Spin Dependence of High-Energy Scattering Amplitudes. I*

A. H. MUELLER AND T. L. TRUEMAN
Brookhaven National Laboratory, Upton, New York
(Received 24 March 1967)

A formalism, useful for discussing spin dependence of scattering amplitudes in terms of complex angular momentum, is given. The question of the spin dependence of elastic scattering amplitudes, its relationship to coupling to Regge poles, fixed poles in the angular-momentum plane, and superconvergence relations are discussed. It is concluded that general principles permit the coupling at a vertex of a Regge pole of integer spin J_0 and two particles whose spin projection is greater than J_0, provided the Regge pole has the wrong signature to be a particle of spin J_0. In particular, the Pomeranchuk trajectory at $t=0$ ($J_0=1$) can flip helicities by two or more units and hence yield spin-dependent cross sections. Several models which support this conclusion are given. The possibility that a particular Regge pole's coupling does vanish is not ruled out, however.

I. INTRODUCTION

THIS is the first in a series of papers devoted to the spin dependence of scattering amplitudes at high energy. The subject of this paper is the general formalism for relating high-energy scattering to complex angular-momentum properties for particles with spin, and, specifically, the spin dependence of elastic scattering. Subsequent papers will be devoted to the special problems in the scattering of massless particles and in inelastic scattering.

One way of asking the question that stimulated our interest in this work is: To what extent does a Regge pole behave like a particle? When its trajectory function $\alpha(t)$ takes on an integral value J_0, can it couple to vertices which involve helicity flip greater than J_0? In particular, can the Pomeranchuk trajectory flip helicities by more than one unit at $t=0$? This last question is especially interesting since, if the answer is no, then high-energy forward elastic scattering must become spin-independent.[1] Naive angular-momentum arguments would yield a negative answer since anything that flips helicity by $\Delta\mu$ must have an equal angular-momentum projection and so must have angular momentum at least $\Delta\mu$. Naive angular-momentum arguments can be very misleading, as is well known. For example, the coupling of a spin-zero to a spin-one meson is not forbidden by angular-momentum conservation alone, but requires, in addition, the assumption of coupling of the vector meson by a conserved current. (Consider the virtual process $\pi \rightarrow W \rightarrow \mu+\nu$.) Since a Regge pole is evidently a rather complex object, it is clear that considerable caution is needed in using simple arguments. One thing is certain: If the total amplitude involves helicity flip $\Delta\mu$, it must not have a pole at some value of t unless there is a bound state or resonance with spin of $\Delta\mu$ or greater with $(\text{mass})^2=t$.

With this question in mind, we summarize the contents of the following sections. Section II: Some properties of spin dependence from general arguments are reviewed. The assertion that helicity flip of one unit or less in the cross channel implies spin independence of forward elastic scattering is supported (Hara's theorem).[1] Section III: Complex angular-momentum formalism for spinning particles is presented. This section is not limited to elastic scattering. Although this subject has been discussed previously by other authors,[2–4] we believe this presentation is more complete and easier to use than the previous ones. In any case, some redundance is useful in this complicated subject.

Section IV applies the machinery of Sec. III to our question. It is shown that the coupling of a Regge pole to helicity flip $\Delta\mu$ at $\alpha(t)=J_0<\Delta\mu$ need not vanish. It will never lead to a pole in the total amplitude while it can contribute significantly to the asymptotic behavior. Gell-Mann pointed out long ago that such nonsensical couplings could exist.[5] However, for potential scattering or for the "right signature" [$(-1)^{J_0}=P_j$ $=$signature]—the case of interest to him—the trajectory had to choose either "sense" or "nonsense." That is, if there was a pole in partial-wave amplitudes with $J_0<\Delta\mu$, then there would be no pole in partial-wave amplitudes with $J_0\geq\Delta\mu$, and vice versa. It is shown here that such is not the case for "wrong-signature" amplitudes. It is shown that the last point is intimately connected with superconvergence relations and fixed poles in the angular-momentum plane. This feature permits the Pomeranchuk trajectory ($P_j=+1$) to couple both to sense and nonsense amplitudes and, hence, permits high-energy scattering to be spin dependent.

Some simple models are investigated in Sec. V, the purpose being to see if we can find any more specific dynamical reason for such couplings to vanish. The result is negative.

* Work performed under the auspices of U. S. Atomic Energy Commission.
[1] Y. Hara, Phys. Letters 23, 696 (1966).

[2] S. Mandelstam, Nuovo Cimento 30, 1113 (1963).
[3] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. 133, B145 (1964).
[4] J. M. Charap and E. J. Squires, Ann. Phys. (N. Y.) 21, 8 (1963).
[5] M. Gell-Mann, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962).

We may summarize the result of the investigation of our question: There is no reason from general principles or from specific models that forbids a Regge trajectory, when it passes through an integer value J_0, from coupling to amplitudes of helicity flip of any magnitude when $P_j=-(-1)^{J_0}$ (wrong signature).[2] When applied to the Pomeranchuk trajectory, this leads to the conclusion that one must expect that high-energy scattering of high-spin particles is spin-dependent. Unfortunately, we have not been able to prove that the coupling cannot vanish "accidentally"—i.e., as a result of the detailed dynamics actually governing strong interactions. One final "unfortunately"—with spin $\frac{1}{2}$ particles, the maximum helicity flip is 1 and so Hara's theorem applies immediately and forward NN scattering must be spin-independent if dominated by the Pomeranchuk trajectory. We hope in the following papers to present results which can more reasonably be tested than the spin dependence of, say, high-energy πD elastic scattering.

We conclude this introduction with a few definitions. The amplitude[6] $G_{\lambda_3\lambda_4,\lambda_1\lambda_2}(s,t)$ will be the center-of-mass amplitude for the reaction for which $s^{1/2}$ is the center-of-mass energy and whose asymptotic behavior in s we are interested in. $F_{\mu_2\mu_4,\mu_1\mu_3}(s,t)$ is the center of mass amplitude for the reaction obtained by crossing particles 2 and 3. The quantity $t^{1/2}$ is the center-of-mass energy for this reaction, whose partial-wave amplitudes will be extended into the complex angular-momentum plane. The indices λ and μ denote the helicities (see Fig. 1). F and G are related by the equation[7]

$$G_{\lambda_3\lambda_4,\lambda_1\lambda_2}(s,t)=\sum_{\mu's} F_{\mu_2\mu_4,\mu_1\mu_3}(s,t)d_{\mu_1\lambda_1}{}^{s_1}(\chi_1)d_{\mu_2\lambda_2}{}^{s_2}(\chi_2)\times d_{\mu_3\lambda_3}{}^{s_3}(\chi_3)d_{\mu_4\lambda_4}{}^{s_4}(\chi_4). \quad (1.1)$$

s_i denotes the spin of each particle. χ_i are angles given in Ref. 7 and are real for s and t in the physical region for either reaction. For $t=0$ and elastic scattering (of nonzero-mass particles) $\chi_i=\pi/2$ for all i.

II. BACKGROUND—HARA'S THEOREM

It is interesting to see how much can be said about the spin dependence of the scattering amplitude $G_{\lambda_3\lambda_4,\lambda_1\lambda_2}(s,t)$ by making fairly weak assumptions. There are well-known relations between amplitudes with different indices which follow from parity or time-reversal invariance or from statistics in the case of identical particles.[6] Do "reasonable assumptions" about asymptotic behavior lead to further relations? It was shown by Foldy and Peierls[8] that if one assumes that high-energy elastic scattering is dominated by the exchange of a unique isotopic spin—Regge pole, elementary particle, or what-have-you—unitarity alone

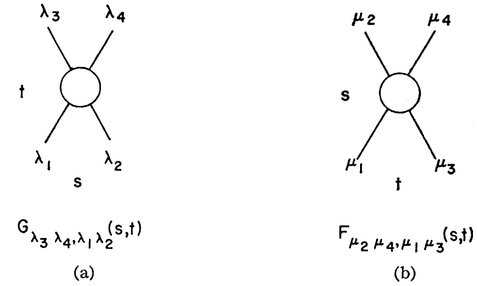


FIG. 1. Center-of-mass scattering amplitudes for (a) s-channel and (b) t-channel scattering.

required that this have isotopic spin zero. When, in turn, they assumed such dominance actually occurred, they found an amplitude diagonal in, and independent of, the charges of the particles. It is then natural to ask the same question about spin. Of course, the problem is substantially different because of the coupling between spin and kinematics. It seems unreasonable to assume that scattering be dominated by the exchange of a system of definite angular momentum J, and so one cannot expect to obtain such strong restrictions from reasonable assumptions as are obtained for an internal symmetry. Guided by the notions of complex angular momentum, Peierls and Trueman[9] asked: Suppose forward high-energy elastic scattering is dominated by the exchange of a system of definite parity (P), charge conjugation parity (C), and signature (P_j). Does unitarity give a restriction on these quantum numbers? The answer was yes, with the obvious set

$$P=C=P_j=+1$$

allowed. Assuming that such a system did, in fact, dominate, they showed that the scattering amplitude was diagonal in spins but not necessarily independent of spin:

$$G_{\lambda_3\lambda_4,\lambda_1\lambda_2}(s,0)\sim G_{\lambda_1\lambda_2}(s)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \quad (2.1)$$

but $G_{\lambda_1\lambda_2}(s)\neq G_{\lambda_1'\lambda_2'}(s)$, in general. It was also shown that if, in addition, helicity flip in the crossed channel was negligible [$F_{\mu_2\mu_4,\mu_1\mu_3}(s,0)\sim\delta_{\mu_1\mu_3}\delta_{\mu_2\mu_4}F_{\mu_1\mu_2}(s)$ with $F_{\mu_1\mu_2}(s)$ possibly depending on μ_1 and μ_2], then $G_{\lambda_1\lambda_2}(s)=G(s)$ independent of λ_1 and λ_2.

In a recent paper Hara[1] has argued that, if forward elastic scattering is dominated by the exchange of the Pomeranchuk trajectory whose trajectory function $\alpha(t)$ satisfies $\alpha(0)=1$, these conditions on $F_{\mu_2\mu_4,\mu_1\mu_3}(s,0)$ must hold. There are only two essential steps to the argument, which we present in a slightly simplified form here[10]: (a) Because at $t=0$ the Pomeranchuk trajectory behaves like a spin 1 particle, it cannot couple to helicity flips greater than 1; this step will

[6] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).
[7] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964); I. J. Muzinich, J. Math. Phys. **5**, 1481 (1964).
[8] L. L. Foldy and R. F. Peierls, Phys. Rev. **130**, 1585 (1963).
[9] R. F. Peierls and T. L. Trueman, Phys. Rev. **134**, B1365 (1964).
[10] Dr. K. Y. Lin of Cornell has also simplified the assumptions needed to prove Hara's theorem [K. Y. Lin, Phys. Rev. **159**, 1363 (1967)]. We thank Dr. Lin for sending us a copy of his work prior to publication.

be discussed at length in the following sections. (b) Because of its quantum numbers, the contributions of helicity flips of ± 1 vanish asymptotically compared to those of helicity flip zero. To see that (b) is true, use the crossing relation (1.1) with Eq. (1) above to show that

$$F_{\mu_2\mu_4,\mu_1\mu_3}(s,0)\sim F_{\mu_2\mu_4,\mu_3\mu_1}(s,0)\,, \tag{2.2}$$

a restriction stronger than that implied by the quantum number alone. Dominance by the Pomeranchuk trajectory implies

$$F_{\mu_2\mu_4,\mu_1\mu_3}(s,0)\sim\beta_{\mu_2\mu_4,\mu_1\mu_3}(0)d_{\mu\mu'}{}^1(z) \tag{2.3}$$

with

$$\beta_{\mu_2\mu_4,\mu_1\mu_3}(0)=\beta_{\mu_2\mu_4,\mu_3\mu_1}(0)\,; \tag{2.4}$$

but

$$d_{-\mu,\mu'}{}^1(z)\sim(-1)^\mu d_{\mu,\mu'}{}^1(z) \tag{2.5}$$

for large z and so we must have

$$\beta_{\mu_2\mu_4,\mu_1\mu_3}(0)=0 \text{ for } |\mu_1-\mu_3|=1 \text{ or } |\mu_2-\mu_4|=1. \tag{2.6}$$

[See the Appendix for the steps in (1) and (2). The strong result (6) relies very heavily on conservation of angular momentum in the forward direction.]

It will be apparent from the following sections that step (a) involves an assumption of dubious validity. In particular, we shall show that, even for a trajectory of the right signature ($P_j=-1$) to be a particle with $J=1$, (a) does not follow from angular-momentum conservation but from high-energy bounds and a shrinking diffraction peak. For a trajectory of the wrong signature, like the Pomeranchuk trajectory, it does not seem to follow from any reasonably general assumptions at all. Models will be given for which it does not hold.

III. FORMALISM

This section contains the basic formulas that are needed in discussing the scattering amplitudes in the complex J plane for particles with spin.[2–4] The formalism presented here is general and not limited to elastic scattering, but we do assume that the reaction described by F contains an even number of fermions in the initial and final states.

The starting point is the partial-wave expansion for helicity amplitudes[6] and its inversion formula:

$$F_{\mu_2\mu_4,\mu_1\mu_3}(s,t)=\sum_{J=n}^{\infty}(2J+1)F_{\mu_2\mu_4,\mu_1\mu_3}(t)d_{\mu\mu'}{}^J(\theta)\,, \tag{3.1}$$

$$\mu=\mu_1-\mu_3\,,$$

$$\mu'=\mu_2-\mu_4\,,$$

$$n=\max\{|\mu|,|\mu'|\}\,;$$

$$F_{\mu_2\mu_4,\mu_1\mu_3}{}^J(t)=\frac{1}{2}\int_{-1}^{1}dz\,d_{\mu\mu'}{}^J(\theta)F_{\mu_2\mu_4,\mu_1\mu_3}(s,t)\,, \tag{3.2}$$

where

$$z=\cos\theta\,.$$

It will be assumed that the amplitudes

$$\tilde{F}_{\mu_2\mu_4,\mu_1\mu_3}(s,t)=\frac{F_{\mu_2\mu_4,\mu_1\mu_3}(s,t)}{\chi_{\mu,\mu'}(z)}\,,$$

$$\chi_{\mu,\mu'}(z)=(1-z)^{|\mu-\mu'|/2}(1+z)^{|\mu+\mu'|/2} \tag{3.3}$$

are analytic in s for fixed t, free from kinematic singularities in s.[11,12] If $F(s,t)$ is bounded by $s^{\alpha(t)}$ for large s, then $\tilde{F}(s,t)$ will satisfy a dispersion relation with $\alpha(t)-n+1$ subtractions.[13] (When this number is negative, we have negatively subtracted or "superconvergent" dispersion relations.[14])

This assumption, in conjunction with Eq. (3.2), may be used to construct the generalization of the Froissart-Gribov continuation.[15] According to our assumption

$$F_{\mu_2\mu_4,\mu_1\mu_3}(s,t)=\chi_{\mu,\mu'}(z)\left\{\sum_{m=0}^{M}a_m(t)z^m+\frac{z^m}{\pi}\int_{-\infty}^{\infty}dz'\frac{\mathrm{Im}\tilde{F}_{\mu_2\mu_4,\mu_1\mu_3}(z,t)}{z'^m(z'-z)}\right\}; \tag{3.4}$$

M denotes the smallest non-negative integer greater than or equal to $\alpha(t)-n$. (One could explicitly use a negatively subtracted dispersion relation when that number is negative. For our purposes there is no advantage to this.) If this is substituted into (3.2) for $J-n<M$, the subtractions play no role. Before proceeding further, it will be convenient to express the rotation matrices in terms of Legendre functions and Clebsch-Gordan coefficients, primarily because the properties of the latter are much better known than the former. The relation we will use extensively is

$$(1-z)^{(\mu'-\mu)/2}(1+z)^{(\mu'+\mu)/2}d_{\mu\mu'}{}^J(\theta)=\sum_{l=J-n}^{J+n}C_l(\mu,\mu',J)P_l(z)\,, \tag{3.5}$$

with

$$C_l(\mu,\mu',J)=(2l+1)2^{\mu'}\left(\frac{(\mu'-\mu)!(\mu'+\mu)!}{(2\mu')!}\right)^{1/2}\times\begin{pmatrix}J & l & \mu'\\ \mu' & 0 & -\mu'\end{pmatrix}\begin{pmatrix}J & l & \mu'\\ \mu & 0 & -\mu\end{pmatrix},$$

$$\mu'\geq 0 \quad\text{and}\quad \mu'\geq|\mu|\,. \tag{3.6}$$

Those which do not satisfy these inequalities may be obtained by using symmetry properties of $d_{\mu\mu'}{}^J$. We

[11] Y. Hara, Phys. Rev. **136**, B507 (1964).
[12] L. L. C. Wang, Phys. Rev. **142**, 1187 (1966).
[13] T. L. Trueman, Phys. Rev. Letters **17**, 1198 (1966).
[14] V. DeAlfaro, S. Fubini, G. Rossetti, and G. Furlan, Phys. Letters **21**, 576 (1966).
[15] M. Froissart (unpublished talk at La Jolla Conference on Theory of Strong Interactions, 1961); V. N. Gribov, Zh. Eskperim. i Teor. Fiz. **42**, 1260 (1962) [English transl: Soviet Phys.—JETP **15**, 873 (1962)]; F. Calogero and J. M. Charap, Ann. Phys. (N. Y.) **26**, 44 (1964).

will choose our amplitudes so that the above inequalities hold. This relation is a special case of the addition theorem for rotation matrices in which we have used[16]

$$d_{\mu\mu'}{}^{\mu'}(\theta)=\left(\frac{(2\mu')!}{(\mu'+\mu)!(\mu'-\mu)!}\right)^{1/2}\times\left(\frac{1+z}{2}\right)^{(\mu'+\mu)/2}\left(\frac{1-z}{2}\right)^{(\mu'-\mu)/2}(-1)^{\mu'-\mu}.$$

Then,

$$F_{\mu_2\mu_4,\mu_1\mu_3}{}^{J\pm}(t)=\frac{1}{\pi}\sum_{p=0}^{2n}C_{p+J-n}(\mu,\mu',J)\times\int_{z_0}^{\infty}dz\{\mathrm{Im}\tilde{F}_{\mu_2\mu_4,\mu_1\mu_3}(z+i\epsilon,t)\mp(-1)^{p-n}\,\mathrm{Im}\tilde{F}_{\mu_2\mu_4,\mu_1\mu_3}(-z+i\epsilon,t)\}Q_{p+J-n}(z)\quad(3.7)$$

provides the equation which allows $F^{J\pm}$ to be continued to complex J. Note that the amplitude has already been separated into positive and negative signatures $F^{J\pm}(t)$ which agree with the physical amplitudes for $(-1)^J$ equal to $+1$ and -1, respectively. The reason for this is to remove the $(-1)^J$ factor that naturally appears between the two terms in (3.7), just the same as in the spinless case.

Because $Q_l(z)$ behaves asymptotically like $(1/z)^{l+1}$ the integrals will converge for

$$\mathrm{Re}\{[\alpha(t)-n]-(p+J-n)\}<0\,,$$

i.e.,

$$\mathrm{Re}J>\alpha(t)\,,$$

is the condition that J must satisfy in order that the continuation $F^{J\pm}(t)$ be given by the integral (3.7).

In order to use Eq. (3.7), it is necessary to know some properties of the $C_l(\mu,\mu',J)$. The explicit expression we have used is

$$C_{p+J-\mu'}(\mu,\mu',J)=2^{\mu'}(2[p+J-\mu']+1)\frac{(\mu'-\mu)!(\mu'+\mu)!(2J+p-2\mu')!(J+p-\mu')!}{(2J+p+1)!}(-1)^{\mu'-\mu}\times\left(\frac{(J+\mu')!(J-\mu)!(J+\mu)!}{(J-\mu')!}\right)^{1/2}\sum_{k=0}\frac{(-1)^k}{k!(p-k)!(\mu'-\mu-k)!(J+p-\mu'-k)!(\mu'+\mu-p+k)!(J-\mu'+k)!}.\quad(3.8)$$

The reader may verify the following properties:

(a) $C_{p+J-\mu'}(\mu,\mu',J)$ goes to a constant as $|J|\to\infty$; this is important in allowing (3.7) to be used for complex J and is also important in the Sommerfeld-Watson transformation.

(b) For J near an integer $0\leqslant J_0<\mu'$, $C_{p+J-\mu'}(\mu,\mu',J)$ behaves as follows:

For

$$|\mu|\leqslant J_0<\mu'\,(\text{sense-nonsense}^5)\,,$$
$$C_{p+J-\mu'}(\mu,\mu',J)\propto(J-J_0)^{1/2};\quad(3.9)$$

or

$0\leqslant J_0<|\mu|\leqslant\mu$, ("nonsense-nonsense"),

for $p\leqslant-1+2(\mu'-J_0)$, $C_{p+J-\mu'}(\mu,\mu',J)\propto$ constant;

for $p>-1+2(\mu'-J_0)$,

$$C_{p+J-\mu'}(\mu,\mu,,J)\propto(J-J_0)\,.\quad(3.10)$$

(c) $$C_{p+J-\mu'}(\mu,\mu',J)=(-)^{\mu-\mu'}C_{-p-J+\mu'-1}(\mu,\mu',J-1)\,,\quad(3.11)$$

even[4] for complex J. Properties (b) and (c) will be important in the discussion of the next section.

(d) $$C_{p+J-\mu'}(\mu,\mu',J)=(-1)^pC_{p+J-\mu'}(-\mu,\mu',J)\,.\quad(3.12)$$

(e) The C's have poles at positive half-integer values of J from the factor $(2J+p-2\mu')!$. [Note that $C_l(\mu,\mu',J)$ is finite at $l=-\frac{1}{2}$ because of $(2l+1)$ factor in (3.8).] These are not passed on to $F^J(t)$. To see this, observe that $d_{\mu\mu'}{}^J$ is regular at the half integers (cf. the hypergeometric expression for it)[4] and that Eq. (3.5) is valid for all z. In particular for large z, using $P_l(z)=P_{-l-1}(z)$, the coefficient of $z^{2(n-J)-1}$ in Eq. (3.5) at the half integers is

$$C_{-1+n-J}(\mu,\mu',J)+C_{J-n}(\mu,\mu',J)\,,$$

which must evidently be finite. Turning to Eq. (3.7) and recalling that $Q_l(z)=Q_{-l-1}(z)$ for l half integral, we see that the poles will cancel out in the sum in pairs:

$$C_{p+J-n}(\mu,\mu',J)Q_{p+J-n}(z)+C_{-(J-n)-1-p}(\mu,\mu',J)Q_{-(J-n)-1-p}$$

is finite for J equal to a positive half integer less than n.

The scattering amplitude $F(s,t)$ may be expressed in terms of the continued partial-wave amplitude $F^J(t)$

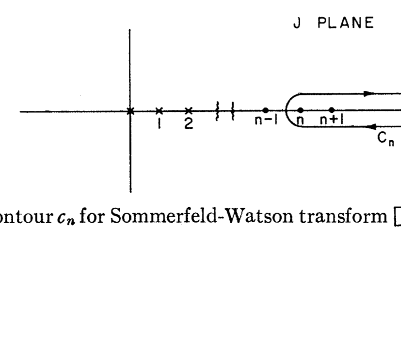


FIG. 2. Contour c_n for Sommerfeld-Watson transform [Eq. (3.13)].

[16] M. Hammermesh, *Group Theory* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1962).

by means of the Sommerfeld-Watson transformation: (see Fig. 2)

$$F_{\mu_2\mu_4,\mu_1\mu_3}(s,t)=\sum_{J=n}^{\infty}\frac{(2J+1)}{2}\{F_{\mu_2\mu_4,\mu_1\mu_3}{}^{J+}(t)[d_{\mu\mu'}{}^J(z)+(-1)^{\mu'}d_{-\mu,\mu'}{}^J(-z)]+F_{\mu_2\mu_4,\mu_1\mu_3}{}^{J-}(t)$$
$$\times[d_{\mu\mu'}{}^J(z)-(-1)^{\mu'}d_{-\mu,\mu'}{}^J(-z)]\}=-\frac{1}{4i}\int_{c_n}dJ(2J+1)\{F_{\mu_2\mu_4,\mu_1\mu_3}{}^{J+}(t)[d_{\mu\mu'}{}^J(z)+(-1)^{\mu'}d_{-\mu,\mu'}{}^J(-z)]$$
$$-F_{\mu_2\mu_4,\mu_1\mu_3}{}^{J-}(t)[d_{\mu\mu'}{}^J(z)-(-1)^{\mu'}d_{-\mu,\mu'}{}^J(-z)]\}(\sin\pi J)^{-1}, \quad (3.13)$$

the relation, valid for integral J, $(-1)^J d_{\mu\mu'}{}^J(z) = (-1)^{\mu'}d_{-\mu,\mu'}{}^J(-z)$ has been used here. The next step is the opening of the contour c_n and the movement of it to the left. Note that in the process poles will be crossed at integers $J<n$, from the factor $(\sin\pi J)^{-1}$, in addition to the singularities of $F^J(t)$. (If one is interested in other than the leading asymptotic terms, one must keep the background integral under control. This can be done by means of Mandelstam's transformation.[17,4]) Finally notice that the sense-nonsense amplitudes F^J with apparent square-root singularities at integers $|\mu|\leq J_0<\mu'$ are always multiplied by $C_l(\mu,\mu',J)$ which behave like $(J-J_0)^{1/2}$ and so there is no branch point of the integrand of (3.13) at those points.

IV. FIXED POLES, SUPERCONVERGENCE RELATIONS, AND NONSENSE TRANSITONS

In this section we shall discuss the relationship between fixed poles in the angular-momentum plane and superconvergence relations, along with the form of the sense-nonsense and nonsense-nonsense transitions in the presence of fixed poles.

Equation (3.7) expresses the Froissart-Gribov continuation of F^J as a sum of integrals over Legendre functions of the second kind. For ReJ sufficiently large these integrals necessarily exist since $Q_{J+p-\mu'}(z)$ goes as $z^{-p-J+\mu'-1}$ as z tends to infinity. If we call $\alpha(t)$ the trajectory function farthest to the right, it is clear that the integrals over the Q functions will converge for Re$J>\alpha(t)$ since Im$\tilde{F}\sim z^{\alpha(t)-\mu'}$. Now consider only the term in (3.7) which has $p=0$ and so leads to the singularity farthest to the right in the J plane.

$$F_{\mu_2\mu_4,\mu_1\mu_3}{}^{J\pm}(t)=C_{J-\mu'}(\mu,\mu',J)-\frac{1}{\pi}\int_{z_0}^{\infty}dz$$
$$\times\{\text{Im}\tilde{F}_{\mu_2\mu_4,\mu_1\mu_3}(z+i\epsilon,t)\mp(-1)^{\mu'}$$
$$\times\text{Im}\tilde{F}_{\mu_2\mu_4,\mu_1\mu_3}(-z+i\epsilon,t)\}Q_{J-\mu'}(z)+\cdots. \quad (4.1)$$

At integral $J_0<\mu'$, $Q_{J-\mu'}$ has a pole in J giving the equation

$$F_{\mu_2\mu_4,\mu_1\mu_3}{}^{J\pm}(t)=\frac{C_{J-\mu'}(\mu,\mu',J)}{(J-J_0)}\frac{1}{\pi}\int_{z_0}^{\infty}dz$$
$$\times\{\text{Im}\tilde{F}_{\mu_2\mu_4,\mu_1\mu_3}(z+i\epsilon,t)\mp(-1)^{\mu'}\,\text{Im}\tilde{F}_{\mu_2\mu_4,\mu_1\mu_3}$$
$$\times(-z+i\epsilon,t)\}P_{\mu'-J-1}(z)+\cdots \quad (4.2)$$

for $J\approx J_0$ and $J_0<\mu'$. The upper sign indicates positive signature while lower sign indicates negative signature.

For integral J in the range $\alpha(t)<J<\mu'$ the function

$$P_{\mu'-J-1}(z)\tilde{F}_{\mu_2\mu_4,\mu_1\mu_3}(z,t)$$

goes to zero faster than z^{-1} at large z which requires that[13]

$$0=\int_{-\infty}^{\infty}dz\,P_{\mu'-J-1}(z)\tilde{F}_{\mu_2\mu_4,\mu_1\mu_3}(z,t). \quad (4.3)$$

Writing the superconvergence relations expressed in (4.3) as integrals over positive z, one obtains

$$0=\int_{z_0}^{\infty}dz'\{\text{Im}\tilde{F}_{\mu_2\mu_4,\mu_1\mu_3}(z+i\epsilon,t)+(-1)^{\mu'-J-1}$$
$$\times\text{Im}\tilde{F}_{\mu_2\mu_4,\mu_1\mu_3}(-z+i\epsilon,t)\}P_{\mu'-J-1}(z). \quad (4.4)$$

Comparing (4.4) with (4.2), we separate the discussion into the following cases:

(i) $J_0<|\mu|$. In this case $C_{J-\mu'}(\mu,\mu',J)$ is finite at $J=J_0$, and we see that for amplitudes of the right signature $[(-1)^{J_0}=P_j]F^{J\pm}$ is regular in J at $J=J_0$, since the superconvergence relations (4.4) require the residue of the pole to be zero. However, (4.4) gives no restriction on amplitudes of the wrong signature so that high-energy behavior does not require that the residues of such poles vanish.

(ii) $\mu'>J_0>|\mu|$. Now $C_{J-\mu'}(\mu,\mu',J)$ goes to zero like $(J-J_0)^{1/2}$ near $J=J_0$. For the right signature Eq. (4.4) requires $F^J\propto(J-J_0)^{1/2}$, while for the wrong signature $F^J\propto 1/(J-J_0)^{1/2}$ is possible.

We can also see some of the differences between right and wrong signature amplitudes by looking at the expansion of the full amplitude given by (3.13). Using (3.5) we obtain the representation

$$(1+z)^{(\mu'+\mu)/2}(1-z)^{(\mu'-\mu)/2}F_{\mu_2\mu_4,\mu_1\mu_3}(z,t)$$
$$=\sum_{J=\mu'}^{\infty}\sum_{p=0}^{2\mu'}(J+\tfrac{1}{2})C_{J-\mu'+p}(\mu,\mu',J)\times\{F_{\mu_2\mu_4,\mu_1\mu_3}{}^{J+}(t)$$
$$\times[P_{J-\mu'+p}(z)+(-1)^{p+\mu'}$$
$$\times P_{J-\mu'+p}(-z)]+F_{\mu_2\mu_4,\mu_1\mu_3}{}^{J-}(t)$$
$$\times[P_{J-\mu'+p}(z)-(-1)^{p+\mu'}P_{J-\mu'+p}(-z)]\}. \quad (4.5)$$

[17] S. Mandelstam, Ann. Phys. (N. Y.) 19, 254 (1962).

Suppose F^J has a fixed pole of the form

$$F_{\mu_2\mu_4,\mu_1\mu_3}{}^{J\pm}(t)\approx\frac{\beta(t)}{J-J_0} \tag{4.6}$$

for some $0\leq J_0<|\mu|$, μ'. This is precisely the form we found for amplitudes of the wrong signature in the previous discussion, but now we wish to see what contribution to the full amplitude such a pole gives. We shall take only the leading contribution at large z to (4.5). Writing (4.5) as a Sommerfeld-Watson contour and opening up the contour to obtain the leading contribution of (4.6), $p=2\mu'$, we find

$$(1+z)^{(\mu'+\mu)/2}(1-z)^{(\mu'-\mu)/2}F_{\mu_2\mu_4,\mu_1\mu_3}(z,t)\underset{z\to\infty}{\longrightarrow} -\beta(t)(J_0+\tfrac{1}{2})z^{J_0+\mu'}[1\pm(-1)^{J_0}] \times a(J_0+\mu')\left[\frac{d}{dJ}C_{J+\mu'}(\mu,\mu',J)\right]\bigg|_{J=J_0},$$

where $P_\alpha(z)\sim a(\alpha)z^\alpha$ as $z\to\infty$. $C_{J+\mu}(\mu,\mu',J)$ has a simple zero at $J=J_0$ so that the contribution to the asymptotic behavior of F is proportional to z^{J_0} for the right signature while the contribution vanishes for the wrong signature. Thus if F^J did have a fixed pole at $J_0<|\mu|,\mu'$ in an amplitude of the right signature, this fixed pole would violate the unitarity bound on F if $J_0>1$. The superconvergence relations have also shown us that no such fixed poles exist. However, there is no reason why F^J cannot have a fixed pole at $J_0<|\mu|,\mu'$ in amplitudes of the wrong signature. Such a pole gives no contribution to the asymptotic form of the full amplitude as we have just seen. In fact, one can consider the complete contribution of the fixed pole, (4.6), to the full amplitude if the Mandelstam version[17] of the Sommerfeld-Watson transformation is used and then pushed back into the left half J plane. Such a calculation shows that a fixed pole of the type (4.6) in a wrong-signature amplitude contributes asymptotically only as z^{-J_0-1} to the full amplitude. We can carry out a similar analysis for a fixed pole at J_0 for which $|\mu|\leq J_0<\mu'$ taking the fixed pole as

$$\frac{1}{(J-J_0)^{1/2}}F_{\mu_2\mu_4,\mu_1\mu_3}{}^{J\pm}(t)\approx\frac{\beta(t)}{J-J_0}. \tag{4.7}$$

Equation (4.7) gives a contribution to the full amplitude at large z

$$(1+z)^{(\mu'+\mu)/2}(1-z)^{(\mu'-\mu)/2}F_{\mu_2\mu_4,\mu_1\mu_3}(z,t)\underset{z\to\infty}{\longrightarrow} -(J_0+\tfrac{1}{2})z^{J_0+\mu'}(1\pm(-1)^{J_0})\beta(t)\times a(J_0+\mu') \times\left[\frac{d}{dJ}(J-J_0)^{1/2}C_{J+\mu'}(\mu,\mu',J)\right]\bigg|_{J=J_0}.$$

Since $(J-J_0)^{1/2}C_{J+\mu'}(\mu,\mu',J)$ has a simple zero at $J=J_0$ for $|\mu|\leq J_0<\mu'$ the conclusions are essentially the same as for $J_0<\mu,\mu'$. These conclusions are in agreement with those found by Mandelstam.[18]

We now come to the question of what happens to a trajectory when α goes through an integral value $J_0<\mu'$. At J_0 there are three types of transition amplitudes: sense-sense, sense-nonsense, and nonsense-nonsense. It was shown in Ref. 5 that in potential theory or, indeed, in any theory not having a third double spectral function, a Regge trajectory chooses either sense or nonsense at J_0. We shall review the argument of Gell-Mann for potential theory[7] and then consider theories possessing third double spectral functions. For simplicity only the relatively simple example of nucleon-nucleon scattering will be considered.

In a potential theory of nucleon-nucleon scattering the coupled triplet amplitudes[19] would have the form

$$F^J(t)=\begin{pmatrix} f_{00}{}^J(t) & f_{01}{}^J(t)\\ f_{01}{}^J(t) & f_{11}{}^J(t)\end{pmatrix} \approx\frac{1}{J-\alpha(t)}\begin{pmatrix}\beta_{00}(t) & J^{1/2}\beta_{01}(t)\\ J^{1/2}\beta_{01}(t) & \beta_{11}(t)\end{pmatrix}$$

near $J=0$ and near $t=t_0$ if $\alpha(t_0)=0$. The $J^{1/2}$ is simply the square root due to a Clebsch-Gordan coefficient as we have seen in the analysis presented earlier in this section.

Factorization of residues gives the relation $\beta_{00}(t)\beta_{11}(t)=\alpha(t)[\beta_{01}(t)]^2$. If $\beta_{00}(t_0)$ is finite the pole at t_0 represents a physical particle. Then $\beta_{11}(t_0)$ must be zero, the pole does not appear in the nonsense $\to$ nonsense transition, and one says that the trajectory chooses sense at $J=0$. On the other hand, if $\beta_{11}(t_0)$ is finite the trajectory is said to choose nonsense. In this case the pole at t_0 does not represent a physical particle.

In the presence of a third double spectral function the above conclusions are valid at a nonsense value of J in an amplitude of the right signature, since such an amplitude has no fixed poles. For amplitudes of the wrong signature, the situation is somewhat different because of the presence of fixed poles. Nucleon-nucleon scattering at $J=0$ in the coupled triplet amplitudes of negative signature illustrates what now occurs when a trajectory passes through a nonsense value of J. F^J has the form

$$F^J(t)=\begin{pmatrix} f_{00}{}^J & f_{01}{}^J\\ f_{01}{}^J & f_{11}{}^J\end{pmatrix} \approx\frac{1}{J-\alpha(t)}\begin{pmatrix}\gamma_{00}(t) & J^{-1/2}\gamma_{01}(t)\\ J^{-1/2}\gamma_{01}(t) & J^{-1}\gamma_{11}(t)\end{pmatrix}. \tag{4.8}$$

We discuss the general structure of such amplitudes in

[18] S. Mandelstam, Nuovo Cimento **30**, 1128 (1963); **30**, 1148 (1963).

[19] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

detail since this structure has a heavy bearing on the question of spin independence at high energies. The J^{-1} factor in $f_{11}{}^J$ comes from the Q function as we have already seen in Eq. (4.2). The $J^{-1/2}$ in $f_{01}{}^J$ occurs as the product of a pole at $J=0$ from a Q function and a $J^{1/2}$ from a Clebsch-Gordan coefficient. Factorization gives the relation $\gamma_{00}(t)\gamma_{11}(t)=(\gamma_{01}(t))^2$. In general, γ_{00}, γ_{11}, and γ_{01} will all be nonzero as will be argued on the basis of two strong interaction models in the next section, and so the pole at $\alpha=0$ will appear in $f_{00}{}^0$ suggesting that there is now no such a thing as choosing sense or choosing nonsense. Of course the pole at $\alpha=0$ does not correspond to a physical particle, since it occurs in an amplitude which itself is not physical.

We shall now examine the implications which an amplitude of the form shown in Eq. (4.8) has for the full amplitude. More explicitly, suppose

$$F_{\mu_2\mu_4,\mu_1\mu_3}{}^{J\pm}(t)\approx\frac{1}{J-\alpha(t)}\frac{\gamma(t)}{J-J_0} \tag{4.9}$$

for $J_0<|\mu|,\mu'$ where $F^{J\pm}$ is of the *wrong* signature at J_0. Using (4.5), writing a Sommerfeld-Watson transformation to pick up the leading term at large z, we obtain

$$\begin{aligned}(1+z)^{(\mu'+\mu)/2}&(1-z)^{(\mu'-\mu)/2}F_{\mu_3\mu_4,\mu_1\mu_2}(z,t)\\ &=-\pi[\alpha(t)+\tfrac{1}{2}]\frac{\gamma(t)C_{\alpha(t)+\mu'}(\mu,\mu',\alpha(t))}{[\alpha(t)-J_0]\sin\pi\alpha(t)}\\ &\quad\times[(-1)^{\mu'}P_{\alpha(t)+\mu'}(-z)\pm P_{\alpha(t)+\mu'}(z)]+\cdots.\end{aligned} \tag{4.10}$$

Since F^J is of the wrong signature at J_0, the $\pm$ in (4.10) is such that

$$1\pm(-1)^{J_0}=0. \tag{4.11}$$

Also,

$$C_{J_0+\mu'}(\mu,\mu',J_0)=0 \tag{4.12}$$

as we have seen previously. Equations (4.11) and (4.12) show that F does not have a singularity as α goes through J_0 since the zeros of $\alpha(t)-J_0$ and $\sin\pi\alpha(t)$ are cancelled by the zeros of the signature factor (4.11) and the Clebsch-Gordan coefficient (4.12). We can explicitly calculate the asymptotic form of (4.10) at $\alpha(t_0)-J_0$ as

$$\begin{aligned}F_{\mu_3\mu_4,\mu_1\mu_2}&(z,t)\\ &=(-1)^{(\mu-\mu')/2}a(J_0+\mu')i\pi(\alpha(t_0)+\tfrac{1}{2})\gamma(t_0)z^{\alpha(t_0)}\\ &\quad\times\left[\frac{d}{dJ}C_{J+\mu'}(\mu,\mu',J)\right]\Bigg|_{J=J_0}+\cdots.\end{aligned} \tag{4.13}$$

If μ is such that $|\mu|\leq J_0<\mu'$ the form

$$\frac{1}{(J-J_0)^{1/2}}F_{\mu_3\mu_4,\mu_1\mu_3}{}^{J\pm}(t)\approx\frac{\gamma(t)}{[J-\alpha(t)](J-J_0)} \tag{4.14}$$

gives the analog of (4.13) as

$$\begin{aligned}F_{\mu_3\mu_4,\mu_1\mu_2}(z,t)&=(1-)^{(\mu-\mu')/2}a(J_0+\mu')i\pi(\alpha+\tfrac{1}{2})\gamma z^{\alpha}\\ &\quad\times\frac{d}{dJ}[(J-J_0)^{1/2}C_{J+\mu'}(\mu,\mu',J)]\Bigg|_{J=J_0}+\cdots,\end{aligned} \tag{4.15}$$

and if $J_0>|\mu|,\mu'$

$$F_{\mu_3\mu_4,\mu_1\mu_2}{}^{J\pm}(t)\approx\frac{\gamma(t)}{J-\alpha(t)}$$

gives

$$\begin{aligned}F_{\mu_3\mu_4,\mu_1\mu_2}(z,t)=i\pi(\alpha+\tfrac{1}{2})\gamma z^{\alpha}C_{\alpha+\mu'}(\mu,\mu',\alpha)\\ \times(-1)^{(\mu-\mu')/2}a(J_0+\mu')\end{aligned}$$

at $\alpha=J_0$. All of the nonleading terms can be calculated by using Mandelstam's version of the Sommerfeld-Watson transformation. It may be directly verified that no singularities in the amplitude F arise from the assumed forms (4.9) and (4.14). The interested reader may verify that even if $F^{J\pm}(t)$ has the form (4.9) or (4.14) for the *right* signature, no singularity is produced in $F(s,t)$ for nonsensical values of the helicities.[20,21] It is thus apparent that the naive angular-momentum arguments are not applicable.

The significance of Eqs. (4.13) and (4.15) is now apparent. A Regge trajectory passing through a nonsense value J_0 can contribute to helicity flip larger than J_0 in the Regge-pole channel. In the light of this result, there would seem to be no reason why helicity flip larger than 1 should not be allowed to couple to the Pomeranchuk trajectory as it passes through $J=1$.

V. MODELS

The fact that fixed poles in the J plane are present at nonsense values of J in strong interaction amplitudes of the wrong signature is a direct consequence of the presence of a third double spectral function.[2,18] In a certain region of small negative values of t, the residues of the fixed poles in $\mathrm{Im}F_{\lambda_1\lambda_2}{}^J(t)$ can be calculated explicitly in terms of a finite integral over a product of two on-shell Born terms and a known function.[22] When two bosons having spins σ_1 and σ_2 scatter into two identical spinless bosons, the fixed pole farthest to the right in the J plane lies at $J=\sigma_1+\sigma_2-1$ if $\sigma_1+\sigma_2$ is even or at $\sigma_1+\sigma_2-2$ if $\sigma_1+\sigma_2$ is an odd integer.[23] This fixed pole farthest to the right is on that sheet in the J plane reached by continuing from large real values of J along the real axis to the integer in question. In weak-interaction amplitudes there are fixed poles at nonsense values of J in both right- and wrong-signature amplitudes.[20,21]

[20] J. B. Bronzan, I. S. Gerstein, B. W. Lee, and F. E. Low, Phys. Rev. Letters, 18, 32 (1967).
[21] V. Singh, Phys. Rev. Letters 18, 36 (1967).
[22] J. M. Charap, Nuovo Cimento 31, 459 (1964).
[23] Ya. I. Azimov, Phys. Letters 3, 195 (1963).

The existence of fixed poles raises the possibility that when a Regge trajectory passes through an integer, J_0, which is a nonsense value of J the form of the amplitude of the wrong signature for the process described above is

$$\frac{1}{(J-J_0)^{1/2}}F_{\lambda_1\lambda_2}{}^J(t)\approx\frac{1}{J-J_0}\frac{\gamma(t)}{J-\alpha(t)}, \tag{5.1}$$

where $\gamma(t_0)=\gamma[\alpha^{-1}(J_0)]$ would not be zero in general. If the above form is realized, it would mean that at t_0 the Regge pole contributes to high-energy scattering amplitudes involving helicity flip greater than J_0 in the t channel. There are several models which indicate that Eq. (5.1) is the correct form when a trajectory passes through a nonsense value of J in a strong interaction amplitude of the wrong signature. The form shown in Eq. (5.1) was found by Bronzan, Gerstein, Lee, and Low[20] for weak amplitudes in the right signature and will be extended in this section to wrong-signature weak amplitudes.

A. Weak Amplitudes

Let $F_{+-}{}^J$ be the Froissart-Gribov continuation of the helicity amplitude for the scattering of two isovector photons of mass m, one of helicity $+1$ and the other of helicity -1 into two π mesons of mass μ. Then with threshold factors explicitly removed

$$\hat{F}_{+-}{}^J(t)=\frac{1}{(\frac{1}{4}t-\mu^2)}\frac{1}{[(\frac{1}{4}t-m^2)(\frac{1}{4}t-\mu^2)]^{(J-2)/2}}F_{+-}{}^J(t) \tag{5.2}$$

has the usual unitarity and left-hand cuts. If we call $\hat{f}_{+-}{}^J(t)$ the contribution of the left-hand cut to $\hat{F}_{+-}{}^J(t)$, we can impose two particle unitarity by means of the Omnes' solution

$$\hat{F}_{+-}{}^J(t)=\hat{f}_{+-}{}^J(t)+\frac{1}{D_J(t)}\frac{1}{\pi}\int_R\frac{\rho_J(t')N_J(t')\hat{f}_{+-}{}^J(t')dt'}{t'-t}, \tag{5.3}$$

where R signifies an integration along the two π unitarity cut. N_J and D_J are the continued numerator and denominator functions for $\pi\pi$ scattering.

If we take $f_{+-}{}^J$ as given by the Born terms in Fig. 3, then near $J=1$ we have

$$\frac{1}{(J-1)^{1/2}}\hat{f}_{+-}{}^J(t)\approx\frac{C}{J-1},$$

where C is a known constant which depends on the isotopic spin of the two pion system. This gives for

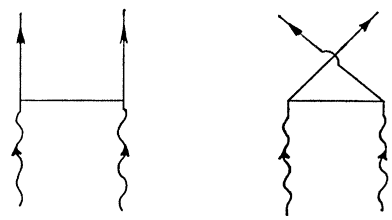


FIG. 3. Born terms for $f_{+-}{}^J$.

$\hat{F}_{+-}{}^J(t)$ near $J=1$

$$\frac{1}{(J-1)^{1/2}}\hat{F}_{+-}{}^J(t)\approx\frac{c}{J-1}\frac{1}{D_J(t)}\left[D_J(t)+\frac{1}{\pi}\int\frac{\rho_J(t')N_J(t')dt'}{t'-t}\right]$$

$$\frac{1}{(J-1)^{1/2}}\hat{F}_{+-}{}^J(t)\approx\frac{c}{J-1}\frac{1}{D_J(t)}.$$

[We have taken $D_J(t)\to 1$ as $t\to\infty$.]

If the two pions in the final state have isotopic spin 1, $\hat{F}_{+-}{}^J$ is of the right signature at $J=1$, and the residue of the fixed pole in $(J-1)^{-1/2}\hat{F}_{+-}{}^J$ is proportional to the electromagnetic form factor, $D_1(0)/D_1(t)$. We are unable, however, to calculate the scale factor between the residue of the fixed pole in $(J-1)^{-1/2}\hat{F}_{+-}{}^J$ and the form factor in this simple model. Now suppose that the two pions in the final state have isotopic spin 0 or 2. Then $\hat{F}_{+-}{}^J$ is of the wrong signature at $J=1$, and the residue of the fixed pole in $(J-1)^{-1/2}\hat{F}_{+-}{}^J$ is proportional to the inverse of an unphysical $\pi\pi$ denominator function.

Further suppose, for any of the possible isospin states, that

$$D_J(t)\approx\mu(t)[J-\alpha(t)] \tag{5.4}$$

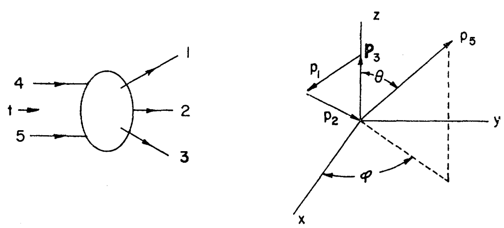
near $J=1$ for some trajectory function α. Then

$$\frac{1}{(J-1)^{1/2}}\hat{F}_{+-}{}^J(t)\approx\frac{C}{J-1}\frac{1}{\mu(t)}\frac{1}{J-\alpha(t)},$$

which gives $(J-1)^{-1/2}\hat{F}_{+-}{}^J(t)$ a singular residue function for a trajectory passing through $J=1$. The point to be emphasized here is that fixed poles and singular residues can occur in weak amplitudes of both the right signature and the wrong signature in contrast to the strong-interaction amplitudes where fixed poles and singular residues only occur for the wrong signature.

B. A Simple Model for Strong Amplitudes

Let $F_{+-}{}^J$ now be the Froissart-Gribov continuation of the helicity amplitude for the scattering of two stable ρ mesons, one of helicity $+1$, the other of helicity -1, into two π mesons. Again define $\hat{F}_{+-}{}^J$ by Eq. (5.1), and call $\hat{f}_{+-}{}^J$ the left-hand-cut contribution to $\hat{F}_{+-}{}^J$. If one wishes all of the cuts except the two-



(a) (b)

FIG. 4. (a) Production process $4+5\rightarrow 1+2+3$. (b) Coordinate system for process (4a).

pion unitarity cut can be included formally in $\hat f_{+-}^{J}$. The full amplitude, $\hat F_{+-}^{J}$, is given by (5.3) with N and D as before.

We can now illustrate one of the basic differences between amplitudes of the right signature and those of the wrong signature. Suppose F^J refers to scattering in an isotopic spin state $I=1$. Then near $J=1$ [see (3.9)], one has[18]

$$\hat f_{+-}^{J}(t)\approx (J-1)^{1/2} r(t),$$

since $J=1$ is of the right signature for $\hat f^J$. A pole in F^J due to a zero in D_J near $J=1$ appears as

$$\frac{1}{(J-1)^{1/2}}\hat F_{+-}^{J}(t)\approx \frac{1}{\mu(t)}\frac{1}{J-\alpha(t)}\frac{1}{\pi}\times\int_R \frac{\rho_1(t')N_1(t')r(t')dt'}{t'-t},$$

where D is defined by Eq. (5.4). One sees that the Regge residue of $(J-1)^{-1/2}\hat F_{+-}^{J}(t)$ is a nonsingular function at the value, t_0, for which $\alpha(t_0)=1$. However if F^J refers to scattering in an $I=0$ or an $I=2$ state, then near $J=1$

$$\hat f_{+-}^{J}(t)\approx \frac{1}{(J-1)^{1/2}} r(t),\quad{}^{18}$$

since $\hat f_{+-}^{J}$ is of the wrong signature at $J=1$, and the form of F^J near a zero of D is

$$\frac{1}{(J-1)^{1/2}}\hat F_{+-}^{J}(t)\approx \frac{1}{\mu(t)}\frac{1}{J-1}\frac{1}{J-\alpha(t)}\frac{1}{\pi}\times\int_R \frac{\rho_1(t')N_1(t')r(t')}{t'-t}dt'.$$

Now the Regge residue function $\beta(t)$ of $(J-1)^{-1/2}\hat F^J$ is singular near t_0 since

$$\beta(t)\approx \frac{1}{\alpha(t)-1}\frac{1}{\mu(t)}\frac{1}{\pi}\int_R \frac{N_1(t')\rho_1(t')r(t')}{(t'-t)}dt'.$$

Of course, it is possible that

$$\int_R \frac{N_1(t')\rho_1(t')r(t')}{t'-t_0}dt'=0,$$

but this would appear to be purely accidental. The point which we have tried to emphasize is that singularities of $\hat f^J$ in J at wrong-signature nonsense values of J can lead to singular residues when Regge trajectories pass through these values of J. This phenomenon of singular residues also occurs for production amplitudes as we shall now demonstrate.

C. A Production Model

The above arguments can also be illustrated in a production model in which two scalar mesons (interacting through a ϕ^3 Lagrangian) scatter into three of the same scalar mesons. Much of the development is taken from Anselm, Azimov, Danilov, Dyatlov, and Gribov[24] and the notation is theirs. The process under consideration and the kinematics are illustrated in Fig. 4. Figure 4(b) refers to the center-of-mass system of 4 and 5. $z=\cos\theta$ is the cosine of the angle between p_5 and p_3 in the center-of-mass system of 4 and 5. φ is the azimuthal angle between p_1 and p_5 with p_3 taken as the pole, and x is the angle between the momenta of particles 1 and 3 in the center-of-mass system of 1 and 2.

The amplitudes at positive integral J and m are defined by

$$f_{Jm}(t,t_{12},x)=\int\frac{d\Omega}{4\pi}P_{Jm}(z)e^{-im\varphi}A(t_1t_{12},x,z,\phi),$$

where $d\Omega=dzd\phi$ and A is the production amplitude. f_{Jm} is nonvanishing only for even values of J such that $m\leq J$. In order to get the correct continuation in J, we write, following Ref. 24,

$$P_{Jm}(z)=\frac{ie^{i\pi m/2}}{\pi}[Q_{Jm}(z+i\epsilon)-(-1)^m Q_{Jm}(z-i\epsilon)],\qquad -1<z<1$$

from which

$$f_{Jm}=\frac{ie^{i\pi m/2}}{\pi}\left\{\int_{-1}^{1}\frac{dz}{2}Q_{Jm}(z+i\epsilon)\int_0^{2\pi}\frac{d\varphi}{2\pi}e^{-im\varphi}A(z,\varphi)-\int_{-1}^{1}\frac{dz}{2}Q_{Jm}(z-i\epsilon)\int_0^{2\pi}\frac{d\varphi}{2\pi}e^{-im(\varphi+\pi)}A(z,\varphi)\right\}.$$

The above expression can be written as an integral over the contour, c, around the interval $(-1,1)$ in the clockwise sense.

$$f_{Jm}=\frac{ie^{i\pi m/2}}{\pi}\int_c\frac{dz}{2}Q_{Jm}(z)A_m(z),\tag{5.5}$$

where

$$A_m(z)=\int_0^{2\pi}\frac{d\varphi}{2\pi}e^{-im\varphi}A(z,\varphi).$$

The contour c can now be expanded to a contour c'

[24] A. A. Anselm, Ya. I. Azimov, G. S. Danilov, I. T. Dyatlov, and V. N. Gribov, Ann. Phys. (N. Y.) 37, 227 (1966).

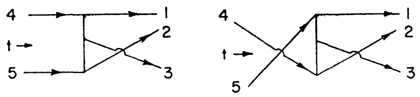


FIG. 5. Born terms for production amplitude f_{Jm}.

over the other singularities in the complex z plane. For ReJ sufficiently large the part of the contour at infinity gives zero contribution. We shall now take the particular case of the graphs indicated in Fig. 5.

The helicity amplitude, $f_{Jm}{}^0(t_1,t_{12},x)$, defined by Eq. (5.5) for the diagrams shown in Fig. 5, is given explicitly by Ref. 24. c' includes regions in both the right- and left-half z plane. The signature is introduced at this point in order to define an integral in which the argument of Q_{Jm} lies entirely in the right-half z plane. This is achieved by using

$$Q_{Jm}(z)=(-1)^{m+1}Q_{Jm}(-z)$$

in the integral over the cuts in the left-half z plane. Now that the signature has been introduced, the continued amplitude $f_{Jm}{}^{0+}$ does not vanish for odd $J\geq m$ nor does it vanish for odd $J<m$. $f_{Jm}{}^{0+}$ does vanish for even $J<m$, however. (Odd values of J correspond to the wrong signature, and odd $J<m$ are nonsense values corresponding to the wrong signature.)

Taking $f_{Jm}{}^{0+}(t,t_{12},x)$ as the left-hand-cut contribution, we can impose two-particle unitarity in the t channel by the Omnes solution

$$f_{Jm}{}^{+}(t,t_{12},x)=f_{Jm}{}^{0}(t,t_{12},x)+\frac{1}{D_J}\frac{1}{2\pi i}\times\int_R\frac{f_{Jm}{}^{0+}(t,'t_{12},x)N_J(t')}{t'-t}\rho_J(t')dt',$$

where N_J/D_J is the two-particle elastic scattering amplitude.

Suppose $D_{J_0}(t)\approx\mu(t)[J_0-\alpha(t)]$ near $t=t_0$ for some even J_0. Then

$$f_{Jm}{}^{+}\approx\frac{1}{\mu(t)}\frac{1}{J_0-\alpha(t)}\frac{1}{2\pi i}\times\int_R\frac{f_{J_0m}{}^{0+}(t',t_{12},x)N_{J_0}(t')}{t'-t_0}\rho_{J_0}(t')dt'.$$

If $m>J_0$ this expression is identically zero since $f_{J_0m}{}^{0+}=0$ implying that the residue of the Regge pole in $f_{Jm}{}^{+}$ vanishes for nonsense values of J in amplitudes of the right signature. However if J_0 is odd, the residue does not vanish for $m>J_0$.

It can be verified that

$$\int_R\frac{\rho_{J_0}(t)f_{J_0m}{}^{0+}(t,t_{12},x)N_{J_0}(t)dt}{t-t_0}$$

cannot vanish for all t_{12} and x. One can see this explicitly by forcing the singularity (A14) of Ref. 24 to move in the t plane around the branch point at $t=4\mu^2$ of the integrand to pinch with the singularity at $t=t_0$. The resulting singularity will not be zero if $N_{J_0}(t_0)$ is not zero.

APPENDIX

Here we give a derivation of the result of Peierls and Trueman, Eq. (2.1). Although the derivation is quite simple, it has caused some confusion to readers of that paper.

Assume that only $P=C=+1$ contributes ($PC=+1$ is sufficient). Then

$$F_{\mu_2\mu_4,\mu_1\mu_3}(s,t)=F_{\mu_2\mu_4,-\mu_3,-\mu_1}(s,t).\tag{A1}$$

For $t=0$, the crossing relation

$$\begin{aligned}G_{\lambda_3\lambda_4,\lambda_1\lambda_2}(s,0)&=\sum_{\mu's}d_{\mu_1\lambda_1}{}^{s_1}(\pi/2)d_{\mu_3\lambda_3}{}^{s_1}(\pi/2)d_{\mu_2\lambda_2}{}^{s_2}(\pi/2)\\&\times d_{\mu_4\lambda_3}{}^{s_2}(\pi/2)F_{\mu_2\mu_4,\mu_1\mu_3}(s,0)=\sum_{\mu's}d_{-\mu_3\lambda_1}{}^{s_1}(\pi/2)\\&\times d_{-\mu_1\lambda_3}{}^{s_1}(\pi/2)d_{\mu_2\lambda_2}{}^{s_2}(\pi/2)d_{\mu_4\lambda_4}{}^{s_2}(\pi/2)\\&\times F_{\mu_2\mu_4,\mu_1\mu_3}(s,0)=G_{\lambda_1\lambda_4,\lambda_3\lambda_2}(s,0)(-1)^{\lambda_1-\lambda_3}.\end{aligned}\tag{A2}$$

Conservation of angular momentum in the forward direction requires

$$\lambda_1-\lambda_2=\lambda_3-\lambda_4$$

and

$$\lambda_3-\lambda_2=\lambda_1-\lambda_4$$

or

$$\lambda_1=\lambda_3,\lambda_2=\lambda_4.$$

Thus, $P=C=+1$ implies

$$G_{\lambda_3\lambda_4,\lambda_1\lambda_2}(s,0)=G_{\lambda_1\lambda_2}(s)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}.\tag{A3}$$

One may use this in the crossing relation to calculate $F_{\mu_2\mu_4,\mu_1\mu_3}(s,0)$ and obtain Eq. (2.2):

$$\begin{aligned}F_{\mu_2\mu_4,\mu_1\mu_3}(s,0)&=\sum_{\lambda_1\lambda_2}G_{\lambda_1\lambda_2}(s)d_{\mu_1\lambda_1}{}^{s_1}(\pi/2)d_{\mu_2\lambda_2}{}^{s_2}(\pi/2)\\&\times d_{\mu_3\lambda_1}{}^{s_1}(\pi/2)d_{\mu_4\lambda_2}{}^{s_2}(\pi/2)=F_{\mu_2\mu_4,\mu_3\mu_1}(s,0),\end{aligned}\tag{A.4}$$

This condition is stronger than the symmetry implied by $P=C=+1$ from which we started since angular-momentum conservation has been used. This is closely related to the "conspirator" condition.[25] For example, in NN scattering the amplitudes

$$f_{00}=F_{++,++}+F_{++,--}$$

and

$$f_{11}=F_{++,++}-F_{++,--},$$

both have $CP=+1$. Obviously, from (A4), if f_{11} is not identically zero, there must be present amplitudes with $CP\neq+1$ of equal importance.

[25] D. V. Volkov and V. N. Gribov, Zh. Eskperim. i Teor. Fiz. 44, 1068 (1963) [English transl: Soviet Phys.—JETP 17, 720 (1963)]; M. Gell-Mann and E. Leader, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967); E. Abers and V. Teplitz, Phys. Rev. 158, 1365 (1967). L. Durand, III, Phys. Rev. Letters 18, 58 (1967).