underset{s\to\infty}{\longrightarrow}5\pi(2\alpha+1)\left(\frac{1+e^{-i\pi\alpha}}{2\sin\pi\alpha}\right)\times\left(\frac{C_{\alpha+2}(0,2,\alpha)(\alpha-1)^{1/2}}{t}\right)\beta(t)s^{\alpha(t)}, \quad (4.6)$$

where $C_{\alpha+2}(0,2,\alpha)(\alpha-1)^{1/2}\approx\frac{1}{5}(\frac{2}{3})^{1/2}(\alpha-1)$ near $\alpha=1$. Thus, the total high-energy cross section for the scattering of photons off pions is given by

$$\sigma_{\pi\gamma}(\infty)=-\frac{\pi\beta(0)}{\sqrt{2}}\alpha'(0),\quad \alpha'(0)=\frac{d}{dt}\alpha(t)\bigg|_{t=0}. \quad (4.7)$$

Now we would like to point out that if one is willing to make some additional assumptions a very strong result can be obtained, a particular case of which gives the numerical value of $\beta(0)$ in (4.7). If we use elastic unitarity for the process $\gamma\gamma\to\pi\pi$ in the $I=0$ state we can write the amplitude $\hat{F}_{+-}{}^{J}$ as

$$\hat{F}_{+-}{}^{J}(t)=\frac{1}{D_J(t)}\left[D_J(t)\hat{f}^J(t)+\frac{t}{\pi}\int\frac{\rho_J(t')N_J(t')\hat{f}^J(t')dt'}{t'(t'-t)}\right]. \quad (4.8)$$

The integral in (4.8) goes from $4\mu^2$ to ∞. (There is a branch point in the t plane which moves with J and which reaches $t=0$ at $J=1$.[13] If we neglect the third double spectral function, which we must do in order to proceed further, this branch point is absent.) N_J and D_J are the continued numerator and denominator functions for $\pi\pi$ scattering in an isotopic spin-zero state. The factor t multiplying the integral in (4.8) guarantees the behavior of $\hat{F}_{+-}{}^{J}(t)$ near $t=0$ which is required by (4.2). $\hat{f}^J(t)$ is the left-hand cut contribution to $\hat{F}_{+-}{}^{J}(t)$. If we now suppose that the Pomeranchuk trajectory appears through a zero of D_J according to

$$D_J(t)\approx\mu(t)[J-\alpha(t)],$$

then the β of (4.5) becomes

$$\beta(t)=\frac{1}{[\alpha(t)-1]^{1/2}}\frac{t}{\mu(t)\pi}\int\frac{\rho_{\alpha(t)}(t')N_{\alpha(t)}(t')\hat{f}^{\alpha(t)}(t')}{t'(t'-t)}dt'.$$

As $t\to0$, or equivalently as $\alpha\to1$,

$$\hat{f}^{\alpha(t)}(t')\longrightarrow-\tfrac{4}{3}\sqrt{2}\frac{e^2}{(\alpha-1)^{1/2}}. \quad (4.9)$$

[Without the neglect of the third double spectral function we would not have (4.9).[14] Thus, the neglect of the third double spectral function appears to be essential.]

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[12] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. 133, B145 (1964).

[13] S. Mandelstam, Nuovo Cimento **30**, 1128 (1963); **30**, 1148 (1963).

[14] S. Mandelstam, Nuovo Cimento **30**, 1113 (1963).

Using (4.9) we obtain

$$\beta(t)\approx-\tfrac{4}{3}\sqrt{2}e^2\frac{t}{\alpha-1}\frac{1}{\mu(0)}\frac{1}{\pi}\int\frac{\rho_1(t')N_1(t')dt'}{(t')^2}$$

near $t=0$. We can identify the integral as

$$\frac{1}{\pi}\int\frac{\rho_1(t')N_1(t')dt'}{(t')^2}=-\frac{d}{dt}D_1(t)\bigg|_{t=0}=\mu(0)\alpha'(0)$$

so that

$$\beta(0)=-\tfrac{4}{3}\sqrt{2}e^2. \tag{4.10}$$

Combining (4.10) with (4.7), we obtain[15]

$$\sigma_{\pi\gamma}(\infty)=(16\pi^2/3)(e^2/4\pi)\alpha'(0), \tag{4.11}$$

where $e^2/4\pi\approx 1/137$.

According to the factorization of residues,

$$\sigma_{N\gamma}(\infty)=\frac{\sigma_{NN}(\infty)}{\sigma_{N\pi}(\infty)}\left(\frac{16\pi^2}{3}\right)\left(\frac{e^2}{4\pi}\right)\alpha'(0). \tag{4.12}$$

If one takes[16] $\alpha'(0)\approx\frac{1}{3}$ BeV^{-2}, and[17]

$$\sigma_{NN}(\infty)/\sigma_{N\pi}(\infty)\approx 38/21,$$

then $\sigma_{N\gamma}(\infty)\approx 95\ \mu$b. This is within the experimental limits of 75–120 μb[18] and is consistent with values found by Stodolsky using vector-meson dominance.[19] However, the experimental errors and theoretical approximations are such that it is difficult to know what significance to attach to such agreement. If the Regge *pole* picture is approximately correct and cuts are not very important for the predictions of high-energy scattering, then our neglect of the third double-spectral function may be reasonable. As far as justifying elastic unitarity is concerned, we are unable to give any strong arguments. We have carried out a two-channel calculation ($\gamma\gamma\to\pi\pi$ and $\gamma\gamma\to K\bar{K}$) and find the same result if all the integrals which begin at the $K\bar{K}$ threshold are slowly varying compared with those beginning at the $\pi\pi$ threshold. Whether or not this is true is a difficult dynamical question which we will not attempt to answer.

Returning once more to the question of the existence of fixed poles, we see from (4.8) that near $J=1$ for arbitrary t,

$$\frac{\hat{F}_{+-}{}^J(t)}{(J-1)^{1/2}}\approx-\frac{1}{D_1(t)}\frac{4\sqrt{2}}{3}\frac{e^2}{J-1}D_1(0).$$

But $D_1(0)=0$ so that $(J-1)^{-1/2}\hat{F}_{+-}{}^J(t)$ no longer has a fixed pole at $J=1$ and a new type of superconvergence relation is obtained.[1,10,20] We remark that none of these results are sensitively dependent on $\alpha(0)=1$. The reader may verify that if $\alpha(t_0)=1$ for t_0 small and positive, the Regge residue and $\alpha'(0)$ are related in the same way that we have found here; $\sigma(s)\propto s^{\alpha(0)-1}$ with the proportionality constant determined; the new superconvergence relation becomes

$$\left\{\frac{e^2}{\sqrt{3}}-\frac{t}{4\pi}\int_{s_0}^{\infty}\frac{\mathrm{Im}F_{+-}(s,t)}{(s-\mu^2)^2+st}\right\}=\frac{e^2}{\sqrt{3}}\frac{D_1(0)}{D_1(t)}. \tag{4.13}$$

It is apparent now that the relation between $\alpha'(0)$ and the cross section is more a property of the approximate dynamical calculation than of the presence or absence of fixed poles. It is probably illuminating to remark that *if* one insisted there be *no fixed poles* in *any* Compton amplitude one would obtain the astounding result that *the asymptotic cross section would be proportional to the square of the particle's charge*, e.g., the total cross section for γ's on neutrons or on π^0's would vanish, while those for γ's on protons and $\pi^{\pm}$'s would not.

Before leaving this subject, we might mention that if one replaces the $SU(2)$-symmetric calculation by an $SU(3)$-symmetric calculation, with the Pomeranchuk a singlet, the calculation is trivially modified and elastic unitarity over the two-meson states gives

$$\sigma_{\pi\gamma}(\infty)=\sigma_{K\gamma}(\infty)=\sigma_{\eta\gamma}(\infty)=4\pi^2(e^2/4\pi)\alpha'(0), \tag{4.14}$$

instead of (4.11). The difference is about 25%. Aside from noting that the Pomeranchuk does not appear to be a pure singlet in $SU(3)$,[17] we have no comment on (4.11) versus (4.14).

V. DRELL-HEARN SUM RULE

In this section, we discuss the question of the convergence of the Drell-Hearn sum rule[7]:

$$\frac{2\pi^2\alpha}{m^2}\kappa_p{}^2=\int_0^{\infty}\frac{d\nu}{\nu}[\sigma_P(\nu)-\sigma_A(\nu)]. \tag{5.1}$$

$\kappa_p=1.79$ is the anomalous magnetic moment of the proton, m the proton mass, ν the lab energy, and $\alpha=1/137$. $\sigma_P(\nu)$ and $\sigma_A(\nu)$ are, respectively, the total cross sections for the absorption of photons on protons with parallel and antiparallel helicities. In terms of the scattering amplitudes, and by use of the crossing

[15] This same result has been obtained independently by H. D. I. Abarbanel, F. E. Low, I. J. Muzinich, S. Nussinov, and J. H. Schwarz, Phys. Rev. **160**, 1329 (1967). We thank B. W. Lee and M. L. Goldberger for telling us about it.

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relations, (5.1) reads

$$\frac{2\pi^2\alpha}{m^2}\kappa_p^2=-\int_{m^2}^{\infty}ds\left[\frac{\mathrm{Im}G_{++,++}(s,0)-\mathrm{Im}G_{+-,+-}(s,0)}{(s-m^2)^2}\right],$$

$$=\int_{m^2}^{\infty}ds\left[\frac{\mathrm{Im}F_{-+,-+}(s,0)+\mathrm{Im}F_{+-,-+}(s,0)}{(s-m^2)^2}\right] \quad (5.2)$$

(see Fig. 2). Conservation of angular momentum at $t=0$ requires that

$$G_{++,-+}(s,0)=G_{-+,++}(s,0)=0 \quad (5.3)$$

or

$$F_{++,-+}(s,0)=F_{--,-+}(s,0)\,,$$
$$F_{-+,-+}(s,0)=F_{+-,-+}(s,0)\,. \quad (5.4)$$

Hence

$$\frac{2\pi^2\alpha}{m^2}\kappa_p^2=2\int_{m^2}^{\infty}ds\,\frac{\mathrm{Im}F_{-+,-+}(s,0)}{(s-m^2)^2}\,. \quad (5.5)$$

For the sake of comparison, note that

$$\sigma_P(\nu)+\sigma_A(\nu)=\frac{2}{(s-m^2)}\,\mathrm{Im}F_{++,-+}(s,0)\,.$$

As we have seen in the last section, the fact that helicity flip of two is involved at the photon vertex does not imply that the Pomeranchuk trajectory cannot contribute to the amplitude in the integral of Eq. (5.5). In view of the results of Sec. III, however, we do not expect $\mathrm{Im}F_{-+,-+}(s,0)\sim s$ but we must be careful about behavior such as $s/\ln s$. Such behavior could be induced by cuts in the angular-momentum plane and would yield a divergent integral in (5.5). We will now argue that such behavior is not compatible with the common picture of Regge poles and cuts. The argument is based on quantum numbers.

First, notice that under CP the proton-antiproton state is

$$|p\lambda_1\lambda_2\rangle\rightarrow CP|p\lambda_1\lambda_2\rangle=|p\ -\lambda_2\ -\lambda_1\rangle\,.$$

Hence, the amplitude $F_{-+,-+}(s,t)$ is a pure $CP=+1$ amplitude. That is, the final photon state must have $CP=+1$, too. Since $C=+1$ for a two photon state, we learn that the amplitude has positive parity. Since[3]

$$P|J\lambda_1\lambda_2\rangle=(-1)^J|J\ -\lambda_1\ -\lambda_2\rangle\,,$$

the partial-wave amplitudes satisfy

$$F_{\lambda,-\lambda,-+}{}^{\alpha P_j}(t)=P_jF_{-\lambda,\lambda,-+}{}^{\alpha P_j}(t)\,,$$

where P_j denotes the signature of this amplitude. The total amplitude has the form

$$F_{\lambda,-\lambda,-+}(s,t)\propto\Big\{\sum_{\alpha_i^+}\beta_{\lambda,-\lambda;-+}{}^{\alpha_i^+}(t)d_{2,2\lambda}{}^{\alpha_i^+}(\theta_t)+\sum_{\alpha_i^-}\beta_{\lambda,-\lambda;-+}{}^{\alpha_i^-}(t)d_{2,2\lambda}{}^{\alpha_i^-}(\theta_t)\Big\}\,,$$

where

$$\beta_{\lambda,-\lambda;-+}{}^{\alpha_i^\pm}(t)=\pm\beta_{-\lambda,\lambda;-+}{}^{\alpha_i^\pm}(t)\,.$$

(The sum contains integrals over the cuts.) Asymptotically, however,

$$d_{2,2\lambda}{}^{\alpha_i}(\theta_t)\sim-d_{2,-2\lambda}{}^{\alpha_i}(\theta_t)\,.$$

Thus, if the amplitude were dominated by the exchange of $P=C=P_j=+1$ we would have

$$F_{+,-;-+}(s,0)\sim-F_{-+;-+}(s,0)$$

in contradiction to (5.4). This does not mean that such trajectories cannot contribute. They simply cannot dominate. [Presumably, one can have a "conspiracy"[21] between amplitudes of opposite signature, just as in NN scattering where essentially the same argument[1] proves that $P=C=P_j=+1$ cannot dominate the amplitude $f_{11}=F_{++,++}(s,0)-F_{++,--}(s,0)$.]

Since the Pomeranchuk trajectory has $P=C=P_j=+1$, we see that it cannot contribute directly to the Drell-Hearn sum rule, as anticipated.[22] What about the associated cuts? These might be expected to produce a branch point at $J=1$ when $t=0$ and hence a $(1/\ln s)$ behavior for $[\sigma_P(s)-\sigma_A(s)]$. It is clear from the analysis just presented that the leading term of the cuts which reach $J=1$ at $t=0$ cannot contribute if they are of positive signature. The only source of trouble then is cuts of negative signature which reach $J=1$ at $t=0$. Such a cut is expected to have the form

$$\frac{1}{[\frac{1}{4}t(\frac{1}{4}t-m^2)]^{(J-2)/2}}\,\frac{1}{(\frac{1}{4}t-m^2)}\,\frac{1}{(J-1)^{1/2}}F_{-+,+-}{}^{J}(t)=\frac{i}{J-1}\int^{\alpha_c(t)}\frac{\rho(\gamma,t)}{J-\gamma}d\gamma\,\cot(\pi J/2)\,, \quad (5.6)$$

where $\alpha_c(0)=1$, and $\rho(\gamma,t)$ is a spectral function which

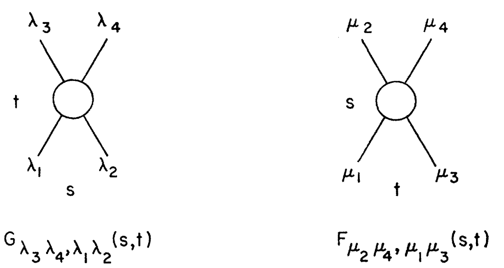

FIG. 2. The processes described by the amplitudes F and G of Eq. (5.2). Particles 1 and 3 are photons; 2 and 4 are protons.

[21] D. V. Volkov and V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **44**, 1068 (1963) [English transl.: Soviet Phys.—JETP **17**, 720 (1963)]; M. Gell-Mann and E. Leader, in *Proceedings of the Thirteenth International Conference on High-Energy Physics Berkeley, 1966* (University of California Press, Berkeley, California, 1967).

[22] This particular result was obtained already by V. D. Mur, Ref. 9, using the factorization of Regge residues. We are grateful to Dr. H. Pagels for drawing our attention to this part of Mur's work.

depends on the details of the dynamics. Near $t=0$, $\rho(\gamma,t)\approx t\rho(\gamma)$ from the fact that $F_{-+,+-}(t,s)$ is finite at $t=0$ and, furthermore, F^J has the reality property $F_{-+,+-}{}^{J*}(t^*)=-F_{-+,+-}{}^J(t)$. The lower limit of the integral in (5.6) is somewhere in the left-hand J plane. The above form has been found for cuts arising from three-particle intermediate states by Anselm, Azimov, Danilov, Dyatlov, and Gribov[23] and would seem to be reasonable for cuts arising from intermediate states of four or more particles. The $\cot(\pi J/2)$ guarantees that the discontinuity across the cut in the t plane vanishes for $J=$odd (right signature).

Expression (5.6) contributes to the full amplitude a leading term

$$(1-z)^{1/2}(1+z)^{3/2}F_{-+,+-}(t,s)$$
$$=-\pi i\int^{\alpha_c(t)}d\gamma\Big\{\rho(\gamma,t)(\tfrac{1}{4}t-m^2)[\tfrac{1}{4}t(\tfrac{1}{4}t-m^2)]^{(\gamma-2)/2}$$
$$\times\frac{P_{\gamma+2}(-z)-P_{\gamma+2}(z)}{2\sin\pi\gamma}\cot(\pi\gamma/2)(2\gamma+1)$$
$$\times\left(\frac{C_{\gamma+2}(1,2,\gamma)}{(\gamma-1)^{1/2}}\right)\Big\}+\cdots \quad (5.7)$$

or

$$F_{-+,+-}(0,s)\underset{s\to\infty}{\longrightarrow}-\pi\rho(0)\frac{\sqrt{3}}{2}\frac{s}{\ln s}.$$

The reality property of F^J requires that $\rho(0)$ be real so that the $s/\ln s$ term does not contribute to the Drell-Hearn integral (5.5). The leading imaginary term obtained from (5.7) is

$$\mathrm{Im}F_{-+,+-}(0,s)\underset{s\to\infty}{\longrightarrow}-\pi^2\rho(0)\frac{\sqrt{3}}{4}\frac{s}{(\ln s)^2},$$

which allows the integral in (5.5) to converge. This result, based on (5.6) and finite $\rho(0)$, serves as a concrete illustration of a more general result. Since the Regge cuts cannot move past $J=1$ at $t=0$, and since the Pomeranchuk pole itself cannot contribute, it seems reasonable to assume that $\lim_{s\to\infty}[F_{-+,+-}(s,0)/s]$ exists. Given this assumption, it follows that the integral converges.[24]

It is also of some interest to ask how fast it converges and so obtain some idea of the accuracy of saturation calculations. From a pure Regge pole point of view, the lowest-mass candidates for the allowed set of quantum numbers are the A_1 at 1080 MeV and the D at 1285 MeV.[25] With slopes of the order of 1 BeV^{-2}, both of these trajectories would be negative at $t=0$ and so lead to an integrand dropping off faster than $1/s^2$. In general, the rate of convergence of the sum rule will depend critically, however, on the strength of the cuts involved, i.e., on the value of $\rho(0)$.

In closing, let us remark that similar arguments give no support to the hope that a number of sum rules which have appeared in the literature will converge. These are the sum rules which have no energy denominators; for example, Eq. (15) of Bég,[26] and also Eq. (2) of Harari,[27] and Eq. (6) of Pagels.[28] Since these involve large isospin and/or helicity flip, a simple Regge *pole* model would lead one to conclude that they might converge. On the other hand, even if the integrals do converge, these amplitudes all have fixed poles at $J=0$ (right signature) to the lowest order in e^2, which contribute asymptotically to the real parts and would vitiate the no-subtraction hypothesis that is needed in the derivation of the sum rules. In fact, the difference between the two sides of these sum rules is then just the residue of the fixed pole at $J=0$, which might vanish accidentally but must depend on the details of the dynamics.

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