Infinite-Component Wave Equations with Hydrogenlike Mass Spectra*

YOICHIRO NAMBU
Enrico Fermi Institute for Nuclear Studies and the Department of Physics, University of Chicago, Chicago, Illinois

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Continuing a previous work, various models of relativistic wave equations are considered which have an infinite number of components. By combining a unitary representation of $SO(4,2)$ and the ordinary finite (Dirac) representation of the Lorentz group, it is possible to construct equations which produce hydrogenlike mass spectra. However, they are also accompanied by redundant, or unphysical, solutions. In the nonrelativistic limit, on the other hand, the equations obtained can be shown to be mathematically equivalent to the Schrödinger equation for the hydrogen atom. This suggests that the method of infinite-component wave equations may be a useful tool in exploring the physics of strong interactions. A general discussion is made about the principles and problems that will be relevant in pursuing such a program.

I. INTRODUCTION

IN the exploration of the nature of strongly interacting particles we are always plagued by our ignorance of the basic dynamical laws and an adequate mathematical tool to handle them. These two difficulties reinforce each other and make it hard to achieve a quantitative description of hadron properties after we have come to a fairly satisfactory qualitative understanding in terms of models, such as the quark model, and simple group-theoretical arguments.

For this reason, setting up elaborate dynamical equations, for example a Bethe-Salpeter equation using the quark model, may not be very useful. First of all, we do not know the precise interquark dynamics, and secondly, it is difficult to solve the equation. Inasmuch as we are mainly interested in the properties of the solution, namely the mass spectrum, the internal structure revealed by form factors and scattering cross sections, etc., this is a roundabout approach.

In a recent paper[1] we tried to strike an intermediate path by considering simple relativistic wave equations having an infinite number of components, which are easily soluble and yield information about the mass spectrum and the matrix elements of observables, and therefore serve as dynamical equations for describing the properties of complex systems. Two model equations were examined. One is based on a set of multispinor fields $\psi_n{}^m(x)$, each regarded as an irreducible-spinor representation of the homogeneous Lorentz group, while the other uses the set as a basis for a unitary infinite-dimensional representation of the Lorentz group, of such a nature that it actually constitutes an irreducible unitary representation of the larger group $SO(4,2)$. In the former, the mass-spectrum and degeneracy structure simulate the real hydrogenlike atom; but otherwise the equation is essentially unphysical because it does not admit a positive-definite probability density. In the latter, we do not seem to run into obvious conflict with quantum-mechanical interpretation, but there are still some odd features; for example, the discrete hydrogenlike mass spectrum is inverted in order, and allows only one sign of frequency, contrary to ordinary relativistic wave equations, which always admit both signs.

The concept of infinite-component wave equations is not new. Even in 1932 Majorana[2] was led to this type of equation, which was later rediscovered by Gelfand and Yaglom.[3] Our second example is a slight generalization of this Majorana-Gelfand-Yaglom equation. As representations of the Lorentz and Poincaré groups, the general structure of infinite-component wave equations was studied in detail by Gelfand, Naimark, and others.[4] An alternative approach to this problem, dating back to Dirac,[5,6] is to use continuous variables rather than discrete ones. This looks more directly related to the description of composite particles, whether the continuous variables are regarded as relative internal coordinates of a bound system, or taken in a more abstract sense, as in the bilocal theory of Yukawa[7] and subsequent works of many others. In particular, in a series of papers Takabayashi[8] has made a transition from the continuous to the discrete basis of representation, and considered simple wave equations and their mass spectra. His equations are based on a harmonic-oscillator-type model, which is viewed as a geometrical interpretation of $SU(3)$ symmetry.[9]

With the apparent success of the quark model and $SU(6)$ symmetry, people have been led to appreciate the use of noncompact groups such as $SL(6)$ and $U(6,6)$

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[1] Y. Nambu, Progr. Theoret. Phys. (Kyoto), Suppls. **37** & **38**, 368 (1966), referred to as A hereafter.

[2] E. Majorana, Nuovo Cimento **9**, 335 (1932).

[3] I. M. Gelfand and A. M. Yaglom, Zh. Experim. i Teor. Fiz. **18**, 703 (1948).

[4] See M. A. Naimark, *Linear Representations of the Lorentz Group*, (translated by A. Swinsen and A. J. Marstand, transl. edited by H. K. Faratat Pergamon Press, Inc., London, 1964).

[5] P. A. M. Dirac, Proc. Roy. Soc. (London) **A182**, 284 (1944); E. P. Wigner, Z. Physik **124**, 665 (1947).

[6] See also E. M. Corson, *Introduction to Tensors, Spinors and Relativistic Wave Equations* (Blackie and Son Ltd., London, 1953).

[7] H. Yukawa, Phys. Rev. **77**, 219 (1953); **91**, 415 (1953).

[8] T. Takabayashi, Progr. Theoret. Phys. (Kyoto) Suppl., extra number, 339 (1965) (earlier papers are quoted therein); **36**, 185 (1966). Kase and Takabayashi, Progr. Theoret. Phys. (Kyoto) **36**, 187 (1966).

[9] As a somewhat related attempt, one may quote also H. C. Corben, Phys. Rev. Letters **15**, 268 (1965).

and their representations as a means of characterizing and systematizing the properties of hadrons without reference to their dynamical origin. This program has been vigorously pursued from a formal mathematical point of view by a number of people. For the justification of the substitution of dynamics with group theory one usually quotes two conspicuous examples, the isotropic-harmonic oscillator and the hydrogen atom. However, the demonstration has been less than complete in the case of the hydrogen atom.

Our example equations, although written down merely on the basis of simplicity, seem to possess many features of the hydrogen atom, at least on the surface. This accident encourages us to embark on a systematic study of infinite-component wave equations. Our main purposes in the present paper are to search for wave equations which simulate more closely the real hydrogenlike atom, and to determine how deep the similarities are. Our attempt is only partially successful as far as translating the hydrogen atom into a relativistic form is concerned, since the class of equations we find still suffers from various diseases if we want to regard them as a *bona fide* field theory. These diseases include unphysical, redundant solutions, especially those with spacelike four-momenta; a tendency to lack antiparticle (negative-frequency) solutions (thereby spoiling the *CPT* theorem); and indefinite sign of energy, either in *c* number or quantized theory. The loosening of spin-statistics connection has been pointed out by Fronsdal,[10] and by Feldman and Matthews[11] recently. But a systematic examination of these problems is not our concern here.

In the case of the nonrelativistic Schrödinger hydrogen atom, on the other hand, the mathematical equivalence of our formalism to the conventional one is actually almost complete, as has recently been shown by Barut and Kleinert,[12] and by Fronsdal[13]. In fact, we can carry out the transition from one to the other by finding a mapping of $SO(4,2)$ algebra onto functions of continuous variables. These same mapping formulas also serve us for such purposes as computing analytically the form factors in relativistic models, and casting the Bethe-Salpeter equation into our discrete form.

II. GENERALIZED MAJORANA EQUATION WITH $SO(4)$ SYMMETRY

In this section we recapitulate and discuss the results of A. Since we shall be dealing with a number of model equations, it is convenient to label each of them by a serial number. Thus the two examples in A will be called No. 1 and No. 2.

[10] C. Fronsdal, Phys. Rev. **156**, 1653 (1967).
[11] G. Feldman and P. T. Matthews, Phys. Rev. **154**, 1241 (1967).
[12] A. O. Barut and H. Kleinert, Phys. Rev. **156**, 1541 (1967).
[13] C. Fronsdal Phys. Rev. **156**, 1665 (1967).

We take a set of spinors $\{\psi_n{}^m\}$, $n, m=1, 2, \cdots$, with n and m lower and upper spin indices, each being symmetric under interchange of indices among themselves. Thus each $\psi_n{}^m$ is an irreducible representation $D(n/2,m/2)$ of the group $SU(2)\times SU(2)$ or $SO(4)$. This choice corresponds to the fact that we shall be dealing with a system which exhibits an $SO(4)$ degeneracy structure.

The spin part of the rotation group $SO(3)$ in real space is identified with a subgroup of the above $SO(4)$. We then extend this to $SO(4,1)$ [which contains the Lorentz group $SO(3,1)$] in such a way that it can be realized within $\{\psi_n{}^m\}$. One way to do this is by regarding each $\psi_n{}^m$ as the usual spinor representation of $SO(3,1)$ $\sim SL(2,C)$. Another way is to take a unitary representation. In either case, the set is large enough to accommodate not only the generators of $SO(3,1)$ but also Lorentz scalar S and vector (current) Γ_μ operators, as well as space reflection. In terms of these one can write down a wave equation

$$[\Gamma_\mu p^\mu+(S-\alpha)\kappa]\Psi(x)=0\,, \qquad (1)$$

where $\Psi(x)=\{\psi_n{}^m(x)\}$, and α and κ are c-number parameters.

The first example (model No. 1), based on an infinite sum of representations, need not be discussed here for the reason already mentioned. In the case of unitary representations, we have considered a special case realized on a subset

$$S_{\pm k}=\{\psi_n{}^m\}\,,\quad n-m=\text{const}=\pm k\,.$$

It turns out that this set constitutes not only an irreducible unitary representation of $SO(4,1)$ but that of $SO(4,2)$, or more precisely, its covering group $SU(2,2)$. Here we will arrange the six dimensions 0, 1, 2, $\cdots$, 5 with the metric $+----+$, and identify the Minkowski subspace with 123 (space) and 0 (time).

The 15 generators $M_{\alpha\beta}$ of $SO(4,2)$ are defined in such a way that $1+i\epsilon^{\alpha\beta}M_{\alpha\beta}$ is the infinitesimal transformation, and

$$\begin{aligned}
&M_{\alpha\beta}=-M_{\beta\alpha}\,,\\
&M_\alpha{}^\beta=M_{\alpha\gamma}g^{\gamma\beta}\,,\quad g^{\alpha\beta}=g_{\alpha\beta}=\delta_{\alpha\beta}\\
&\qquad\qquad\times(1,-1,-1,-1,-1,1)\,, \qquad (2)\\
&M_\alpha{}^\beta=-M_\beta{}^\alpha\,,\quad \alpha,\beta\epsilon(1,\cdots,4)\quad\text{or}\quad \alpha,\beta\epsilon(0,5)\,,\\
&M_\alpha{}^\beta=M_\beta{}^\alpha\,,\quad \alpha(\beta)\epsilon(1,\cdots,4),\beta(\alpha)\epsilon(0,5)\,.
\end{aligned}$$

They satisfy the commutation relations

$$-i[M_{\alpha\beta},M_{\gamma\delta}]=g_{\beta\gamma}M_{\alpha\delta}+g_{\alpha\delta}M_{\beta\gamma}-g_{\alpha\gamma}M_{\beta\delta}-g_{\beta\delta}M_{\alpha\gamma}\,. \qquad (3)$$

Under the restriction to $SO(3,1)$, these 15 generators breakup into a 6 vector (M_{ik}, M_{0i}, $i=1, 2, 3$), a 4-vector (M_{5i},M_{50}), a 4-vector (M_{4i},M_{40}), and a scalar (M_{54}).

In our representation space S_k, the generators have the following expressions:

$$\begin{aligned} 2M_{ij}&=\epsilon_{ijk}(a^\dagger\sigma_k a+b^\dagger\sigma_k b)\,,\\ 2M_{4i}&=(a^\dagger\sigma_i a-b^\dagger\sigma_i b)\,,\\ 2M_{0i}&=(a^\dagger\sigma_i Cb^\dagger-bC\sigma_i a)\,,\\ 2M_{5i}&=-i(a^\dagger\sigma_i Cb+bC\sigma_i a)\,,\\ 2M_{54}&=(a^\dagger Cb^\dagger-bCa)\,,\\ 2M_{40}&=-i(a^\dagger Cb^\dagger+bCa)\,,\\ 2M_{50}&=(a^\dagger a+b^\dagger b+2)\equiv N\,,\quad C=\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \end{aligned}\tag{4}$$

in terms of the two-component Bose operators $a^\dagger$, a and $b^\dagger$, b, which change $\psi_n{}^m$ into $\psi_{n\pm1}{}^m$ and $\psi_n{}^{m\pm1}$, respectively. If we combine a and b into a 4-component Dirac spinor $\xi=(a,b)$, and use two sets of Pauli matrices σ_i and ρ_i, Eq. (4) can be seen to consist of 7 bilinear forms $\xi^\dagger\xi$, $\xi^\dagger\sigma_i\xi$, $\xi^\dagger\rho_3\sigma_i\xi$ and 8 quadratic (and necessarily symmetric) forms $\xi^\dagger\bar{C}\xi^\dagger$, $\xi^\dagger\rho_3\sigma_i\bar{C}\xi^\dagger$, and H.c. ($\bar{C}=\rho_2\sigma_2$), whose algebra is equivalent to that of the noncompact group $SU(2,2)$ of 4×4 matrices (the algebra of 15 Dirac matrices) that leaves $\xi^\dagger\rho_3\xi=\xi^\dagger\gamma_0\xi$ invariant. The parity, or reflection in the subspace (123), may be defined within the pair $S_{\pm k}$ by the operation $S_{\pm k}\rightarrow RS_{\pm k}\sim S_{\mp k}$ such that

$$\begin{aligned} &a\rightarrow RaR^{-1}=-ib\,,\quad b\rightarrow RbR^{-1}=-ia\,,\\ &a^\dagger\rightarrow ib^\dagger\,,\quad b^\dagger\rightarrow ia^\dagger\,,\\ &R=\exp[\tfrac{1}{2}i\pi(a^\dagger b+b^\dagger a)]\,,\\ &R^2=(-i)^{n+m}. \end{aligned}\tag{5}$$

The case $k=0$ is unique in that it does not require doubling of the representation.

An important property of this degenerate unitary representation S_k is that, with respect to the compact subgroup $SO(4)\times SO(2)$, it reduces to a sum

$$\sum_{n-m=k} D_{SO(4)}(n/2,m/2)\times D_{SO(2)}(n/2+m/2+1)\,.$$

Thus, to a given eigenvalue of $M_{50}\sim N$, there corresponds a unique irreducible representation in the complementary space $(1,\cdots,4)$.

Our model equation No. 2 was obtained by equating in Eq. (1)

$$\begin{aligned} &\Gamma_\mu=2M_{5\mu}\,,\quad \mu=1,2,3,0\,,\\ &S=2M_{54}. \end{aligned}\tag{6}$$

(In general, one could take for Γ_μ a linear combination of $M_{5\mu}$ and $M_{4\mu}$. But, as will be clear from what follows, it can be reduced to pure $M_{5\mu}$, pure $M_{4\mu}$, or $M_{5\mu}\pm M_{4\mu}$ by a rotation. The last two do not give a discrete spectrum.) Introducing the 6-dimensional notation $q^\alpha=(p_0,\mathbf{p},\kappa,0)$ and $\Gamma_\alpha=2M_{5\alpha}$, we have then

$$(\Gamma_\alpha q^\alpha-\kappa\alpha)\Psi=0\,.\tag{7}$$

Since the spaces $S_{\pm k}$ are decoupled, we first note that all solutions come in pairs of opposite parities except when $k=0$. Now, if q^α is timelike, i.e.,

$$q_\alpha q^\alpha=p_0{}^2-\mathbf{p}^2-\kappa^2=m^2-\kappa^2>0\,,$$

we can go to a "rest frame" in which

$$q^\alpha=((\mathrm{sgn}m)(m^2-\kappa^2)^{1/2},\mathbf{0},0)\,,$$

and obtain the eigenvalues

$$\Gamma_0(m^2-\kappa^2)^{1/2}\times\mathrm{sgn}(m)=N(m^2-\kappa^2)^{1/2}=\kappa\alpha$$

or

$$m=|\kappa|(1+\alpha^2/N^2)^{1/2}\,\mathrm{sgn}(\kappa\alpha)\,.\tag{8}$$

When $q_\alpha q^\alpha<0$ we can take a frame in which only one of the spacelike components, e.g., Γ_4, survives, so that

$$\pm\Gamma_4(\kappa^2-m^2)^{1/2}=\kappa\alpha\,.\tag{9}$$

Since Γ_4 is a noncompact generator, its engenvalues are continuous, running between $-\infty$ and $+\infty$, which means $\kappa^2>m^2>-\infty$. The corresponding eigenfunctions are non-normalizable. Formally, Eq. (9) is obtained from Eq. (8) by analytic continuation of N^2 to negative values. [The special case $q_\alpha q^\alpha=0$ belongs to the end of the spectrum (8), and likewise has only one sign, although the eigenfunction is non-normalizable.]

Thus, Eq. (7) possesses a discrete inverted hydrogen-like spectrum $m>\kappa$ (if $\kappa\alpha>0$), followed by a continuous part of both signs $|m|<\kappa$, which extends into imaginary m, or spacelike 4-momentum.[14] The continuous part must be regarded as physically relevant, because, under an external field, transition between discrete and continuous parts will take place in general.

We see from the foregoing that this model equation still possesses many unphysical features, which it shares with the original Majorana equation. The only improvement is that the discrete mass spectrum does not come down to zero. The difficulties are: (1) The mass spectrum is inverted; (2) it is not symmetric between positive and negative values; (3) it contains possibly dangerous spacelike solutions. On the other hand, it has also some interesting features, namely, the hydrogen-like degeneracy, i.e., $SO(4)$ type for the discrete states, and $SO(3,1)$ type for the continuum. This can be seen by noting that in its 6-dimensional rest frame, Eq. (7) contains only the generator of a 2-dimensional subspace (05) or (45), and therefore commutes with the generators of the complementary space. The symmetry is "dynamical," in the sense that the symmetry subspace is cut out from the 6 space in a different way for each mass level. For the same reason, it also follows that the orthogonality of eigenfunctions does not hold with respect to the norm $\Psi^\dagger\Psi$ which forms the basis of the unitary representation, but it holds only with respect to the charge (density) $\Psi^\dagger\Gamma_0\Psi$ which is the physically conserved quantity. These two things are not equivalent; a mass eigenstate is not an eigenstate of Γ_0.

[14] The existence of spacelike solutions in the Majorana equation was pointed out by E. Majorana (Ref. 2), and by V. Bargmann, Math. Rev. 10, 583 (1949).

Nevertheless, Γ_0 has no off-diagonal elements, since the eigenfunctions are not orthogonal in the usual sense.

In A, we computed the magnetic moment under the minimal electromagnetic interaction, and found the g factor to be negative, unlike the hydrogen atom. On the other hand, the form factors show reasonable behavior. For the scalar and vector vertices, we find in fact that

$$\begin{aligned}\Psi^+(p')\Psi(p)&=F(t)=4(m^2-\kappa^2)/[4(m^2-\kappa^2)-t],\\ t&=(p'-p)^2,\\ \Psi(p')\Gamma_\mu\Psi(p)&=[(p_\mu+p_\mu')/2m]F(t)^2,\end{aligned}\tag{10}$$

for the ground state $N=2$. This can be derived by using the techniques of Sec. IV, but its general property may be inferred by observing that $F(t)$ must be a function of

$$q_\alpha'q^\alpha=p_\mu'p^\mu-\kappa^2=-t/2+m^2-\kappa^2=(m^2-\kappa^2)\cosh\vartheta,$$

where ϑ is the hyperbolic angle between q and q'. $F(t)$ is then the elementary spherical function $1/\cosh^2(\vartheta/2)$ associated with our representation.

III. RELATIVISTIC MODELS WITH HYDROGENLIKE SPECTRA

We try now to improve our model No. 2 so as to obtain a correct hydrogenlike behavior with respect to level ordering, form factors, etc. Our previous examples were first-order wave equations. This is not only due to our search for readily solvable equations. The algebra of $SO(4,2)$ happens to contain 4-vector generators so that it is indeed possible to write a first-order equation. We shall therefore restrict ourselves at first to first-order equations. By means of elimination, first-order equations can be converted into second-order equations, if necessary. The search for general first-order equations will be made by taking a product of infinite-dimensional and finite-dimensional representations, i.e., regarding Ψ to have components $\psi_{n,s}{}^m$, where s is a spinor index of finite order. An underlying argument is that the conventional field theory for finite spin is known to work. Especially for spin-0,-$\frac{1}{2}$, and -1 cases the degree of singularity (or growth) of form factors associated with the finite spin is not serious, and we may expect enough compensation from the unitary part. Furthermore, the unitary representation can be interpreted as describing the internal-orbital motion of particles (see later sections). It will then be quite proper to introduce intrinsic spins of the constitutents as separate variables. We can adopt the Dirac and Duffin-Kemmer formalism to handle the finite-spin part.[15,16] We discuss here a few different models as typical examples. Other examples will be found in Appendix A.

[15] The Dirac and Duffin-Kemmer algebras are finite-dimensional realizations of $SU(2,2)$, but probably this fact is accidental to the hydrogen-atom problem.

[16] The Duffin-Kemmer-type equations are not treated in this paper. In general, they lead to additional spurious solutions which are not present (or whose masses are pushed to ∞) in the ordinary Duffin-Kemmer equation.

Model No. 3

$$L\Psi\equiv[\Gamma_\mu p^\mu+S(x\gamma_\mu p^\mu+\kappa\gamma_5N_5)-\kappa\alpha]\Psi=0.\tag{11}$$

Γ_μ and S are the same as in Eq. (6). The pseudoscalar $N_5=n-m=k(\neq0)$ is a constant to within a sign in the representation space $S_{\pm k}$. The Dirac γ's are standard ones, with γ_0 being Hermitian, and γ_i, $\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3$ anti-Hermitian. This equation follows from a Lagrangian $\bar\Psi L\Psi$, where the adjoint function to Ψ must be defined by $\bar\Psi=\Psi^+\gamma_0$. The parameters of the equation are κ, α, and x, which we will take to be positive.

In order to diagonalize Eq. (11) in the rest frame $p^\mu=(m,\mathbf{0})$, we first make a nonunitary rotation in the Dirac space:

$$\begin{aligned}\Psi=\exp[-(\phi_1/2)\gamma_0\gamma_5]\Psi_1=U_1\Psi_1,\quad \bar\Psi=\bar\Psi_1U_1^{-1},\\ \tanh\phi_1=\kappa k/xm,\end{aligned}\tag{12}$$

which leads to

$$[\Gamma_0m+S\gamma_0(x^2m^2-\kappa^2k^2)^{1/2}\operatorname{sgn}(m)-\kappa\alpha]\Psi_1=0.\tag{13}$$

We have assumed that $x^2m^2>\kappa^2k^2$. The next transformation is a unitary one in the (04) plane:

$$\begin{aligned}\Psi_1=\exp[i\phi_2\gamma_0M_{04}]\Psi_2=U_2\Psi_2,\quad \bar\Psi_1=\bar\Psi_2U_2^{-1},\\ \tanh\phi_2=(x^2m^2-\kappa^2k^2)^{1/2}\operatorname{sgn}(xm)/m,\end{aligned}\tag{14}$$

assuming $|m|>(x^2m^2-\kappa^2k^2)^{1/2}$, and we find

$$\{\Gamma_0[\kappa^2k^2-(x^2-1)m^2]^{1/2}\operatorname{sgn}(m)-\kappa\alpha\}\Psi_2=0.\tag{15}$$

Thus

$$\begin{aligned}m&=\kappa(k^2-\alpha^2/N^2)^{1/2}/(x^2-1)^{1/2}\\ &=m_0(1-\alpha^2/k^2N^2)^{1/2},\quad m_0\equiv\kappa k/(x^2-1)^{1/2}.\end{aligned}\tag{16}$$

This equation is correctly hydrogenlike and consistent with the rotations (12) and (14), if we require the values of x, α, and k to be such that

$$x^2-1>0,\quad 1-x^2\alpha^2/k^2(|k|+2)^2>0.\tag{17}$$

[Note that $N=|k|+2,\ |k|+4,\ \cdots$.]

The continuum part of the spectrum above m_0 can be obtained in a similar way by rotating Eq. (13) in the S direction instead of Γ_0. As was already discussed in Sec. II, however, the continuum part consists of both signs, since S takes all real eigenvalues. Unlike model No. 2, on the other hand, there are no spacelike solutions, as we can check easily by diagonalizing the equation under the assumption that p^μ is spacelike.

The difficulties with this equation are that the discrete spectrum has only positive eigenvalues, as in model No. 2, and that each eigenvalue is fourfold degenerate, not counting the $O(4)$ degeneracy of spins. The latter situation is due to the two signs γ_0 and k can take in Eq. (15); this means that half of the solutions have opposite parity to the other.

The current operator that follows from Eq. (11) is

$$j_\mu=\bar\Psi(\Gamma_\mu+xS\gamma_\mu)\Psi.\tag{18}$$

After carrying out transformations (12) and (14), we find that j_0 changes sign with γ_0. Hence, the energy operator also changes sign with γ_0 (for a fixed $m>0$). Certainly this is a serious difficulty in setting up a physically consistent Lagrangian formalism. We find, keeping only the diagonal part,

$$\begin{aligned} j_0 &= \Psi_2{}^\dagger\gamma_0\Gamma_0(\cosh\phi_2 - x\sinh\phi_2\cosh\phi_1)\Psi_2 \\ &= [(1-x^2)mN^2/\kappa\alpha]\Psi_2{}^\dagger\gamma_0\Psi_2. \end{aligned}$$

Hence the "charge" j_0 changes sign with γ_0, and so does the energy $T_{00}=mj_0$ in spite of m being >0.

Equation (11) was meaningful only for $k\neq 0$. When $k=0$, or $\Psi=\{\psi_n{}^n\}$, we simply double the space: $\Psi\to(\Psi^{(1)},\Psi^{(2)})$, and write

$$L\Psi=[\tau_3(\Gamma_\mu p^\mu-\kappa\alpha)+S(x\gamma_\mu p^\mu+\kappa\tau_1)]\Psi=0\,, \quad (19)$$

where the τ matrices operate in the space $(\Psi^{(1)},\Psi^{(2)})$. Again this can be diagonalized by means of two rotations, $\sim\exp(\phi_1\gamma_0\tau_1/2)$ and $\sim\exp(i\phi_2\tau_3\gamma_0 M_{04})$, and leads to the solution (16) (with k replaced by 1). The fourfold degeneracy corresponds to τ_3, $\gamma_0=\pm1$.

The models considered above are closely related to the quadratic equations found by Fronsdal.[13] For example, Eq. (13) can be squared to give

$$\{[S^{-1}(\Gamma_0 m-\kappa\alpha)]^2-x^2m^2+\kappa^2k^2\}\Psi=0$$

or

$$L\Psi=\{[S^{-1}(\Gamma_\mu p^\mu-\kappa\alpha)]^2-x^2p_\mu p^\mu+\kappa^2k^2\}\Psi=0\,. \quad (20)$$

We may then drop the Dirac spin indices, and obtain Fronsdal's equation. The Lagrangian has to be defined by $\Psi^+ SL\Psi$ for reasons of Hermiticity.

Although the Dirac indices may be eliminated in this way, the fourfold degeneracy and associated difficulties of the solution remain unchanged.

Model No. 4

$$L\Psi=[\Gamma_\mu p^\mu+(S-\alpha)(x\gamma_\mu p^\mu+\kappa\gamma_5 N_5)]\Psi=0\,. \quad (21)$$

The discrete solution is

$$\Gamma_0[\kappa^2k^2-(x^2-1)m^2]^{1/2}\,\mathrm{sgn}(m)-\alpha\gamma_0(x^2m^2-\kappa^2)^{1/2}\times\mathrm{sgn}(m)=0\,,$$

or

$$m=\pm\kappa|k|\left[\left(1+\frac{\alpha^2}{N^2}\right)\bigg/\left(x^2-1+\frac{x^2\alpha^2}{N^2}\right)\right]^{1/2}, \quad \gamma_0=+1\,,\quad x^2\geq 1\,. \quad (22)$$

This is also hydrogenlike, and besides has the symmetric degeneracy pattern m, m, $-m$, $-m$.

The continuum part, on the other hand, is found to consist of upper and lower regions:

$$|m|>\kappa|k|/(x^2-1)^{1/2},\quad (x^2m^2-\kappa^2k^2>m^2)$$

and

$$0\leq|m|<\kappa|k|/x\,,\quad (\kappa^2k^2-x^2m^2>0)\,,$$

each m being fourfold degenerate. Beyond this there is also the spacelike solution corresponding to imaginary m.

In spite of the unphysical lower and imaginary continua, this model possesses one good feature. If we evaluate the current

$$j_\mu=\bar\Psi[\Gamma_\mu+x(S-\alpha)\gamma_\mu]\Psi\,,$$

we find that it has a definite sign for the discrete and upper continuum part of the spectrum. This suggests the possibility of quantizing the field according to Fermi statistics, and thus satisfying positive definiteness of energy (for the physical part) irrespective of integer or half-integer spin.

We next consider a second-order equation.

Model No. 5

Take the space S_0 and put[17]

$$(\Gamma_\mu p^\mu+(1/\kappa)Sp_\mu p^\mu-\alpha\gamma_\mu p^\mu)\Psi=0\,. \quad (23)$$

The massive solution has the spectrum

$$\begin{aligned} m&=\pm\kappa(1-\alpha^2/N^2)^{1/2}, \quad (\gamma_0=+1) \\ |m|&>\kappa\,, \quad (Sm\gamma_0>0) \end{aligned} \quad (24)$$

for the discrete and continuum states, which is symmetric in sign, but otherwise similar to model No. 3. In addition, however, there exists the obvious massless solution $p_\mu=0$. In a typical case, this means that

$$[(\Gamma_0-\Gamma_3)-\alpha(\gamma_0-\gamma_3)]\Psi=0$$

or

$$(\gamma_0-\gamma_3)\Psi=(\Gamma_0-\Gamma_3)\Psi=0\,, \quad (25)$$

as can be seen by squaring the first equation. The eigenvalues of $\Gamma_0-\Gamma_3$ are continuous and ≥ 0,[18] so Eq. (25) corresponds to the end point of a continuum.

Except for the massless solution, the spectrum is right for a hydrogenlike system: The degeneracy is m, $-m$ and m, m, $-m$, $-m$ for the discrete and continuous part, respectively. It may be allowed to interpret this as corresponding to the discrete and continuous parts of the system $p^\pm e^\pm$ and $p^\pm e^\mp$ combined (where p is a scalar particle in this case).

As for the current

$$j_\mu=\bar\Psi(\Gamma_\mu-\alpha\gamma_\mu+(2/\kappa)Sp_\mu)\Psi\,, \quad (26)$$

it turns out that the charge has a definite sign for the discrete and continuous parts.[19] It would seem possible,

[17] A somewhat related equation was considered by A. O. Barut and H. Kleinert (unpublished). They discuss only physically relevant parts of the spectrum.

[18] This can be seen from Eq. (40). $\Gamma_0-\Gamma_3\sim M_{50}-M_{53}\sim M_{50}-M_{54}\sim r$ as far as the spectrum is concerned.

[19] We need some care in the continuum case. Instead of a pure eigenstate of S (after rotation), a wave packet of finite norm should be considered. Though both S and Γ_0 have diagonal elements, we may eliminate S in j_0 by means of the wave equation. The coefficient of Γ_0 then determines the sign, as $\langle\Gamma_0\rangle$ can be made arbitrarily large.

therefore, to quantize the field correctly according to Fermi statistics and make the energy positive-definite.

Summarizing this section, we have considered linear as well as quadratic equations in the product space of S_k and Dirac spinor. All of them have hydrogenlike discrete and continuous spectra, but suffer from some unwanted features such as the lack of symmetry in positive and negative frequencies, the presence of redundant solutions (including spacelike solutions[20]), and the indefiniteness of energy sign. Relatively speaking, model 5 may be the most satisfactory, but the significance of massless solutions remains to be clarified.

IV. ALTERNATIVE REPRESENTATIONS OF $SO(4,2)$ ALGEBRA

We develop here representations of the $SO(4,2)$ algebra in terms of continuous variables. First we note that the set $S_{\pm k}$ can be generated from the ground-state $\psi_k{}^0$ or $\psi_0{}^k$ by repeated applications of $X_\mu{}^\dagger$, where

$$X_\mu{}^\dagger = a^\dagger\sigma_\mu C b^\dagger\,, \quad X^\mu = aC\sigma^\mu b = (X_\mu{}^\dagger)^\dagger\,,$$
$$\sigma_\mu = (\sigma_0 = 1, \sigma_1, \sigma_2, \sigma_3)\,, \quad \sigma^\mu = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3)\,, \tag{27}$$

which satisfy

$$X_\mu{}^\dagger X^{\dagger\mu} = X_\mu X^\mu = 0\,. \tag{28}$$

In view of Eq. (4), this means also

$$\sum_{i=1}^{4} (M_{0i}M_0{}^i - M_{5i}M_5{}^i) = 0\,,$$
$$\sum_{i=1}^{4} (M_{0i}M_5{}^i + M_{5i}M_0{}^i) = 0\,. \tag{29}$$

For simplicity we restrict ourselves to the case $k=0$ below.

Let us set up a correspondence between S_0 and a space $F = \{f(x_\mu)\}$ of functions of four variables such that

$$\Psi_0{}^0 = \psi_0{}^0 \to 1\,, \quad X_\mu{}^\dagger \to x_\mu\,. \tag{30}$$

F is then a set of polynomials in x_μ.

For the time being, we assume x_μ to be independent. $X_\mu{}^\dagger$ and X_μ generate the algebra

$$[X^\mu, X_\mu{}^\dagger] = N = a^\dagger a + b^\dagger b + 2 \text{ (no summation)}\,,$$
$$[X_0, X_i{}^\dagger] = 2K_i = a^\dagger\sigma_i a - b^\dagger\sigma_i b\,, \tag{31}$$
$$[X_i, X_j{}^\dagger] = -2i\epsilon_{ijk}L_k = -i\epsilon_{ijk}(a^\dagger\sigma_k a + b^\dagger\sigma_k b)\,, \quad (i\neq j)$$
$$[X_\mu{}^\dagger, N] = -X_\mu{}^\dagger\,, \quad [X_\mu, N] = X_\mu\,.$$

These relations can be translated into the space F by the assignment

$$\tfrac{1}{2}N \to x_\mu D^\mu + \alpha = \textstyle\sum x_\mu \partial/\partial x_\mu + \alpha\,,$$
$$X_\mu \to -x_\mu D_\lambda D^\lambda + 2x_\lambda D^\lambda D_\mu + 2\alpha D_\mu\,,$$
$$K_i \to x_i D_0 - x_0 D_i = x_i\partial/\partial x_0 + x_0 \partial/\partial x_i\,, \tag{32}$$
$$L_i \to -i\epsilon_{ijk}x^j D^k = i\epsilon_{ijk}(x_j\partial/\partial x_k - x_k\partial/\partial x_j)\,.$$

Here α is an arbitrary constant. If, however, we impose the condition (29) or that $f(x_\mu)$ is a function of a null vector:

$$x_\mu x^\mu = x_0{}^2 - x_i{}^2 = 0\,,$$

such a condition must be compatible with Eq. (32). In other words, for any operator O in (32), we must have $[x_\mu x^\mu, O]f = 0$ if $x_\mu x^\mu = 0$. This is true only if

$$\alpha = 1\,, \tag{33}$$

which will be assumed from now on. With Eqs. (4), (28), (30), (33), we can complete the mapping of the space S_0 onto F. For computing the norm of a vector in F we essentially follow the prescription: We write first

$$(\Psi,\Psi) = (f(X^\dagger)\Psi_0, f(X^\dagger)\Psi_0) = (\Psi_0, f^*(X)f(X^\dagger)\Psi_0)\,, \tag{34}$$

then translate X, $X^\dagger$, and Ψ_0 using Eqs. (30) and (31). An analytic definition is then

$$(\Psi,\Psi) \to \frac{1}{(2\pi i)^4}\oint\cdots\oint \frac{dz_0\cdots dz_3}{z_0\cdots z_3} f^*(X)f(X^\dagger)\,, \tag{35}$$

where the z's are complex extensions of the x's, and the contours taken around zero. This formula is often convenient in practical calculations.

We next show that there exists another mapping,

$$S_0 \to G = \{f(x_\mu)\}\,,$$

in which the norm can be defined as an integral over the real axis with a simple weight function

$$(\Psi,\Psi) \to (f,f) = \int\cdots\int f^*(x_\mu)f(x_\mu)\frac{dx_0\cdots dx_3}{x_\lambda x^\lambda}\,. \tag{36}$$

It is realized by retaining the correspondence (32) but redefining the generators as

$$\begin{aligned}
\tfrac{1}{2}(X_0{}^\dagger + X_0) &= M_{40} \quad\text{or}\quad M_{45}\,,\\
\tfrac{1}{2}(X_0{}^\dagger - X_0) &= M_{50} \quad\text{or}\quad M_{50}\,,\\
\tfrac{1}{2}(X_i{}^\dagger + X_i) &= M_{4i} \quad\text{or}\quad M_{4i}\,,\\
\tfrac{1}{2}(X_i{}^\dagger - X_i) &= M_{5i} \quad\text{or}\quad -M_{0i}\,,\\
-iK_i &= M_{0i} \quad\text{or}\quad M_{5i}\,,\\
L_i &= \epsilon_{ijk}M_{jk}\,,\\
-\tfrac{1}{2}iN &= M_{54} \quad\text{or}\quad -M_{04}\,.
\end{aligned} \tag{37}$$

(The two choices are mathematically equivalent as they correspond to the interchange $0\leftrightarrow 5$, but can make a difference in physical interpretation.) It can be readily verified that these operators are self-adjoint in the sense $(g,Of) = (Og,f)$ when the scalar product is taken according to Eq. (36).

[20] In general this is inevitable. It derives from the fact that the Hamiltonian defined (from a linear equation) by $i\partial\Psi/\partial t = H\Psi$ is not Hermitian, as $\Gamma_0{}^{-1}$ does not commute with $\mathbf{\Gamma}\cdot\mathbf{p}$. Thus there may be complex energy for some p.

We have actually not made the restriction (29) on the generators. Therefore we are dealing with a different representation from Eq. (4). From Eqs. (32) and (37) we derive the formula

$$x_\mu x^\mu D_\lambda D^\lambda = \sum_\alpha (M_{5\alpha}M^{5\alpha}+M_{4\alpha}M^{4\alpha})+2\,,$$
$$x_\mu x^\mu = \sum_\alpha (M_{5\alpha}+M_{4\alpha})(M_5{}^\alpha+M_4{}^\alpha)\,, \qquad (38)$$
$$x_\mu x^\mu (D_\lambda D^\lambda)^2 = \sum_\alpha (M_{5\alpha}-M_{4\alpha})(M_5{}^\alpha-M_4{}^\alpha)\,.$$

Next we proceed to impose the condition

$$x_\mu x^\mu = 0\,,$$

or

$$x_0 = \pm (x_i x_i)^{1/2} = \pm r\,. \qquad (39)$$

Since this is consistent with the commutator algebra, we can make the substitution $f(x_\mu)\to f(x_i,x_0(x_i)) = g(x_i)$. Accordingly we can also drop D^0 and replace x_0 by $\pm r$ in Eq. (32) without affecting the commutation relation. The result is

$$\begin{aligned}
&X_0^\dagger = \pm r = M_{40}+M_{50} \quad\text{or}\quad M_{45}+M_{50}\,,\\
&X_0 = \pm r\Delta = M_{40}-M_{50} \quad\text{or}\quad M_{45}-M_{50}\,,\\
&X_i^\dagger = x_i = M_{4i}+M_{5i} \quad\text{or}\quad M_{4i}-M_{0i}\,,\\
&X_i = x_i\Delta + 2x_kD^kD_i + 2D_i = M_{4i}-M_{5i} \quad\text{or}\quad M_{0i}+M_{4i}\,,\\
&-iK_i = \pm irD_i = M_{0i} \quad\text{or}\quad M_{5i}\,,\\
&-\tfrac{1}{2}iN = -i(x_iD^i+1) = M_{54} \quad\text{or}\quad -M_{04}\,,\\
&(\Delta = D_iD_i = -D_iD^i)\,.
\end{aligned} \qquad (40)$$

The metric that replaces Eq. (36) for $g(x_i)$ is

$$(g,g) = \int\cdots\int g^*(x_i)g(x_i)\frac{dx_1dx_2dx_3}{r}\,, \qquad (41)$$

as may be understood from the fact

$$\theta(\pm x_0)\delta(x_\mu x^\mu)d^4x \sim d^3x/2r\,.$$

Finally, we observe that

$$(g,M_{50}g) = \pm\tfrac{1}{2}\int\cdots\int g^*(1-\Delta)g\,d^3x$$
$$= \pm\tfrac{1}{2}\int\cdots\int (g^*g + \nabla_i g^*\nabla_i g)d^3x\,.$$

Remembering that $2M_{50}=N$ defined by Eq. (4) has positive eigenvalues, we conclude that the + sign has to be taken for Eqs. (40) and (41) to be equivalent to the original representation (4).

The representation (40) is essentially the same as that found by Fronsdal.[13] He arrived at this result by starting from the Fock (stereographic) representation of the hydrogenic Schrödinger equation. Barut and Kleinert[12] had also discovered the use of $SO(4,2)$ in the hydrogen problem, identified the relevant representation, and written down observables in terms of the generators. The next section deals with a re-examination of this problem.

V. NONRELATIVISTIC HYDROGEN ATOM

Equation (40) contains all the necessary operators for writing down a Schrödinger equation for the hydrogen atom if we identify the variables x_i with the actual spatial coordinates. To exhibit the significance of our procedure clearly, it is better to treat the case as a two-body problem. We will thus write down the Hamiltonian

$$H = \frac{1}{2m_1}[\mathbf{p}^{(1)}-e_1\mathbf{A}(\mathbf{r}^{(1)})]^2 + \frac{1}{2m_2}[\mathbf{p}^{(2)}-e_2\mathbf{A}(\mathbf{r}^{(2)})]^2 + \frac{e_1e_2}{|\mathbf{r}^{(1)}-\mathbf{r}^{(2)}|} + e_1\varphi(\mathbf{r}^{(1)}) + e_2\varphi(\mathbf{r}^{(2)}) \qquad (42)$$

for a generalized hydrogenlike system in an external electromagnetic field. Introducing the new coordinates and momenta

$$\begin{aligned}
&C_1\mathbf{r}^{(1)}+C_2\mathbf{r}^{(2)} = \mathbf{X}\,, \quad C_1+C_2=1\,,\\
&\mathbf{r}^{(1)}-\mathbf{r}^{(2)} = \mathbf{r}\,,\\
&\mathbf{p}^{(1)}+\mathbf{p}^{(2)} = \mathbf{P}\,,\\
&C_2\mathbf{p}^{(1)}-C_1\mathbf{p}^{(2)} = \mathbf{p}\,,
\end{aligned} \qquad (43)$$

we transform Eq. (42) into

$$H = \frac{1}{2m_1}[C_1\mathbf{P}+\mathbf{p}-e_1\mathbf{A}(\mathbf{X}+C_2\mathbf{r})]^2 + \frac{1}{2m_2}[C_2\mathbf{P}-\mathbf{p}-e_2\mathbf{A}(\mathbf{X}-C_1\mathbf{r})]^2 + e_1\varphi(\mathbf{X}+C_2\mathbf{r}) + e_2\varphi(\mathbf{X}-C_1\mathbf{r}) + \frac{e_1e_2}{r}\,. \qquad (44)$$

The relative coordinates $\mathbf{r}$ may now be identified with the representation variables of Eq. (40), and the operators $\mathbf{r}$, r, $\mathbf{p}$, p^2 that appear in Eq. (44) replaced by the generators of $SO(4,2)$. An awkward point, however, is that $\mathbf{r}$ appears in the external fields $\mathbf{A}$ and φ. This cannot be avoided in general, but in the limiting case $m_2 \gg m$, and $\varphi=0$ we can choose $C_1=1$, $C_2=0$ to obtain

$$H = \frac{1}{2m_1}[\mathbf{P}+\mathbf{p}-e_1\mathbf{A}(\mathbf{X})]^2 + \frac{e_1e_2}{r}\,. \qquad (45)$$

[Note that the usual center-of-mass transformation would correspond to $C_1=0$, $C_2=1$.] Multiplying (45) by r/r_0 from the left,

$$\frac{r}{r_0}H=\frac{1}{2m_1}\frac{r}{r_0}[\mathbf{P}-e_1\mathbf{A}(\mathbf{x})]^2+\frac{1}{m_1r_0}r\mathbf{p}\cdot[\mathbf{P}-e_1\mathbf{A}(\mathbf{x})]+\frac{1}{2m_1}\frac{r}{r_0}p^2+\frac{e_1e_2}{r_0}, \quad (46)$$

where r_0 is an arbitrary scale factor. Identifying r_i/r_0 with the dimensionless variables in Eq. (40), this leads to the eigenvalue equation

$$\left\{(M_{50}+M_{45})\left[E-\frac{1}{2m_1}[\mathbf{P}-e_1\mathbf{A}(\mathbf{X})]^2\right]-\frac{1}{2m_1r_0^2}(M_{50}-M_{45})-\frac{1}{m_1r_0}M_{5i}[P_i-e_1A_i(\mathbf{X})]-\frac{e_1e_2}{r_0}\right\}\Psi=0, \quad (47)$$

adopting the second assignment of generators.

We have thus succeeded in transforming the Schrödinger equation into the form of an infinite-component equation in which only the generators M_{50}, M_{40}, M_{0i} appear. Clearly, it also preserves the norm in view of Eq. (41) and $H=(1/r)(rH)$.

Equation (47) was first obtained by Fronsdal. In case $\mathbf{A}=0$, it can be easily diagonalized by the previous techniques and yields the well-known energy spectrum

$$E=\frac{P^2}{2m_1}-\frac{4(e_1e_2)^2m_1}{N^2}, \quad N=2,4,6,\cdots, \quad (48)$$

(r_0 drops out of the result). The first term is the kinetic energy of the whole system in motion, and the second term is the binding energy. Of course the continuum $E-P^2/2m_1>0$ also follows from the equation. Besides, Eq. (47) satisfies the gauge principle when a transverse field is switched on. Thus the system behaves as if it were a single particle and its internal dynamics had been replaced by the abstract algebra of $SO(4,2)$.

It is instructive to re-examine the previous relativistic models in the light of the present results. For example, model 2, Eqs. (6) and (7), can be translated into the Schrödinger representation

$$(2M_{50}P^0+2M_{5i}P^i+2M_{54}\kappa-\kappa\alpha)\Psi\rightarrow[(P_0-\kappa)r/r_0+(P_0+\kappa)r_0rp^2-2r\mathbf{P}\cdot\mathbf{p}-\kappa\alpha]\Psi(\mathbf{r}),$$

or

$$[(P_0-\kappa)/r_0+(P_0+\kappa)r_0p^2-2\mathbf{P}\cdot\mathbf{p}-\kappa\alpha/r]\Psi(\mathbf{r}). \quad (49)$$

Setting $r_0=1/(P_0+\kappa)$ and $p-P\rightarrow p'$, this becomes

$$[P_0^2-P^2-\kappa^2+(p'^2-\kappa\alpha/r)]\psi(r)=0. \quad (50)$$

It is a Klein-Gordon–type equation, where the internal energy is given by the hydrogenic Hamiltonian (the last term) but with the wrong sign. In the small binding nonrelativistic limit, the system looks as if made up of two particles with total mass $=\kappa$ and reduced mass $=-\kappa$.

In a similar way, other models may be treated. In the nonrelativistic limit of model No. 3 we may start from Eq. (13). (However, the formulas in this section apply only to the space S_0.) We write

$$L\Psi=[\Gamma_0P_0+S\{x^2(P_0^2-P^2)-\kappa^2\}^{1/2}-\kappa\alpha-\mathbf{\Gamma}\cdot\mathbf{P}]\Psi=0, \quad (51)$$

and expand it

$$[\Gamma_0(m_0+\epsilon)+S(m_0+x^2\epsilon-x^2P^2/2m_0)\times(x^2-1)^{1/2}m_0\alpha-\mathbf{\Gamma}\cdot\mathbf{P}]\Psi=0, \quad (52)$$
$$\kappa=(x^2-1)^{1/2}m_0,\quad P_0=m_0+\epsilon,\quad \epsilon\ll m_0.$$

This becomes, in the Schrödinger representation,

$$\left[-\epsilon+\frac{x^2}{x^2-1}\frac{1}{2m_0}\left(\mathbf{P}-\frac{2m_0r_0}{x^2}\mathbf{p}\right)^2+\frac{2m_0r_0}{x^2}p^2-\frac{m_0r_0}{(x^2-1)^{1/2}}\frac{\alpha}{r}\right]\psi(\mathbf{r})=0. \quad (53)$$

In the presence of an external field, we replace $\mathbf{P}$ and ϵ by $\mathbf{P}-eA(\mathbf{X})$ and $\epsilon-e\varphi(\mathbf{X})$. Comparing Eq. (53) with Eq. (44) we find then that they are equivalent if we choose

$$\begin{aligned}&C_1=0,\quad C_2=2m_0r_0/x^2=1,\\ &m_1=m_0/x^2,\quad m_2=m_0(x^2-1)/x^2,\quad m_1+m_2=m_0,\\ &e_2=e,\quad e_1=0,\quad \alpha=2e_1e_2(m_1/m_2)^{1/2}\\ &\qquad=2e_1e_2/(x^2-1)^{1/2}.\end{aligned} \quad (54)$$

We can keep α finite if we regard this as a limit $e_1\rightarrow 0$ and $m_2\rightarrow 0$, keeping e_1^2/m_2 finite.

In the case of model No. 5, the equation corresponding to (49) is

$$[2M_{50}P^0+2M_{5i}P^i+2M_{54}(P_0^2-P^2)/\kappa-\alpha(\gamma_0P^0+\gamma_iP^i)]\Psi=0. \quad (55)$$

Replacing the last term by $-\alpha(P_0^2-P^2)^{1/2}$, we find in the lowest approximation

$$\left(\epsilon+\frac{1}{\kappa}(\mathbf{P}-\mathbf{p})^2+\frac{1}{\kappa}\mathbf{p}^2-\frac{\alpha}{r}\right)\psi(\mathbf{r})=0, \quad (56)$$

so that

$$m_1=m_2=\kappa/2,\quad e_1=0,\quad e_2=e,$$

but

$$\alpha=e_1e_2\neq 0, \quad (57)$$

which is not exactly a hydrogenlike situation.

We close this section with a reference to Bethe-Salpeter (BS) equations. It is a natural idea that the Bethe-Salpeter equation might also be amenable to transformation into a discrete representation. In fact, Eqs. (32) and (38) provide the necessary formulas for this purpose. Consider, for example, two scalar particles interacting via a scalar photon. The BS equation reads, in the differential form,

$$\{(\tfrac{1}{2}P+p)^2-\mu^2\}\{\tfrac{1}{2}P-p)^2-\mu^2\}-\frac{g^2}{4\pi^2x^2}\psi(x_\mu)=0\,, \quad (58)$$

or in the rest frame,

$$\Big[x^2(p^2)^2+2x^2p^2(\tfrac{1}{4}m^2-\mu^2)+x^2(\tfrac{1}{4}m^2-\mu^2)^2 -m^2x^2p_0{}^2-\frac{g^2}{4\pi^2}\Big]\psi(x_\mu)=0\,, \quad (59)$$

$$(x^2=x_\mu x^\mu,\ p^2=p_\mu p^\mu=-D_\mu D^\mu)\,.$$

For the first three terms we have the ready-made formulas (38). Equating the necessary scale parameter $r_0{}^2$ with $1/(\frac{1}{4}m^2-\mu^2)$, the sum of them simply becomes

$$-(m^2-4\mu^2)(M_{5\alpha}M^{5\alpha}+1)\,,\quad (m^2-4\mu^2<0,\ \text{bound states})$$

or

$$-(m^2-4\mu^2)(M_{4\alpha}M^{4\alpha}+1)\,,\quad (m^2-4\mu^2>0,\ \text{continuum}). \quad (60)$$

The fourth term, on the other hand, cannot be translated so simply. We have to express D_0 from X_0 of Eq. (32):

$$2(x_\lambda D^\lambda+1)D_0-x_0D_\lambda D^\lambda=X_0\,, \quad (61)$$

and make the substitution (37). Although the equation cannot be solved easily except when $m^2=0$, this method provides another way of looking at the BS equation.

VI. FORMULATION OF THE GENERAL PROBLEM

The main lessons we have learned from the results of the previous sections may be summarized as follows. (1) It is possible to set up infinite-component, simple model equations which simulate a composite, hydrogen-like system in many respects. But so far they do not satisfy all the requirements for being a physically meaningful Lagrangian field theory, except possibly model No. 5. (2) In the nonrelativistic sense, however, there are no basic difficulties. In fact, the Schrödinger equation can be transformed into our type of equation and vice versa. One limitation to this is that as a genuine two-body problem the correspondence is only approximate when external fields are introduced.

As a possible field theory, the infinite-component fields need not be regarded as approximate and phenomenological substitutes for dynamical equations for composite systems. They may be fundamental in their own right. But there are many novel properties we have yet to understand in this type of theory. For example, we may ask: (a) What becomes of the spin-statistics connection in general (which seems to have been loosened)? (b) What are the implications of the positive-negative asymmetry of mass spectrum vis-à-vis P, C, and T invariance (or noninvariance)? (c) Are the spacelike solutions really dangerous? (d) Do (b) and (c) imply breakdown of local commutativity and causality between fields? If so, is it possible that the currents (bilinear forms of fields) still satisfy causality? Although some of these problems have been examined by Feldman and Matthews[11] and by Fronsdal,[10] much remains to be done.

Assuming that these difficulties can be overcome or ignored, at least in suitable model equations, we may set down the working principles to be followed in applying our method to other complex systems, in particular to hadron physics and nuclear physics.[21] First we take a definite model (like the quark model or the harmonic-oscillator model) with a certain compact symmetry group G_0 (considered as rest symmetry, i.e., the little group) and degeneracy structure represented by a particular set S of representations. G_0 must contain the $SU(2)$ or $O(3)$ group corresponding to internal rotations of a system. We enlarge the group G_0 to G in such a way that (a) S is its unitary representation (or a product of unitary and simple finite representations), (b) G contains the (internal) Lorentz group, and (c) there exist among the generators of G also Lorentz 4-vectors Γ_μ, symmetric tensors $\Gamma_{\mu\nu}$, etc. We can then couple Γ_μ, $\Gamma_{\mu\nu}$, etc., with the external momentum to write down a first-order, second-order, etc., wave equation. In any case, we may assume (at least as a simplest possibility) that the wave equation is linear or quadratic in all these generators and momenta.

In practice we may try the ladder-operator technique, as we have done here, to generate G out of G_0, which will also determines the type of the representation S. For example, the Majorana equation has $G_0=SO(3)$, $G=SO(3,2)$, and $S=D(0)+D(1)+\cdots$ or $D(\frac{1}{2})+D(\frac{3}{2})+\cdots$ of $SO(3)$ (of course, the reflection group must also be built into this). Our model No. 2 has $G_0=SO(4)$, $G=SO(4,2)$, $S=\sum D(n,\frac{1}{2}k+n)$. In other models, $G_0=SO(4)\times SU(2)$, $G=SO(4,2)\times SL(2,C)$, and S is the product of finite (Dirac) and infinite representations.

When $G_0=SU(3)$, considered as the symmetry group of the harmonic oscillator,[22] the corresponding S consists of $D(n,0)$, $n=0, 1, 2, \cdots$ of $SU(3)$. Use of the three-component ladder operators a_i, $a_i{}^\dagger$ leads to an enlarged group $Sp(6,R)$, with 9 compact generators $a_i{}^\dagger a_k$ and 12 noncompact generators a_ia_k, $a_i{}^\dagger a_k{}^\dagger$. This group, however, does not contain the Lorentz group.

[21] Similar or related views can be found in the literature frequently. P. Budini and C. Fronsdal, Phys. Rev. Letters **14**, 968 (1965); A. O. Barut, in *Non-Compact Groups in Particle Physics*, edited by Y. Chow (W. A. Benjamin, Inc., New York, 1966); Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Letters **17** 148 (1965); T. Takabayashi, Ref. 8.

[22] This problem is being investigated in collaboration with S. P. Rosen.

The correct scheme seems to be $G=SU(3,1)$, with 15 generators $a_i^\dagger a_k$ and $(N+C)^{1/2}a_i$, $a_i^\dagger(N+C)^{1/2}$, $(N=\sum a_i^\dagger a_i,\ C=\text{const})$. These make up the Lorentz generators and a symmetric traceless tensor $\Gamma_{\mu\nu}$ so that a second-order wave equation can be written down.[23]

The existence of vector or tensor operators serves the purposes of giving a mass spectrum and bringing these operators into a well-defined algebra. From the foregoing results it is clear that the infinite representation is suitable for characterizing in an abstract way the internal orbital motion of a system. The rest symmetry group G_0 reflects how the internal orbital motion is organized, i.e., how the rotational and radial degrees of freedom can be excited. The intrinsic spin of the constituents will then be multiplied into this orbital wave function. Thus the observables consist of generators of the orbital as well as the intrinsic-spin algebra. Of course, the strength of our approach lies in that we do not always have to draw a line between orbital and intrinsic spin. The latter may also belong to an infinite-dimensional representation [as in the $SL(6)$ or $U(6,6)$ models]. The orbital representation space, at the same time, need not always be introduced for each relative coordinate or constituent particle. The key factor that determines our space S is the energy-level structure and its degeneracy, exact or approximate, especially for the low-lying excitations.

The method of infinite-component equations, considered in the above sense, is a rather phenomenological and simplified scheme of characterizing a complex system *in toto*, albeit in an approximate fashion but without missing its essential features. It may also be regarded as a way of realizing the algebra of currents (observables) that follows from the conventional field theory. In fact the group G and space S may be chosen in such a way as to satisfy the algebra. On the other hand, it is conceivable that the wave equation has more dynamical contents (since it specifies not only commutators, but also anticommutators, etc., which depends on a specific representation), and yet at the same time may satisfy the current algebra only approximately (since we imbed observables in an algebra of finite dimensions and take only a few irreducible representations).

One of the deficiencies of the wave-equation method is the lack of a clear-cut principle for choosing an equation, except that it is expected to produce at least a qualitatively correct level scheme. Besides, we do not know yet whether such an equation can be realized in general as a field theory, or merely as an S-matrix theory. One may perhaps conjecture that it will fall somewhere in between, in the sense that a Lagrangian formalism can be set up and quasilocal interactions with external as well as among infinite-component fields introduced according to a simple prescription, which can reproduce complicated processes sufficiently well in Born approximation, but it may not quite qualify for being a complete and consistent field theory.

[23] To achieve the familiar equidistant spectrum, one needs a fourth- or third-order equation

$$[\Gamma_{\mu\nu}p^\mu p^\nu+\alpha p^2+\beta(p^2)^2]\Psi=0$$

or

$$[\Gamma_{\mu\nu}p^\mu p^\nu+\alpha p^2+\beta p^2\gamma_\mu p^\mu]\Psi=0.$$

They have spacelike (and discrete) solutions too, but no massless solutions. Other possibilities are under study.

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APPENDIX A: SOME MORE MODELS

Model No. 6

$$[\tau_3\Gamma_\mu p^\mu+S(x\gamma_\mu p^\mu+\tau_1\kappa)-\tau_3\{(x^2-1)^{1/2}\gamma_\mu p^\mu-\kappa\}\alpha]\Psi=0. \quad \text{(A1)}$$

This is a modification of Eq. (19). A new feature is that the spectrum is linear in m:

$$m=\frac{\kappa}{(x^2-1)^{1/2}}\frac{1-\alpha^2/N^2}{1+\alpha^2/N^2},\qquad \gamma_0,\ \tau_3=\pm1, \quad \text{(A2)}$$

for the discrete part, with the degeneracy m, m, m, m. The continuum extends over the ranges $|m|>\kappa/(x^2-1)^{1/2}=m_0$. In addition, there is an infinite degeneracy of all spins at $m=m_0$. The energy takes both signs.

Model No. 7

Modify Eq. (11) as

$$L\Psi=[\Gamma_\mu p^\mu+S(x\gamma_\mu p^\mu+\gamma_5N_5)-\alpha\eta\gamma_\mu p^\mu]\Psi=0 \quad \text{(A3)}$$

for the case $|N_5|=|k|=\text{even}$, where η is the operator with the properties

$$\begin{aligned}&a\to b,\quad b\to a,\\ &[\Gamma_\mu,\eta]=0,\quad \{S,\eta\}=0,\quad \{N_5,\eta\}=0,\\ &R\eta R^{-1}=(-1)^{N_5}\eta,\quad \eta^2=1.\end{aligned} \quad \text{(A4)}$$

When $|k|=\text{odd}$, replace η with $iN_5\eta$. Equation (21) may be diagonalized to give

$$\Gamma_0[\kappa^2-(x^2-1)m^2]^{1/2}\,\mathrm{sgn}(m)-\alpha\gamma_0 m=0$$

or

$$\begin{aligned}&m=\pm\kappa/(x^2-1+\alpha^2/N^2)^{1/2},\\ &\gamma_0\eta=1.\end{aligned} \quad \text{(A5)}$$

Thus we have a fourfold solution m, m, $-m$, $-m$ corresponding to $\eta=\gamma_0=\pm1$. The continuum part now extends above $\kappa/(x^2-1)^{1/2}$ only, though there still exist imaginary solutions. Equation (24) resembles very closely the solution of the relativistic Coulomb problem, except that we do not have spin-orbit splitting. The energy again remains indefinite.

For the case $k=0$, an analogous equation takes the form

$$L\Psi=[\tau_3(\Gamma_\mu p^\mu-\alpha\gamma_\mu p^\mu)+S(x\gamma_\mu p^\mu+\kappa\tau_2]\Psi=0, \quad \text{(A6)}$$

with the same spectrum as Eq. (24), but with

$$\gamma_0=+1, \quad \tau_3=\pm 1.$$

An interesting point in this model is that it admits a Pauli-Gürsey group which accounts for the degeneracy structure. The Pauli-Gürsey transformation for Eq. (A6) is defined by

$$\begin{aligned} &\delta\Psi=u\Psi^\dagger, \quad \delta\Psi^\dagger=u^\dagger\Psi, \\ &u=e^{i\alpha}u_0, \quad (\alpha \text{ arbitrary}) \\ &u_0=\gamma_5\tau_2 G, \quad G=\tau_2 C_\gamma \mathfrak{C}, \\ &u^T=-u, \quad u^\dagger u=1. \end{aligned} \quad \text{(A7)}$$

Here C_γ is the charge conjugation ($\sim\rho_2\sigma_2$) in the Dirac space, and $\mathfrak{C}$ is its analog for the infinite representation:

$$\begin{aligned} &\mathfrak{C}=\exp(\tfrac{1}{2}\pi iN)\exp[\tfrac{1}{2}\pi i(a^\dagger\sigma_2 a-b^\dagger\sigma_2 b)], \\ &\mathfrak{C}^2=1, \quad \mathfrak{C}^T=(-1)^{N_5}\mathfrak{C}=\mathfrak{C}, \\ &\mathfrak{C}a\mathfrak{C}^{-1}=iCa, \quad \mathfrak{C}b\mathfrak{C}^{-1}=-iCb, \\ &\mathfrak{C}\Gamma_\mu{}^T\mathfrak{C}^{-1}=\Gamma_\mu, \quad \mathfrak{C}S^T\mathfrak{C}^{-1}=S. \end{aligned} \quad \text{(A8)}$$

The Lagrangian $\Psi^\dagger L\Psi$ remains invariant under (A7) if Ψ is quantized according to Fermi statistics, because $\delta\mathfrak{L}\sim\Psi uL\Psi+\text{H.c.}$ is a symmetric quadratic form and hence has to vanish. The ordinary gauge transformation and Eq. (26) form an $SU(2)$ group, as can be seen by writing $\chi=(\Psi,\Psi^+)$, and the three generators of this group as

$$\begin{pmatrix} i & \\ & -i\end{pmatrix}, \quad \begin{pmatrix} & u_0 \\ -u_0^{-1} & \end{pmatrix}, \quad \begin{pmatrix} & iu_0 \\ iu_0^{-1} & \end{pmatrix}. \quad \text{(A9)}$$

Similar transformations can be defined for Eq. (A3) with general k, if we maintain the conventional relation between spin and statistics. The symmetry group for the integral-spin case, however, is $SU(1,1)$ instead of $SU(2)$.

Model No. 8

This is a variation of model No. 5, without the Dirac space:

$$[\Gamma_\mu p^\mu\kappa+(S-\alpha)p_\mu p^\mu]\Psi=0. \quad \text{(A10)}$$

When $p_\mu p^\mu\neq 0$, this leads to

$$[\Gamma_0\kappa+(M-\alpha)m]\Psi=0,$$

or

$$m=\kappa\Big/\left(1+\frac{\alpha^2}{n^2}\right)^{1/2}, \quad \text{(A11)}$$

which is similar to the model No. 7 case. In addition, however, there is a massless family of solutions satisfying $\Gamma_\mu p^\mu\Psi=0$, as in model No. 5.

APPENDIX B: COMPUTATION OF FORM FACTORS

We show here a method of computing form factors which is based on Eqs. (4) and (27)–(35). We demonstrate it to derive Eq. (10) in model No. 2. The scalar form factor between the same levels is given by

$$\begin{aligned} F=(\Psi(p),\Psi(0))=(\Psi(0),e^{i\vartheta M_{03}}\Psi(0)), \\ \tanh\vartheta=v=p/E, \end{aligned} \quad \text{(B1)}$$

if the momentum is in the direction 3. Going to the 6-dimensional rest frame, (B1) further reduces to

$$\begin{aligned} &(\Psi_1 e^{-i\theta M_{04}}e^{i\vartheta M_{03}}e^{-i\theta M_{04}}\Psi_1) \\ &\quad=(\Psi_1\exp[i\vartheta(M_{03}\cosh\theta-M_{43}\sinh\theta)]\Psi_1), \\ &\qquad\tanh\theta=\kappa/m. \end{aligned} \quad \text{(B2)}$$

According to Eqs. (4), (27), (31), and (32), we can make the replacement

$$\begin{aligned} &2M_{03}=X_3{}^\dagger+X^3\to x_3+x_3D_\lambda D^\lambda-2x_\lambda D^\lambda D_3-2D_3, \\ &2M_{43}=2K_3\to 2(x_3D^0+x_0D^3). \end{aligned} \quad \text{(B3)}$$

The eigenfunction Ψ_1 is a polynomial $f(x_\mu)$. Let us consider instead the Laplace kernel $\exp[l^\mu x_\mu]$. When applied to it, we may replace (B3) by

$$\begin{aligned} &2M_{03}\to D_3+l_\lambda l^\lambda D_3-2l_3l_\lambda D^\lambda-2l_3, \\ &2M_{43}\to l^0D_3+l^3D_0, \end{aligned} \quad \text{(B4)}$$

acting on $\exp(l^\mu x_\mu)$. Since Eq. (B5) is linear in the derivatives, the exponential form (B2) may be computed explicitly by a suitable change of variables. By expanding the kernel in powers of l^μ before and after the operation, and comparing the coefficients, we find how a monomial x^n is transformed by the operation.

For the ground state, we need keep only the variables $l^0\equiv s$ and $l^3\equiv t$. Thus

$$\begin{aligned} &2M_{03}\cosh\theta-2M_{43}\sinh\theta \\ &\quad=\{(1+s^2+t^2)\cosh\theta+2s\sinh\theta\}D_t+2t\cosh\theta \\ &\qquad\qquad+\{2st\cosh\theta-2t\sinh\theta\}D_s, \\ &\quad=[\{(1+\xi^2)\cosh\theta+2\xi\sinh\theta\}D_\xi+\xi\cosh\theta] \\ &\qquad-[\{(1+\eta^2)\cosh\theta+2\eta\sinh\theta\}D_\eta+\eta\cosh\theta], \end{aligned} \quad \text{(B5)}$$

where $s+t=\xi$, $s-t=\eta$. A further transformation turns this into

$$\begin{aligned} &[D_p-(\cot p+\sinh\theta)]-[D_q-(\cot q+\sinh\theta)] \\ &\quad=\exp[\ln(\sin p\sin q)-(p+q)\sinh\theta](D_p-D_q) \\ &\qquad\times\exp[-\ln(\sin p\sin q)+(p+q)\sinh\theta], \end{aligned}$$

$$\begin{aligned} &p=\frac{1}{2i}\ln[(\xi-\xi_1)/(\xi-\xi_2)], \quad q=\frac{1}{2i}\ln[(\eta-\eta_1)/(\eta-\eta_2)], \\ &\xi_{1,2}=\eta_{1,2}=-\tanh\theta\pm i\operatorname{sech}\theta. \end{aligned} \quad \text{(B6)}$$

The expectation value (B2) is then

$$\exp[i\vartheta(M_{03}\cosh\theta-M_{43}\sinh\theta)]\exp[l^\mu x_\mu]_{l=0,x=0}$$
$$\rightarrow\exp[\ln(\sin p\,\sin q)-(p+q)\sinh\theta]$$
$$\times\exp\left[\tfrac{1}{2}i\vartheta\left(\frac{\partial}{\partial p}-\frac{\partial}{\partial q}\right)\right]$$
$$\times\exp[-\ln(\sin p\,\sin q)+(p+q)\sinh\theta]$$
$$=\exp[\ln(\sin p\,\sin q)-\ln(\sin(p+\tfrac{1}{2}i\vartheta)\,\sin(q-\tfrac{1}{2}i\vartheta))]\quad\text{at}\quad p=q=a,$$

$$a=\frac{1}{2i}\ln(\xi_1/\xi_2)=\tfrac{1}{2}\ln\frac{\sinh\theta-i}{\sinh\theta+i},$$

which is equal to

$$\frac{1}{\cosh^2\theta\,\sinh^2(\vartheta/2)+1}=\frac{1}{1-t/4(m^2-\kappa^2)}.\tag{B7}$$

Similarly, the vector form factor $(\Psi(p),\Gamma_\mu\Psi(0))$ may be calculated, and yields the results (10) of Sec. II.

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Classical Theory of Magnetic Charge*

TUNG-MOW YAN†
Department of Physics, Harvard University, Cambridge, Massachusetts
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A classical theory of magnetic charge is formulated on the basis of an action principle. It is an extension of Schwinger's quantum theory of magnetic charge to the classical level. The action integral is defined by limiting procedure to ensure the equivalence of all singularity lines. The action principle gives correct equations of motion for the particles, Maxwell's equations, and the conservation laws of the Lorentz group. The consistency of the theory demands a charge-quantization condition, and the existence of a constant with the dimensions of action.

IN a recent paper, Rohrlich[1] has considered the problem of constructing a classical theory of magnetic charge. One of his conclusions is that no action integral exists from which both the particle equations and the field equations can be derived. This result thus casts severe doubt on the consistency of the theory of magnetic charge with the Lorentz group. However, Rohrlich's conclusion is based on the use of two independent vector potentials (A_μ and B_μ), which enlarges the number of degrees of freedom of the electromagnetic field and makes all the six components of the field strength $F_{\mu\nu}$ fundamental dynamical variables. In quantum field theory such a system violates physical positiveness requirements,[2] and it may well be that a similar situation holds at the classical level.[3] In the following we present a classical theory of magnetic charge in which no additional degree of freedom is introduced for the electromagnetic field; the vector potential B_μ is considered a given function of the field strength. All the equations of motion for the particles and the Maxwell field equations are derived from a nonlocal action integral. The ten conservation laws of the Lorentz group follow from the relativistic covariance of the theory which is satisfied by a limiting definition of the action integral. As we shall see, consistency of the theory requires a charge quantization condition and the existence of a constant with the dimension of action.

The idea involved is essentially an extension of Schwinger's quantum theory of magnetic charge.[4] We start with the tentative action integral[5]

$$W=\int(dx)\mathcal{L}_{\text{em}}(x)+\sum_{a=1}^{N}W_a+\sum_{b=1}^{*N}W_b,\tag{1}$$

$$\mathcal{L}_{\text{em}}=-\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu},\tag{2a}$$

$$W_a=\int ds_a\left[p_\mu\frac{dx^\mu}{ds}-\frac{1}{2m}(p-eA)^2\right]_a,\tag{2b}$$

$$W_b=\int ds_b\left[p_\mu\frac{dx^\mu}{ds}-\frac{1}{2m}(p-gB)^2\right]_b,\tag{2c}$$

where a and b are the labels of an electrically charged particle and a magnetically charged particle, respectively. The vector potential B_μ is a function of the

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[1] F. Rohrlich, Phys. Rev. **150**, 1104 (1966).
[2] J. Schwinger, Phys. Rev. **130**, 800 (1963).
[3] Footnote 16 of Ref. 1 seems to indicate that this is the case.
[4] J. Schwinger, Phys. Rev. **144**, 1087 (1966). A nonrelativistic quantum-particle theory has also been constructed by J. Schwinger (unpublished).
[5] We use units with $c=1$. Also, $-g_{00}=g_{11}=g_{22}=g_{33}=1$.