Quarks and Magnetic Poles*

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A theory is developed which accounts for the free nonrelativistic motion of fractionally charged quarks within hadrons, and at the same time does not permit quarks to appear as individuals. This is accomplished by modifying Dirac's idea that the quantization of electric charge derives from the existence of a point magnetic pole, to include the situation in which the pole is extended in space and of hadronic size. The needed formalism makes use of Mandelstam's gauge-independent, path-dependent quantum electrodynamics, as extended by Cabibbo and Ferrari to include the existence of point magnetic poles. It is shown that the further extension to a pole of finite size and the use of parallel straight paths are the only new features that are required. In particular, no assumptions need be made with regard to the masses of quarks, the interactions between them, or the existence of a constraining potential.

I. INTRODUCTION

THE existence of a massive triplet of elementary particles, called quarks, from which hadrons (baryons and mesons) are constructed, was postulated three years ago by Gell-Mann and Zweig[1] on the basis of symmetry considerations. Quarks are assumed to have baryon number $\frac{1}{3}$ and electric charge $\frac{2}{3}e$ or $-\frac{1}{3}e$, where e is the proton charge; antiquarks have opposite charges and baryon number. A baryon is supposed to consist of three quarks moving nonrelativistically, and a meson is supposed to consist of a quark and an antiquark. More complicated schemes that make use of integer-charged subparticles are also possible[2] but will not be considered in this paper.

Apart from symmetry considerations,[1,3] there is rather good evidence for the existence of quarks as quasifree particles within hadrons.[4] This evidence is based on the additive quark model, according to which the impulse approximation can be applied to the constituent quarks and antiquarks involved in high-energy hadron-hadron collisions, in much the same way as it can be applied to the nucleons involved in high-energy nuclear collisions. There is, however, the striking difference between the two situations that nucleons are quite easily knocked out of nuclei and observed by themselves, whereas individual quarks have not been found.[5] Attempts have been made to account for this difference in behavior between quarks in hadrons and nucleons in nuclei, in terms of strong interactions between quarks,[3,6] but difficulties have been encountered in maintaining nonrelativistic motion and in stabilizing only those quark-antiquark systems that have integer baryon number.[7]

It was recently ponted out[6] that the observations can be understood in terms of a selection principle that has a range built into it. The selection principle would require that the baryon number for any cluster of quarks and antiquarks that lie within this range of each other be an integer. At the same time, the quarks should be able to move rather freely within each cluster, without being greatly inhibited by the selection principle. The range that is to be incorporated into the selection principle, which we denote by R, cannot be smaller than hadronic size, but it can be very much larger. Since single quarks have not been found, the upper limit would be set by the ability of laboratory apparatus to resolve individual quarks. Thus we expect that 10^{-13} cm$\lesssim R \lesssim 10^{-2}$ cm, although it seems likely that R is much nearer the smaller end of this interval than the larger.

A model for this selection principle was recently proposed that makes no assumptions about quark masses or interactions.[8] It generalized Dirac's idea[9] that the quantization of electric charge derives from the existence of a point magnetic pole of strength g. He showed that the quantum theory of a particle of electric charge q is consistent only if

$$qg/\hbar c = \tfrac{1}{2}n, \tag{1}$$

where n is an integer, so that all charges are integer multiples of the proton charge e provided that $g = (137/2)e$. Our generalization consisted in assuming

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2 T. D. Lee, Nuovo Cimento 35, 933 (1965).
3 G. Morpuro, Physics 2, 95 (1965).
4 E. M. Levin and L. L. Frankfurt, JETP Pis'ma. v Redaktsiyn 2, 105 (1965) [English transl.: Soviet Phys.—JETP Letters 2, 65 (1965)]; V. V. Ansovich, *ibid.* 2, 439 (1965) [English transl.: *ibid.* 2, 272 (1965)]; H. J. Lipkin and F. Scheck, Phys. Rev. Letters 16, 71 (1966). There have been a great many more recent papers, mainly in the Letters journals.
5 L. B. Leipuner, W. T. Chu, R. C. Larsen, and R. K. Adair, Phys. Rev. Letters 12, 423 (1964); V. Hagopian, W. Selove, R. Ehrlich, E. Leboy, R. Lanza, D. Rahm, and M. Webster, *ibid.* 13, 280 (1964); W. Blum, S. Brandt, V. T. Cocconi, O. Czyzewski, J. Danysz, M. Jobes, G. Kellner, D. Miller, D. R. O. Morrison, W. Neale, and J. G. Rushbrooke, *ibid.* 13, 353 (1964); A. Bubler-Broglin, G. Fortunato, T. Massam, T. Muller, and A. Zichichi, Nuovo Cimento 45A, 520 (1966); H. Kasha, L. B. Leipuner, and R. K. Adair, Phys. Rev. 150, 1140 (1966); W. A. Chupka, J. P. Schiffer, and C. M. Stevens, Phys. Rev. Letters 17, 60 (1966); G. Gallinaro and G. Morpurgo, Phys. Letters 23, 609 (1966).
6 L. I. Schiff, Phys. Rev. Letters 17, 612 (1966).
7 O. W. Greenberg and D Zwanziger, Phys. Rev. 150, 1177 (1966); M. V. Barnhill, Bull. Am. Phys. Soc. 11, 901 (1966).
8 L. I. Schiff, Phys. Rev. Letters 17, 714 (1966).
9 P. A. M. Dirac, Proc. Roy. Soc. (London) A133, 60 (1931); Phys. Rev. 74, 817 (1948).

that the magnetic pole has a finite size, which we also denote by R, that is of the order of the range associated with the selection principle. It was then argued that Dirac's approach could be modified so that the total charge of all particles within a distance R of each other is quantized, whereas the individual charges need not be. Thus if quarks have third-integral charge, only those combinations that have integer baryon number are allowed to form isolated clusters of size not greater than R. This provides the desired selection principle, although it does not explain why quarks have third-integral rather than some other fractional charge.

The assumption that the magnetic pole has a finite size is not unreasonable when it is realized that its coupling constant with the electromagnetic field, $g^2/\hbar c = 137/4$, is so large compared to unity that quantum electrodynamics is not likely to be valid. For the present purpose, a classical nonrelativistic treatment of a pole is adequate, and it is then expected to be extended in space because of its very large magnetostatic self-energy. For example, with $R = 10^{-13}$ cm, $g^2/R = 6.7$ BeV. That such an extended pole must behave classically can be seen from a calculation of the angular momentum associated with the electromagnetic field of the pole and a point charge q that lies outside it. This angular momentum, which has the magnitude qg/c for a point pole and is directed[10] from q to g, is equal to $(qg/c) \times [1-(R^2/3r^2)]$ for an extended pole,[8] where R is now the rms radius of the pole and r is the charge-pole separation distance. Thus as a charge moves radially in toward a fixed pole, the field angular momentum decreases, and a corresponding torque is exerted on the pole by the changing electric field of the charge, so that the angular momentum of the pole changes continuously.

Since our selection principle is expressed in terms of the spatial coordinates of the quarks, and we are concerned with their nonrelativistic motion, we shall make use of the many-particle nonrelativistic Schrödinger equation. As usually formulated, this equation contains the vector and scalar potentials of the electromagnetic field in which the charged particles move, and we are particularly interested in potentials that arise from a magnetic pole. It is clearly impossible to represent the magnetic field of a pole everywhere in this way, since the field is the curl of the vector potential, and hence has vanishing divergence. Dirac solved this problem for a point pole by allowing the vector potential to be singular along a line extending from the pole to infinity; he called this line a "string." He also found it necessary to require that "a string must never pass through a charged particle"; this is what Wentzel[11] called the Dirac "veto." In this way Dirac arrived at the relation (1) between the magnitudes of electric charges and magnetic poles.

The question now arises as to what happens if the pole is of finite size. We note first that the generalization of the usual Schrödinger equation to the situation in which a point pole is present can be carried through in an infinite number of ways. The Dirac string can be curved in any way, and there may be more than one string, as proposed by Schwinger.[12] These generalizations may give different results; for example, Schwinger concluded in contrast with (1) that $qg/\hbar c$ is an integer, or possibly an even integer, on the basis of two oppositely directed straight strings. There is a similar ambiguity when the pole is of finite size, since the string can be spread out into a "bundle" of any diameter and shape, diverging or converging, or into several bundles, in an infinite number of ways. In our earlier paper[8] it was argued that the particular assumption of a cylindrical bundle, with diameter R equal to that of the pole, would lead to the desired selection principle. This argument was rejected by Peres,[13] who stated without proof that even an extended pole necessarily leads to a singular string rather than a bundle. Peres's statement is incorrect; an explicit demonstration which refutes it has been given by Wentzel.[11]

Several other authors have dealt in various ways with quantum effects produced by magnetic poles.[14] We shall make use in the present paper of a modification of the formalism of Cabibbo and Ferrari,[14] which is a generalization of Mandelstam's quantum electrodynamics without potentials[15] to the case in which a point magnetic pole is present. In this formalism only the electromagnetic fields appear, so that there are no strings. On the other hand, the wave function is nonlocal since it depends on a set of paths that extend from each charged particle to infinity. In the absence of magnetic poles the choice of paths is unimportant, but when a pole is present different results may be obtained for different sets of paths. There is a close parallelism between the Dirac and the Mandelstam-Cabibbo-Ferrari formulations of the theory. There is an arbitrariness in the choice of the strings in the one case and of the paths in the other, and different choices may lead to different theoretical predictions. Also, as we shall see, the Dirac veto has its counterpart in the requirement that a path must never pass through a pole.

It is more convenient to work with paths than with strings. We shall show that it is possible to choose the paths in such a way that the desired selection principle is obtained. As remarked above, this choice is not

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[11] G. Wentzel, Progr. Theoret. Phys. (Kyoto) Suppl. Nos. 37 & 38, 163 (1966).

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[14] I. Tamm, Z. Physik **71**, 141 (1931); M. Fierz, Helv. Phys. Acta **17**, 27 (1944); P. P. Banderet, *ibid.* **19**, 503 (1946); N. Cabibbo and E. Ferrari, Nuovo Cimento **23**, 1147 (1962); R. A. Ferrell and J. J. Hopfield, Physics **1**, 1 (1964); D. Zwanziger, Phys. Rev. **137**, B647 (1965); S. Weinberg, *ibid.* **138**, B988 (1965); C. R. Hagen, *ibid.* **140**, B804 (1965); A. S. Goldhaber, *ibid.* **140**, B1407 (1965); R. Tevikyan, Zh. Eksperim. i Teor. Fiz. **50**, 911 (1966) [English transl.: Soviet Phys.—JETP **23**, 606 (1966)]; T.-M. Yan, Phys. Rev. **150**, 1349 (1966).

[15] S. Mandelstam, Ann. Phys. (N.Y.) **19**, 1 (1962).

unique, and there are infinitely many other ways of choosing the paths which do not lead to the selection principle. Nevertheless, our choice is a rather simple one, and it seems likely that it can be carried over from the many-particle nonrelativistic Schrödinger equation to field theory. Our theory does not account for the fact that magnetic poles have not been found,[16] although this might be explained by their large mass.

Section II presents Mandelstam's gauge-independent, path-dependent formalism, within the framework of the nonrelativistic Schrödinger equation rather than field theory. The extension of Cabibbo and Ferrari, which takes account of the possible existence of point magnetic poles, is also given in the same framework. In Sec. III, the modification of the Mandelstam-Cabibbo-Ferrari theory that results for an extended pole when the paths are chosen to be parallel straight lines is developed. It is then shown in Sec. IV that this leads to the selection principle proposed[6] and discussed[8] in earlier papers.

II. MANDELSTAM'S PATH-DEPENDENT WAVE FUNCTIONS

The many-particle nonrelativistic Schrödinger equation that describes the motion of a set of particles with masses m_j and charges q_j in the potentials $\mathbf{A}(\mathbf{r},t)$, $\phi(\mathbf{r},t)$ is

$$i\hbar\frac{\partial\psi}{\partial t}=\sum_j\left\{-\frac{\hbar^2}{2m_j}\left[\nabla_j-\frac{iq_j}{\hbar c}\mathbf{A}(\mathbf{r}_j,t)\right]^2+q_j\phi(\mathbf{r}_j,t)\right\}\psi, \quad (2)$$

where the electromagnetic fields are defined by

$$\mathbf{H}=\nabla\times\mathbf{A},\quad \mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}-\nabla\phi. \quad (3)$$

As is well known, the gauge transformation

$$\mathbf{A}\rightarrow\mathbf{A}'=\mathbf{A}+\nabla\chi,\quad \phi\rightarrow\phi'=\phi-\frac{1}{c}\frac{\partial\chi}{\partial t},$$
$$\psi\rightarrow\psi'=\psi\exp\left[\frac{i}{\hbar c}\sum_j q_j\chi(\mathbf{r}_j,t)\right], \quad (4)$$

leaves the fields and the form of the wave equation unchanged.

We now follow the nonrelativistic version of Mandelstam's[15] approach. For any particular choice of the gauge, say that corresponding to the above potentials $\mathbf{A}$, ϕ, we define the path-dependent wave function

$$\psi_P=\psi\exp\left[\frac{i}{\hbar c}\sum_j q_j\int_{\mathbf{r}_j}^{\infty}\mathbf{A}(\mathbf{r},t)\cdot d\mathbf{r}\right], \quad (5)$$

where the jth integral runs from $\mathbf{r}_j$ to ∞ along a path P_j that is yet to be specified. Application of a gauge transformation to this ψ_P gives

$$\psi_P\rightarrow\psi_P'=\psi'\exp\left[\frac{i}{\hbar c}\sum_j q_j\int_{\mathbf{r}_j}^{\infty}\mathbf{A}'(\mathbf{r},t)\cdot d\mathbf{r}\right].$$

Substitution from Eqs. (4) shows that $\psi_P'=\psi_P$ provided that the gauge function χ vanishes at infinity. Thus the path-dependent wave function (5) is gauge-independent.

The paths P_j remain to be specified. Mandelstam chose them in such a way that derivatires of ψ_P are defined by dividing $\psi_{P'}-\psi_P$ by the coordinate difference, where $\psi_{P'}$ is the value of ψ_P at the displaced point calculated with a path P_j' that goes from the displaced point to the original point and then follows the original path P_j to infinity:

$$\frac{\partial\psi_P}{\partial x_j}=\lim_{\Delta x_j\to 0}\frac{\psi_{P'}(x_j+\Delta x_j,\, y_j,z_j)-\psi_P(x_j,y_j,z_j)}{\Delta x_j}.$$

Equation (5) gives for the derivatives of ψ_P

$$\frac{\partial\psi_P}{\partial x_k}=\exp\left[\frac{i}{\hbar c}\sum_j q_j\int_{\mathbf{r}_j}^{\infty}\mathbf{A}(\mathbf{r},t)\cdot d\mathbf{r}\right]\left\{\frac{\partial}{\partial x_k}-\frac{iq_k}{\hbar c}A_x(\mathbf{r}_k,t)\right\}\psi,$$

$$\frac{\partial\psi_P}{\partial t}=\exp\left[\frac{i}{\hbar c}\sum_j q_j\int_{\mathbf{r}_j}^{\infty}\mathbf{A}(\mathbf{r},t)\cdot d\mathbf{r}\right]\times\left\{\frac{\partial}{\partial t}+\frac{i}{\hbar c}\sum_k q_k\int_{\mathbf{r}_k}^{\infty}\frac{\partial\mathbf{A}}{\partial t}\cdot d\mathbf{r}\right\}\psi.$$

The wave equation for ψ_P may then be obtained from Eq. (2) with the help of the second of Eqs. (3), provided that ϕ vanishes at infinity:

$$i\hbar\frac{\partial\psi_P}{\partial t}=\sum_j\left\{-\frac{\hbar^2}{2m_j}\nabla_j^2+q_j\int_{\mathbf{r}_j}^{\infty}\mathbf{E}(\mathbf{r},t)\cdot d\mathbf{r}\right\}\psi_P. \quad (6)$$

Equation (6), which is evidently gauge-independent, must be supplemented by a gauge-independent version of Eq. (5), in order that the path dependence of ψ_P can be determined in terms of the fields alone. Mandelstam accomplished this by imagining that the path P_j is displaced by an infinitesimal amount at some point, and then calculating the corresponding change in ψ_P. More generally, we may follow Cabibbo and Ferrari[14] and use Eq. (5) to write the relation between the ψ_P's calculated with paths P_j and P_j' that differ by finite amounts as

$$\psi_{P'}=\psi_P\exp\left[\frac{iq_j}{\hbar c}\int_{C_j}\mathbf{A}(\mathbf{r},t)\cdot d\mathbf{r}\right],$$

where the cricuit C_j is the closed path $P_j'-P_j$. Stokes's theorem may now be used to express the circuit integral as $\int_{S_j}(\nabla\times\mathbf{A})\cdot d\boldsymbol{\sigma}$, where the integral is over the surface

[16] W. V. R. Malkus, Phys. Rev. **83**, 899 (1951); H. Bradner and W. M. Isbell, *ibid.* **114**, 603 (1959); E. M. Purcell, G. B. Collins, T. Fujii, J. Hornbostel, and F. Turkot, *ibid.* **129**, 2326 (1963); E. Amaldi, G. Baroni, A. Manfredini, H. Bradner, L. Hoffmann, and G. Vanderhaeghe, Nuovo Cimento **28**, 773 (1963); E. Goto, H. H. Kolm, and K. W. Ford, Phys. Rev. **132**, 387 (1963); W. C. Carithers, R. Stefanski, and R. K. Adair, *ibid.* **149**, 1070 (1966).

S_j bounded by the closed path. We thus obtain the gauge-independent equation

$$\psi_{P'}=\psi_P \exp\left[\frac{iq_j}{\hbar c}\int_{S_j} \mathbf{H}\cdot d\boldsymbol{\sigma}\right]. \tag{7}$$

In order that Eq. (7) relate $\psi_{P'}$ and ψ_P uniquely, it is necessary to require that the exponential be independent of the choice of the surface S_j that is bounded by the circuit C_j, so that the way in which P_j is deformed into P_j' is unimportant. The difference in the value of the integral for two such surfaces is, by Gauss's theorem, equal to the integral of $\nabla\cdot\mathbf{H}$ over the volume enclosed by the two surfaces. We thus require that

$$\frac{q_j}{\hbar c}\int_{V_j}(\nabla\cdot\mathbf{H})d\tau=2\pi n, \tag{8}$$

where n is an integer, for any volume V_j that can be formed by displacing the path P_j. Since $\nabla\cdot\mathbf{H}$ is equal to 4π times the magnetic pole density, Eq. (8) leads to the Dirac condition (1) for point poles, provided that a path is never permitted to pass through a pole. This is the gauge-independent counterpart of the Dirac veto. For extended poles, this version of the Dirac veto also holds in the sense that a path must never pass through a region where the pole density is different from zero; then Eq. (1) is still valid.

Equations (6) and (7) suffice to determine ψ_P in its dependence on the paths P_j, the particle coordinates $\mathbf{r}_j$, and the time t, without reference to the original wave function ψ. ψ_P is clearly nonlocal, since it depends on the electric and magnetic fields all along the paths. As Mandelstam has pointed out, such a nonlocality is essential if the Aharanov-Bohm effect[17] is to be understood in terms of fields alone without resort to potentials.

III. CHOICE OF PARALLEL STRAIGHT PATHS

The theory described in the last section does not lead to the selection principle that was discussed in Sec. I. It is, however, possible to obtain this selection principle by choosing the paths to be parallel straight lines that extend from each $\mathbf{r}_j$ to infinity in the direction of some unit vector $\boldsymbol{\varepsilon}$. The integral that appears in the exponent of Eq. (5) may then be written

$$\int_{\mathbf{r}_j}^{\infty}\mathbf{A}(\mathbf{r},t)\cdot d\mathbf{r}=\int_0^{\infty}\boldsymbol{\varepsilon}\cdot\mathbf{A}(\mathbf{r}_j+\boldsymbol{\varepsilon}s,\,t)ds,$$

and ψ_P is a function of the particle coordinates $\mathbf{r}_j$, the time t, and the single unit vector $\boldsymbol{\varepsilon}$. Spatial derivatives of ψ_P are now calculated differently than in Sec. II, since when a coordinate is displaced, the entire path moves with it; thus the wave equation (6) is modified. Further, Eqs. (7) and (8) take somewhat different forms, since now the only way in which the paths can be altered is by changing $\boldsymbol{\varepsilon}$.

The spatial derivatives of the integral

$$I\equiv\int_0^{\infty}\boldsymbol{\varepsilon}\cdot\mathbf{A}(\mathbf{r}+\boldsymbol{\varepsilon}s,\,t)ds$$

are most easily calculated by choosing coordinates such that the positive z axis is along $\boldsymbol{\varepsilon}$. We then have

$$I=\int_0^{\infty}A_z(x,y,\,z+s,\,t)ds,$$

$$\frac{\partial I}{\partial z}=-A_z(x,y,z,t),$$

$$\frac{\partial I}{\partial x}=\int_0^{\infty}\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)ds+\frac{\partial}{\partial z}\int_0^{\infty}A_x ds$$

$$=-A_x(x,y,z,t)-\int_0^{\infty}H_y(x,y,\,z+s,\,t)ds,$$

$$\frac{\partial I}{\partial y}=-A_y(x,y,z,t)+\int_0^{\infty}H_x(x,y,\,z+s,\,t)ds.$$

For a general choice of axes, these equations may be written

$$\nabla I=-\mathbf{A}(\mathbf{r},t)+\int_0^{\infty}\boldsymbol{\varepsilon}\times\mathbf{H}(\mathbf{r}+\boldsymbol{\varepsilon}s,\,t)ds.$$

We thus obtain for the gradient of ψ_P with respect to the coordinate of the kth particle

$$\nabla_k\psi_P=\exp\left[\frac{i}{\hbar c}\sum_j q_j\int_0^{\infty}\boldsymbol{\varepsilon}\cdot\mathbf{A}(\mathbf{r}_j+\boldsymbol{\varepsilon}s,\,t)ds\right]\times\left\{\nabla_k-\frac{iq_k}{\hbar c}\mathbf{A}(\mathbf{r}_k,t)+\frac{iq_k}{\hbar c}\int_0^{\infty}\boldsymbol{\varepsilon}\times\mathbf{H}(\mathbf{r}_k+\boldsymbol{\varepsilon}s,\,t)ds\right\}\psi.$$

Calculation of the second derivatives is slightly more complicated, and leads to

$$\left\{\nabla_k-\frac{iq_k}{\hbar c}\int_0^{\infty}\boldsymbol{\varepsilon}\times\mathbf{H}(\mathbf{r}_k+\boldsymbol{\varepsilon}s,\,t)ds\right\}^2\psi_P=\exp\left[\frac{i}{\hbar c}\sum_j q_j\int_0^{\infty}\boldsymbol{\varepsilon}\cdot\mathbf{A}(\mathbf{r}_j+\boldsymbol{\varepsilon}s,\,t)ds\right]\times\left\{\nabla_k-\frac{iq_k}{\hbar c}\mathbf{A}(\mathbf{r}_k,t)\right\}^2\psi.$$

The time derivative is calculated as before, so that

[17] W. E.Ehrenburg and R. E. Siday, Proc. Phys. Soc. (London) **B62**, 8 (1949); Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959); **123**, 1511 (1961); W. H. Furry and N. F. Ramsey, *ibid.* **118**, 623 (1960). An experiment that verifies the Aharonov-Bohm effect has been performed by G. Möllenstedt and W. Bayh, Naturwiss. **49**, 81 (1962).

the wave equation for ψ_P is

$$i\hbar\frac{\partial\psi_P}{\partial t}=\sum_j\left\{-\frac{\hbar^2}{2m_j}\left[\nabla_j-\frac{iq_j}{\hbar c}\int_0^\infty \boldsymbol{\varepsilon}\times\mathbf{H}(\mathbf{r}_j+\boldsymbol{\varepsilon}s,t)ds\right]^2+q_j\int_0^\infty \boldsymbol{\varepsilon}\cdot\mathbf{E}(\mathbf{r}_j+\boldsymbol{\varepsilon}s,t)ds\right\}\psi_P, \quad (9)$$

which, like Eq. (6), is evidently gauge-independent.

Comparison of Eqs. (2) and (9) shows that our choice of parallel straight paths is formally equivalent to a particular choice of the potentials. We do *not*, however, interpret Eq. (9) in this way. Rather, we adopt an attitude similar to that of Cabibbo and Ferrari, and regard (9) as the correct equation of motion for the wave function ψ_P, expressed in terms of the fields alone, whether or not magnetic poles are present. Even though the integrals that appear in Eq. (9) are not regarded as potentials, it is still possible to make use of the standard theory of the Schrödinger equation to obtain the usual expression for the electric charge and current density in terms of these integrals. Although we shall not do so, it is easy to show that this charge-current is conserved.

In order that Ehrenfest's theorem may be satisfied with the correct expression for the Lorentz force, it is also necessary to show that the electric and magnetic fields are obtained by differentiating these integrals in accordance with Eqs. (3). That is, we must show that

$$\mathbf{H}=\nabla\times\int_0^\infty \boldsymbol{\varepsilon}\times\mathbf{H}(\mathbf{r}+\boldsymbol{\varepsilon}s,t)ds,$$

$$\mathbf{E}=-\frac{1}{c}\frac{\partial}{\partial t}\int_0^\infty \boldsymbol{\varepsilon}\times\mathbf{H}(\mathbf{r}+\boldsymbol{\varepsilon}s,t)ds-\nabla\int_0^\infty \boldsymbol{\varepsilon}\cdot\mathbf{E}(\mathbf{r}+\boldsymbol{\varepsilon}s,t)ds.$$

It is again convenient to choose coordinates so that the positive z axis is along $\boldsymbol{\varepsilon}$. We then require the following equations to be valid:

$$H_x=-\frac{\partial}{\partial z}\int_0^\infty H_x(x,y,z+s,t)ds,$$

$$H_y=\frac{\partial}{\partial z}\int_0^\infty -H_y(x,y,z+s,t)ds,$$

$$H_z=\int_0^\infty\left(\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}\right)ds=-\frac{\partial}{\partial z}\int_0^\infty H_z(x,y,z+s,t)ds+\int_0^\infty(\nabla\cdot\mathbf{H})ds, \quad (10)$$

$$E_x=-\frac{1}{c}\frac{\partial}{\partial t}\int_0^\infty -H_y(x,y,z+s,t)ds-\frac{\partial}{\partial x}\int_0^\infty E_z(x,y,z+s,t)ds$$

$$=-\frac{\partial}{\partial z}\int_0^\infty E_x(x,y,z+s,t)ds+\int_0^\infty\left[(\nabla\times\mathbf{E})+\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right]_y ds,$$

$$E_y=-\frac{\partial}{\partial z}\int_0^\infty E_y(x,y,z+s,t)ds-\int_0^\infty\left[(\nabla\times\mathbf{E})+\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right]_x ds,$$

$$E_z=-\frac{\partial}{\partial z}\int_0^\infty E_z(x,y,z+s,t)ds.$$

They are in fact valid if the homogeneous Maxwell equations are satisfied all along the path, in which case the magnetic-pole density and current must be zero there.

We thus obtain the gauge-independent counterpart of the Dirac veto as the condition for the correctness of Eq. (9): A path must never pass through a region where the magnetic-pole density or current is different from zero.

It is interesting to see what the fields are that are calculated from Eqs. (10) when the field of a fixed magnetic pole is inserted on the right side. For a point pole of strength g located at the origin, we find that the x, y, and z components of $\int_0^\infty \boldsymbol{\varepsilon}\times\mathbf{H}(\mathbf{r}+\boldsymbol{\varepsilon}s,t)ds$ are, respectively,

$$-\frac{gy}{r(r+z)},\quad \frac{gx}{r(r+z)},\quad 0.$$

This is just Dirac's singular vector potential with the string along the negative z axis. Then the magnetic field calculated from Eqs. (10) is the spherically symmetric, radially outgoing magnetic field of the point pole, together with an equal singular return flux along the negative z axis. The electric field is zero since the pole is at rest. In similar fashion, the magnetic field calculated from Eqs. (10) when the field of an extended pole is inserted on the right side is easily seen to be the radially outgoing field of the extended pole, together with an equal return flux in the form of a parallel bundle in a tube along the negative z axis.[11] The electric field is again zero. The return flux, whether singular as in the case of a point pole or finite as in the case of an extended pole, has no physical consequences provided that the Dirac veto is respected and Eq. (1) is satisfied.

IV. UNIQUENESS OF THE WAVE FUNCTION

Equation (9) is the modification of the nonrelativistic Mandelstam-Cabibbo-Ferrari wave equation (6) that is obtained when the paths are parallel straight lines. We must now find the corresponding modifications of Eqs. (7) and (8), which are required in order that the ψ_P associated with $\boldsymbol{\varepsilon}$ be uniquely related to the $\psi_{P'}$ associated with some other unit vector $\boldsymbol{\varepsilon}'$.

Equation (7) was obtained from Eq. (5) by considering first a closed circuit which was the difference between two paths, and then the area swept out by the path as it is deformed from its initial to its final shape.

Since there was no limitation on this procedure other than that the circuit be closed, it can be applied to a change in $\boldsymbol{\varepsilon}$ provided that the initial and final paths do not extend to infinity and are joined at their far ends by some path segment. Then Eq. (7) is still valid if the surface S_j is the conical sheet swept out by the finite straight path attached to the fixed point $\mathbf{r}_j$ as the unit vector is changed in some way from $\boldsymbol{\varepsilon}$ to $\boldsymbol{\varepsilon}'$.

There is, however, an essential difference between the present situation and that of Sec. II. In the Mandelstam formalism each path P_j can be varied independently, whereas in our formalism there is a single $\boldsymbol{\varepsilon}$ for all particles so that all paths remain parallel to each other as $\boldsymbol{\varepsilon}$ changes. This means that Eq. (7) is replaced by

$$\psi_{P'}=\psi_P \exp\left[\frac{i}{\hbar c}\sum_j q_j \int_{S_j} \mathbf{H}\cdot d\boldsymbol{\sigma}\right]. \tag{11}$$

As in the transition from (7) to (8), the relation between ψ_P and $\psi_{P'}$ must be independent of the way in which $\boldsymbol{\varepsilon}$ is changed into $\boldsymbol{\varepsilon}'$. The difference in the value of each integral in the exponent of Eq. (11) for two such changes is equal to the integral of $\nabla\cdot\mathbf{H}$ over the conical volume V_j enclosed by the two surfaces S_j and a cap at the far end. As the cap recedes to infinity, we obtain for the uniqueness condition the requirement that

$$\frac{1}{\hbar c}\sum_j q_j \int_{V_j} (\nabla\cdot\mathbf{H})d\tau = 2\pi n\,, \tag{12}$$

where n is an integer. The volumes V_j are now any set of semi-infinite cones that are congruent to each other in shape and orientation, and differ only in the position of their vertices, the vertex of V_j being at $\mathbf{r}_j$. There is also the restriction implied by the Dirac veto: that none of the conical surfaces must pass through a region where the magnetic-pole density or current is different from zero.

Equation (12) contains the selection principle described in Sec. I. Consider for simplicity a fixed spherical magnetic pole of strength g and diameter R, and a number of particles with electric charges q_j which lie in any cylindrical tube of diameter R whose axis passes through the center of the pole. Then it is impossible for one of the conical volumes V_j to contain the pole without all of them containing it. Thus in accordance with Eq. (12), the Dirac quantization condition (1) is generalized to

$$\frac{g}{\hbar c}\sum_j q_j = \tfrac{1}{2}n\,,$$

where n is an integer. This permits fractionally charged particles to exist, provided that their maximum projected separation distance in the plane perpendicular to the direction of the magnetic pole is less than R, and their total charge is an integer multiple[18] of $e=\hbar c/2g$. The same argument shows that if one of the particles is separated from all of the others (in a perpendicular direction) by more than the distance R, it must have integer charge. It is of course not necessary for the pole to be spherical in the foregoing demonstration; it is however necessary that the pole density vanish beyond some finite distance from its center.

Nearby poles, or a sufficiently high density of remote poles, would on this theory produce peculiar effects on charged particles. The absence of these effects is, however, consistent with present experiments on poles.[16]

V. CONCLUDING REMARKS

We have shown that it is possible to modify the Mandelstam-Cabibbo-Ferrari formalism so that it leads to a limitation on the separation distances of fractionally charged quarks, provided that the Dirac magnetic pole is of finite size. The modification consists only in requiring the paths associated with quarks to be parallel straight lines. (It should be noted that charged particles other than quarks might well have other kinds of paths.) Each extended Dirac pole defines an infinite set of cylindrical tubes which circumscribe it and point in all possible directions. Our theory then requires that the quarks contained in any one of these tubes possess integer total charge and hence integer total baryon number. A single such pole only restrains individual quarks at right angles to the pole direction; but two such poles in different directions would require the quarks contained in any intersection of their tubes to have integer total baryon number. In this case, the production cross section for individual quarks is expected to be zero, in agreement with observation.[5]

It should be emphasized that no assumptions have been made with regard to the masses of quarks, the interactions between them, or the existence of a constraining potential. It is reasonable, then, to suppose that the mass of a quark is a few hundred MeV (roughly a third of a baryon mass or half of a meson mass), and that the forces between them are of the order of the strong interactions. The present theory manifests itself simply as a boundary condition that requires the many-particle wave function to vanish whenever the spatial coordinates $\mathbf{r}_j$ of all quarks and antiquarks fail to satisfy the uniqueness condition (12).

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[18] It shoud also be noted that, under these circumstances, an allowed closed circuit of any of the $\mathbf{r}_j$ does not destroy the uniqueness of ψ_P.