The expectation value (B2) is then

$$\exp[i\vartheta(M_{03}\cosh\theta-M_{43}\sinh\theta)]\exp[l^\mu x_\mu]_{l=0,x=0}$$
$$\rightarrow \exp[\ln(\sin p\sin q)-(p+q)\sinh\theta]$$
$$\times\exp\left[\tfrac{1}{2}i\vartheta\left(\frac{\partial}{\partial p}-\frac{\partial}{\partial q}\right)\right]$$
$$\times\exp[-\ln(\sin p\sin q)+(p+q)\sinh\theta]$$
$$=\exp[\ln(\sin p\sin q)-\ln(\sin(p+\tfrac{1}{2}i\vartheta)\sin(q-\tfrac{1}{2}i\vartheta))]$$
$$\text{at}\quad p=q=a,$$

$$a=\frac{1}{2i}\ln(\xi_1/\xi_2)=\tfrac{1}{2}\ln\frac{\sinh\theta-i}{\sinh\theta+i},$$

which is equal to

$$\frac{1}{\cosh^{2}\theta\sinh^2(\vartheta/2)+1}=\frac{1}{1-t/4(m^2-\kappa^2)}. \tag{B7}$$

Similarly, the vector form factor $(\Psi(p),\Gamma_\mu\Psi(0))$ may be calculated, and yields the results (10) of Sec. II.

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Classical Theory of Magnetic Charge*

Tung-mow Yan†
Department of Physics, Harvard University, Cambridge, Massachusetts
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A classical theory of magnetic charge is formulated on the basis of an action principle. It is an extension of Schwinger's quantum theory of magnetic charge to the classical level. The action integral is defined by limiting procedure to ensure the equivalence of all singularity lines. The action principle gives correct equations of motion for the particles, Maxwell's equations, and the conservation laws of the Lorentz group. The consistency of the theory demands a charge-quantization condition, and the existence of a constant with the dimensions of action.

In a recent paper, Rohrlich[1] has considered the problem of constructing a classical theory of magnetic charge. One of his conclusions is that no action integral exists from which both the particle equations and the field equations can be derived. This result thus casts severe doubt on the consistency of the theory of magnetic charge with the Lorentz group. However, Rohrlich's conclusion is based on the use of two independent vector potentials (A_μ and B_μ), which enlarges the number of degrees of freedom of the electromagnetic field and makes all the six components of the field strength $F_{\mu\nu}$ fundamental dynamical variables. In quantum field theory such a system violates physical positiveness requirements,[2] and it may well be that a similar situation holds at the classical level.[3] In the following we present a classical theory of magnetic charge in which no additional degree of freedom is introduced for the electromagnetic field; the vector potential B_μ is considered a given function of the field strength. All the equations of motion for the particles and the Maxwell field equations are derived from a nonlocal action integral. The ten conservation laws of the Lorentz group follow from the relativistic covariance of the theory which is satisfied by a limiting definition of the action integral. As we shall see, consistency of the theory requires a charge quantization condition and the existence of a constant with the dimension of action.

The idea involved is essentially an extension of Schwinger's quantum theory of magnetic charge.[4] We start with the tentative action integral[5]

$$W=\int(dx)\mathcal{L}_{\rm em}(x)+\sum_{a=1}^{N}W_a+\sum_{b=1}^{*N}W_b, \tag{1}$$

$$\mathcal{L}_{\rm em}=-\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{2a}$$

$$W_a=\int ds_a\left[p_\mu\frac{dx^\mu}{ds}-\frac{1}{2m}(p-eA)^2\right]_a, \tag{2b}$$

$$W_b=\int ds_b\left[p_\mu\frac{dx^\mu}{ds}-\frac{1}{2m}(p-gB)^2\right]_b, \tag{2c}$$

where a and b are the labels of an electrically charged particle and a magnetically charged particle, respectively. The vector potential B_μ is a function of the

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[1] F. Rohrlich, Phys. Rev. **150**, 1104 (1966).

[2] J. Schwinger, Phys. Rev. **130**, 800 (1963).

[3] Footnote 16 of Ref. 1 seems to indicate that this is the case.

[4] J. Schwinger, Phys. Rev. **144**, 1087 (1966). A nonrelativistic quantum-particle theory has also been constructed by J. Schwinger (unpublished).

[5] We use units with $c=1$. Also, $-g_{00}=g_{11}=g_{22}=g_{33}=1$.

field strength $F_{\mu\nu}$, and is given explicitly by[6]

$$B_\mu(x) = -\int (dx')\, {}^*F_{\mu\nu}(x') h_n{}^\nu(x'-x)\,, \tag{3}$$

where ${}^*F_{\mu\nu}$ is the dual tensor of $F_{\mu\nu}$:

$${}^*F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\lambda\kappa}F^{\lambda\kappa}(\epsilon^{0123}=1)\,, \tag{4}$$

and $h_n{}^\nu(x)$ is a three-dimensional distribution localized on the semi-infinite singularity line $x^\mu = n^\mu |x^2|^{1/2}$, where n^μ is an arbitrary constant vector. For definiteness, we may take it to be spacelike. The function $h_n{}^\nu(x)$ satisfies the differential equation

$$-\partial_\nu h_n{}^\nu(x-x') = \delta(x-x')\,. \tag{5}$$

Independent variations of the variables A_μ, $F_{\mu\nu}$, $p_a{}^\mu$, $x_a{}^\mu$, $p_b{}^\mu$, and $x_b{}^\mu$ yield the following equations

$$\partial_\nu F^{\mu\nu} = j^\mu\,, \tag{6}$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + \int (dx')\epsilon_{\mu\nu\lambda\kappa}\, {}^*j^\lambda(x') h_n{}^\kappa(x-x')\,, \tag{7}$$

$$(dx^\mu/ds)_a = (1/m_a)[p_a{}^\mu - e_a A^\mu(x_a)]\,, \tag{8}$$

$$\frac{d}{ds_a}[p_a{}^\mu - e_a A^\mu(x_a)] = \frac{e_a}{m_a}[p_a - e_a A(x_a)]_\nu \times (\partial^\mu A^\nu - \partial^\nu A^\mu)(x_a)\,, \tag{9}$$

$$(dx^\mu/ds)_b = (1/m_b)[p_b{}^\mu - g_b B^\mu(x_b)]\,, \tag{10}$$

$$\frac{d}{ds_b}[p_b{}^\mu - g_b B^\mu(x_b)] = \frac{g_b}{m_b}[p_b - g_b B(x_b)]_\nu \times (\partial^\mu B^\nu - \partial^\nu B^\mu)(x_b)\,, \tag{11}$$

where

$$j^\mu(x) = \sum_{a=1}^{N} e_a \int ds_a \delta(x - x_a(s_a))\left(\frac{dx^\mu}{ds}\right)_a\,, \tag{12}$$

$${}^*j^\mu(x) = \sum_{b=1}^{{}^*N} g_b \int ds_b \delta(x - x_b(s_b))\left(\frac{dx^\mu}{ds}\right)_b\,. \tag{13}$$

These equations can be recast in the more symmetrical version

$$\partial_\nu F^{\mu\nu} = j^\mu\,, \quad \partial_\nu\, {}^*F^{\mu\nu} = {}^*j^\mu\,, \tag{14}$$

$$m_a\left(\frac{d^2x^\mu}{ds^2}\right)_a = e_a\left(\frac{dx_\nu}{ds}\right)_a(\partial^\mu A^\nu - \partial^\nu A^\mu)(x_a)\,, \quad a=1,2,\cdots,N\,, \tag{15}$$

$$m_b\left(\frac{d^2x^\mu}{ds^2}\right)_b = g_b\left(\frac{dx_\nu}{ds}\right)_b(\partial^\mu B^\nu - \partial^\nu B^\mu)(x_b)\,, \quad b=1,2,\cdots,{}^*N\,. \tag{16}$$

[6] The relations between vector potential and field strength are given by J. Schwinger, Phys. Rev. **151**, 1055 (1966). If $h_n{}^\nu(x)$ is chosen to be a purely spatial vector, Eq. (3) then reduces to that given by Schwinger, within a gauge term.

The Maxwell field equations can also be presented in the alternative form

$$\partial_\mu\, {}^*F_{\nu\lambda} + \partial_\nu\, {}^*F_{\lambda\mu} + \partial_\lambda\, {}^*F_{\mu\nu} = -\epsilon_{\mu\nu\lambda\kappa} j^\kappa\,, \tag{17}$$

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = +\epsilon_{\mu\nu\lambda\kappa}\, {}^*j^\kappa\,. \tag{18}$$

Thus, from the definition of B_μ, Eq. (3), one finds the relation

$${}^*F_{\mu\nu}(x) = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x) + \int (dx')\epsilon_{\mu\nu\lambda\kappa} j^\lambda(x') h_n{}^\kappa(x'-x)\,. \tag{19}$$

Equations (7) and (19) show that the action integral (1) produces correct Maxwell field equations as well as correct equations of motion of the particles if we impose the restriction that the singularity line never connect an electrically charged particle and a magnetically charged particle, since in such a case

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,,$$
$${}^*F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,,$$

and Eqs. (15) and (16) become the correct equations describing the motion of the charged particles. In principle, the paths of the classical motion of the charged particles can be followed exactly. It is therefore always possible to choose a singularity line which satisfies this requirement. Difficulty arises only when collision occurs between an electrically charged particle and a magnetically charged particle. Nevertheless, the anomalous position of the singularity line should be removed if the equivalence of all space-time points is to be maintained everywhere, without exception. We now exhibit a limiting definition of the action integral which accomplishes this purpose. The clue to the solution of the problem is provided by the experience with the quantum theory of magnetic charge.[4] The situation here is even simpler, since we deal with numbers, not operators.

A gauge transformation in classical electrodynamics without magnetic charge is described by

$$A_\mu \to A_\mu + \partial_\mu\lambda\,, \quad p_{a\mu} \to p_{a\mu} + e_a\partial_\mu\lambda\,. \tag{20}$$

When magnetically charged particles are present, a gauge transformation on the magnetic quantities is described by

$$B_\mu \to B_\mu + \partial_\mu\lambda\,, \quad p_{b\mu} \to p_{b\mu} + g_\mu\partial_\mu\lambda\,. \tag{21}$$

Let us examine the effect on the vector potentials by the change from one singularity line to another. Consider $B_\mu(x)$. If B^μ and B_μ' are the vector potentials associated with the singularity line n and n', respectively, since ${}^*F_{\mu\nu}$ is a physical quantity unaffected by this change, we have

$$\partial_\mu(B'-B)_\nu(x)-\partial_\nu(B'-B)_\mu(x)=\epsilon_{\mu\nu\lambda\kappa}\int(dx')\times[h_{n'}{}^\lambda(x'-x)-h_n{}^\lambda(x'-x)]j^\kappa(x'),\quad(22)$$

which is zero almost everywhere. Consequently, $(B'-B)_\mu$ is almost expressible in terms of the gradient of a scalar function $\lambda(x)$ given by

$$\lambda(x)=\int_\infty^x dx_\mu'(B'-B)^\mu(x').\quad(23)$$

Therefore, the change from one singularity line to another is a gauge transformation almost everywhere, as described by

$$n\rightarrow n',$$

$$B_\mu(x)\rightarrow B_\mu(x)+(B_\mu'-B_\mu)(x),\quad(24a)$$

$$p_b\rightarrow p_b{}^\mu+g_b\partial\int_\infty^{x_b}dx_\lambda'(B'-B)^\lambda(x'),\quad b=1,2,\cdots,{}^*N.\quad(24b)$$

The expression $(p-gB)^2/2m$ which appears in the action integral is not exactly gauge invariant under (24). A truly gauge-invariant structure is obtained by the following considerations. We observe that $(p-gB)^2/2m$ can be defined by the limiting construction[7]

$$\frac{1}{2m_b}[p_b-g_bB(x_b)]^2=\lim\frac{4}{m_b}\frac{k^2}{\epsilon^2}\times\left\{1-\exp\left[-\frac{i}{k}\int_{x-\frac12\epsilon}^{x+\frac12\epsilon}dx_1{}^\mu(p-gB)_\mu(x_1)\right]_b\right\},\quad(25)$$

where the path of the line integral is a straight line connecting the two end points, and an average over all directions is performed before letting $\epsilon\rightarrow0$ with the aid of the relation

$$\langle(1/\epsilon^2)\epsilon_\mu\epsilon_\nu\rangle=\tfrac14 g_{\mu\nu}.$$

The constant k is introduced for dimensional reasons. It has the dimension of action. Under the transformation (24), the object

$$\exp\left[-\frac{i}{k}\int_{x-\frac12\epsilon}^{x+\frac12\epsilon}dx_1{}^\mu(p-gB)_\mu(x_1)\right]_b$$

is changed by the factor

$$\exp\left[\frac{ig_b}{k}\int_C dx_1{}^\mu(B'-B)_\mu(x_1)\right]=\exp\left[\frac{ig_b}{k}\int(dx'')j^\kappa(x'')\times\frac12\int d\sigma_1{}^{\mu\nu}\epsilon_{\mu\nu\lambda\kappa}(h_{n'}{}^\lambda-h_n{}^\lambda)(x''-x_1)\right],$$

where the contour C begins at infinity, moves successively to $x_b-\frac12\epsilon$, $x_b+\frac12\epsilon$, and then returns to infinity. The line integral has been converted into an integral over the two-dimensional surface defined by the boundary. Consider the expression

$$X_b=\exp\left[\frac{ig_b}{k}\int(dx'')j^\kappa(x'')\times\frac12\int d\sigma_1{}^{\mu\nu}\epsilon_{\mu\nu\lambda\kappa}h_n{}^\lambda(x''-x_1)\right].\quad(26)$$

We first evaluate the integral

$$\frac12\int d\sigma'^{\mu\nu}\epsilon_{\mu\nu\lambda\kappa}h_n{}^\lambda(x''-x')$$

for a fixed point x'', and any surface $\sigma_{\mu\nu}'$. We distinguish two possibilities, whether the boundary encircles the singularity line or not. In either case, in virtue of the singular nature of the function $h_n{}^\nu(x)$, it is always possible to close the surface without altering the value of the integral. In the former case, the point x'' is inside the closed surface, while in the latter, the point x'' is outside. Thus,

$$\frac12\int d\sigma'^{\mu\nu}\epsilon_{\mu\nu\lambda\kappa}h_n{}^\lambda(x''-x')=\frac12\oint d\sigma'^{\mu\nu}\epsilon_{\mu\nu\lambda\kappa}h_n{}^\lambda(x''-x')=-\int(d\sigma_\lambda'\partial_\kappa'-d\sigma_\kappa'\partial_\lambda')h_n{}^\lambda(x''-x').$$

We have converted the two-dimensional surface integral into an integral over the three-dimensional hypersurface which spans it, and the relation

$$\frac12\oint d\sigma^{\mu\nu}\epsilon_{\mu\nu\lambda\kappa}f^\lambda(x)=-\int(d\sigma_\lambda\partial_\kappa-d\sigma_\kappa\partial_\lambda)f^\lambda(x)$$

for an arbitrary f^λ has been used. Should the boundary of $\sigma_{\mu\nu}'$ not encircle the singularity line, but be tangential to it, the value of the integral must be multiplied by $\frac12$.[4,8] A plus or minus sign must also be included depending on the sense of the original contour. With these remarks, Eq. (26) becomes

$$X_b=\exp\left[\frac{ig_b}{k}\sum_a e_a\eta_a\right],\quad(27)$$

where $\eta_a=\pm0,\ \pm\frac12$, or ±1 is to be used in the manner described above. In arriving at Eq. (27), use has been

[7] This is the relativistic generalization of a corresponding structure in Schwinger's nonrelativistic theory.

[8] Generally, such surfaces form a set of measure zero in the average over all directions of ϵ. However, the limiting construction (25) can also be defined alternatively in terms of a discrete averaging process performed symmetrically over four orthogonal directions. In this procedure, such surfaces make finite contributions. The inclusion of the extra factor $\frac12$ makes the result independent of the particular averaging process adopted.

made of the relation

$$\int d\sigma_\lambda \int ds \frac{dx^\lambda}{ds}\delta(x-x(s))=1 \text{ or } 0,$$

which holds for any hypersurface σ_λ The condition that all $(p-gB)^2/2m$ be invariant under the change of singularity line is

$$X_b=1, \quad b=1,2,\cdots,{}^*N \tag{28}$$

for all possible situations, i.e.,

$$e_a g_b/4\pi k=n, \quad n=0, \pm1, \pm2, \cdots; \quad a=1, 2, \cdots, N; \quad b=1, 2, \cdots, {}^*N, \tag{29}$$

which is the classical charge quantization condition. It involves a constant with the dimension of action. Equation (29) is identical to the charge-quantization condition in quantum theory if the constant k is identified with the Planck constant $\hbar$. An analogous discussion can be applied to the expression $(p-eA)^2/2m$, leading to the same charge-quantization condition (29).

When the action integral (1) is understood in terms of the limiting definition just described, it is unambiguously gauge invariant.[9] Independent variations of A_μ, $F_{\mu\nu}$, $p_a{}^\mu$, and $p_b{}^\mu$ will still produce the same Eqs. (6)–(8), and (10). Variations of $x_a{}^\mu$ and $x_b{}^\mu$ will lead to equations different from Eqs. (9) and (11), however. Consider the variation of $x_b{}^\mu$:

$$\delta_{x_b}\left[-\frac{1}{2m}(p-gB)^2\right]_b = \lim \frac{4}{m_b}\frac{k^2}{\epsilon^2}\left[\exp\left[-\frac{i}{k}\int_{x-\frac12\epsilon}^{x+\frac12\epsilon} dx_\mu''(p-gB)^\mu(x'')\right] \times\left(\exp\left\{-\frac{i}{k}\left[\int_{x-\frac12\epsilon+\delta x}^{x+\frac12\epsilon+\delta x} dx_\mu'(p-gB)^\mu(x')-\int_{x-\frac12\epsilon}^{x+\frac12\epsilon} dx_\mu'(p-gB)^\mu(x')\right]\right\}-1\right)\right]_b. \tag{30}$$

We now close the contour appearing in the curly brackets by adding and subtracting two line segments, one of which connects the two points $x_b+\frac12\epsilon+\delta x$ and $x_b+\frac12\epsilon$, the other connects the two points $x_b-\frac12\epsilon+\delta x$ and $x_b-\frac12\epsilon$. Furthermore, we convert the closed contour integral into a surface integral:

$$\exp\left\{\left(-\frac{i}{k}\right)\left[\int_{x-\frac12\epsilon+\delta x}^{x+\frac12\epsilon+\delta x}-\int_{x-\frac12\epsilon}^{x+\frac12\epsilon}\right]dx_\mu'(p-gB)^\mu(x')\right\}=\exp\left[\frac{ig}{k}\frac{1}{2}\int d\sigma_{\mu\nu}'(\partial^\mu B^\nu-\partial^\nu B^\mu)(x')\right] \times\exp\left\{\left(-\frac{i}{k}\right)\left[\int_{x+\frac12\epsilon}^{x+\frac12\epsilon+\delta x}-\int_{x-\frac12\epsilon}^{x+\frac12\epsilon+\delta x}\right]dx_\mu'(p-gB)^\mu(x')\right\}. \tag{31}$$

The charge-quantization condition allows us to replace $\partial^\mu B^\nu-\partial^\nu B^\mu$ by ${}^*F_{\mu\nu}$ in the surface integral. Therefore,

$$\exp\left[\frac{ig}{k}\frac{1}{2}\int d\sigma'_{\mu\nu}(\partial^\mu B^\nu-\partial^\nu B^\mu)(x')\right] =\exp\left[\frac{ig}{k}\frac{1}{2}\int d\sigma'_{\mu\nu}{}^*F^{\mu\nu}(x')\right]=1+\frac{ig}{k}\delta x_\mu\epsilon_\nu{}^*F^{\mu\nu}, \tag{32}$$

to the order of accuracy required. Evaluation of the line integrals finally yields

$$\delta W_b=\int ds_b\left[-\frac{dp^\mu}{ds}+\frac{1}{m}(p-gB)_\nu(g\,{}^*F^{\mu\nu}+g\partial^\nu B^\mu)\right]_b\delta x_\mu =0,$$

or

$$\frac{d}{ds_b}[p_b-g_bB(x_b)]^\mu=\frac{g_b}{m_b}[p_b-g_bB(x_b)]_\nu{}^*F^{\mu\nu}(x_b),$$

i.e.,

$$m_b(d^2x^\mu/ds^2)_b=g_b(dx_\nu/ds)_b{}^*F^{\mu\nu}(x_b), \tag{33}$$

which is the correct equation of motion for a magnetically charged particle. Similar consideration about the variation of $x_a{}^\mu$ leads to the equation

$$m_a(d^2x^\mu/ds^2)_a=e_a(dx_\nu/ds)_aF^{\mu\nu}(x_a). \tag{34}$$

We conclude that the action principle does give the correct equations of motion for the particles as well as the Maxwell equations if the action integral is properly interpreted.

With the limiting definition for the action integral, the theory is independent of the choice of singularity line, and all space-time points are equivalent, since no particular direction is preferred. In other words, the theory is relativistically covariant. Consequently, the usual derivation of the conservation laws follows. Indeed, under the transformation

$$x^\mu \to \bar{x}^\mu=x^\mu+\delta x^\mu,$$
$$F_{\mu\nu}(x)\to\bar{F}_{\mu\nu}(\bar{x})=F_{\mu\nu}(x)-\partial_\mu\delta x^\lambda F_{\lambda\nu}-\partial_\nu\delta x^\lambda F_{\mu\lambda},$$
$$A_\mu(x)\to\bar{A}_\mu(\bar{x})=A_\mu(x)-\partial_\mu\delta x^\lambda A_\lambda, \text{ etc.},$$

[9] The combination $(\partial_\mu A_\nu-\partial_\nu A_\mu)$ which appears in $\mathcal{L}_{em}$ can also be given a formally gauge-invariant limiting definition, but since difficulty does not arise from this term, this has not been done explicitly.

one finds

$$\delta W=\epsilon_\mu[P^\mu(\sigma_1)-P^\mu(\sigma_2)]+\tfrac{1}{2}\omega_{\mu\nu}[J^{\mu\nu}(\sigma_1)-J^{\mu\nu}(\sigma_2)]=0,$$

when the coordinate variation is specialized to a Lorentz transformation

$$\delta x^\mu=\epsilon^\mu+\omega^{\mu\nu}x_\nu,\quad \omega^{\mu\nu}=-\omega^{\nu\mu}.$$

Thus

$$P_\mu(\sigma)=\text{const},\quad J^{\mu\nu}(\sigma)=\text{const},\tag{35}$$

where

$$P_\mu(\sigma)=\int_\sigma d\sigma_\nu T^{\mu\nu}(x),$$

$$J^{\mu\nu}(\sigma)=\int_\sigma d\sigma_\lambda[x^\mu T^{\lambda\nu}(x)-x^\nu T^{\lambda\mu}(x)],$$

and

$$T^{\mu\nu}=-F^{\mu\lambda}F_\lambda{}^\nu-\tfrac{1}{4}g^{\mu\nu}F^{\lambda\kappa}F_{\lambda\kappa}+\sum_{a=1}^{N}\int ds_a m_a\frac{dx_a{}^\mu}{ds_a}\frac{dx_a{}^\nu}{ds_a}\delta(x-x_a(s_a))+\sum_{b=1}^{*N}\int ds_b m_b\frac{dx_b{}^\mu}{ds_b}\frac{dx_b{}^\nu}{ds_b}\delta(x-x_b(s_b)).$$

These are recognized to be the conservation laws of total energy momentum and generalized angular momentum. The ten conservation laws (35) also follow from the equation

$$\partial_\nu T^{\mu\nu}(x)=0,$$

which can be verified directly by using Eqs. (33), (34), and (14).

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Longitudinal Distribution of Čerenkov Light from Extensive Air Showers

Carlo Castagnoli, Mauro Dardo, and Piero Penengo
Gruppo Italiano di Fisica Cosmica G.I.F.C.O. del C.N.R., Istituto di Fisica Generale, Università di Torino, Torino, Italia
(Received 12 April 1967)

The longitudinal development of extensive air showers of $\sim 10^{13}$ eV is studied both experimentally and theoretically. The risetime and duration of Čerenkov pulses in uv light at 3500 m above sea level are studied experimentally with high-speed techniques. The results are in quite good agreement with an accurate Monte Carlo estimate in the "*B*" approximation of electromagnetic shower theory.

INTRODUCTION

The development of extensive air showers (EAS) has been extensively studied from a statistical point of view. The information on the different stages of growth of an individual EAS is instead scanty and not very precise.

The methods to obtain such information are based on (a) measurements of the distribution of EAS particles, and (b) detection of the Čerenkov light emitted by the electronic component of the EAS in the atmosphere.

In case (a), the information is obtained through the longitudinal distribution of the particles in the shower by measuring the relative delays between the particles with the method used by Bassi *et al.*[1] This method is today extensively used in an experiment at Haverah Park.[2] A second method is based on the observation of the directions of motion of the muons, these data being related to the distribution of particle production along the shower axis.[3]

In case (b), it is possible to use two methods: the first one based on the comparison between the amount of Čerenkov light and the number of particles reaching the ground[4]; the second one based on the temporal analysis of the Čerenkov light pulse[5] produced by the

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