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Statistical Time Symmetry and Two-Time Boundary Conditions in Physics and Cosmology

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The hypothesis of complete time symmetry in oscillating cosmologies in which there exist locally irreversible processes is examined. A time-symmetric formalism for purely statistical processes is developed, and all aspects of quantum and statistical mechanics are shown to be time-symmetric within this framework. We conclude from analysis of a simple example that in a completely time-symmetric oscillating cosmology, statistical processes which produce entropy in the expanding phase will reverse themselves in the contracting phase, although microscopic reversal of motion need not occur. The analysis provides a self-consistent formalism for general problems involving the coexistence of different entropic directions of time in the observable universe. Boltzmann's H theorem is discussed in this framework.

I. INTRODUCTION

A TIME-HONORED problem of statistical mechanics and thermodynamics has been to explain the general validity of the second law of thermodynamics: If all microscopic equations of motion are invariant under time-inversion,[1,2] why do the vast majority of natural processes operate only in one time sense? Why does entropy always increase with time in these processes?

It is understood, of course, that if entropy changes in the same time sense everywhere, it is purely a matter of definition whether one says that entropy always increases or always decreases.[3] On the other hand, it is not trivial to inquire why the entropic or statistical directions of time can be chosen consistently to be the same over the observable universe. One can specify cosmologically a time axis, in the sense of setting up an ordinary cosmological coordinate system, and then compare the local statistical directions of time with this cosmological direction. Observationally, the result is well known: Either each local direction coincides with the cosmological direction, or each local direction is opposite to the cosmological direction.

A clear explanation of this coincidence has been given by Gold.[1] Gold advances the argument that the expansion of the universe with the cosmological red-shift allows space to swallow up electromagnetic radiation *ad libitum*. Hence, whenever radiation is formed during a physical process, the chances that it will escape without a compensating return from space are very good. Empty space is thus effectively a low-temperature thermal reservoir, and one might say that entropy increases result from temperature equalization with this cold body.

But there are other, more difficult questions, which are connected with the idea of opposing directions of time in different parts of space-time. For example, it is apparent that there is no difference between expanding and contracting open cosmological world models[3]; all statistical processes unwinding in the former wind back up again in the latter. But consider a closed, *oscillating* universe. If one postulates complete time symmetry, then the statistical direction of time in the contracting phase would have to be opposite to that in the expanding phase. Now suppose an observer were to survive the transition between the two directions of time through the moment of maximum expansion, perhaps by shutting himself in a vault, so that his own time sense remained unchanged. On emerging from the vault in the contracting phase, would he then be able to interfere with local "irreversible" processes and spoil their convergence down to states of lower entropy? Since such convergence is notoriously sensitive to very small perturbations, he might disrupt things to such an extent that the direction of time over large regions of the universe would be eventually changed!

Of more immediate interest is the discussion of time-symmetric oscillating cosmologies that have periods comparable with the normal relaxation times of certain otherwise irreversible processes: e.g., galactic evolution. If complete time symmetry is postulated (we emphasize that it is only a postulate), we would ask how to describe evolutionary processes close to the moment of maximum expansion. How are the rates of these statistical processes affected by the fore-knowledge that the direction of time must soon reverse itself? Since it is not known whether our own universe is oscillating or temporally open, these questions may be of more than academic interest.

We shall be led to formulate these problems in terms of "two-time statistical boundary conditions," and shall discuss closed statistical systems which begin in improbable, low entropy states and are somehow fated to end up in improbable states again after the passage of a specified period of time. Of course, in specifying the *final* states of closed or quasiclosed statistical systems by

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[1] T. Gold, Am. J. Phys. **30**, 403 (1962).

[2] However, there is some feeling that weak interactions may not be time-symmetric. See, for example, T. D. Lee and C. S. Wu, Ann. Rev. Nucl. Sci. **16**, 471 (1966). It is doubtful that any asymmetry of this sort would influence the macroscopic irreversibilities discussed here.

[3] D. L. Schumacher, Proc. Cambridge Phil. Soc. **60**, 575 (1964).

means of arbitrary boundary conditions, one sets up unphysical situations which are not met with in ordinary physics. When performing an experiment, one can consider systems which might have been prepared in a special way in the past (e.g., initially prepared in a low-entropy state), but never systems which are "certain" to finish up in an improbable, low-entropy state at a specified later time.

For example, one might take a box divided in half by a partition, fill one half of the box with a gas, and then jerk out the partition, performing no work on the gas in the process. The gas will then expand to fill the rest of the box. The occurrence of the reverse thermodynamic process would be extremely improbable, though dynamically possible. It is just such reverse processes that we wish to consider, and, more generally, such processes in which both initial and final states are "extremely improbable."

There are, however, experiments in which one considers ensembles of a great many statistical systems and then selects for study only those which have by chance ended up in highly improbable final states. This is a somewhat different concept, although the mathematical formalism is similar.

Aharonov, Bergmann, and Lebowitz[4] and Penfield[5] have considered ensembles of quantum systems which satisfy "selection" conditions in the past and future and show that quantum mechanics may be cast in both predictive and retrodictive formalisms. Aharonov, Bergmann, and Lebowitz show, in particular, that the quantum process of measurement is time-symmetric, provided that the eigenfunctions corresponding to the measured observables are themselves stationary states; i.e., that the observables measured are constants of the motion. We generalize their results to include measurement of observables that are not constants of the motion.

In Sec. II we introduce the idea of two-time statistical boundary conditions by considering the Ehrenfest urn model as an elementary application. We show in what sense the urn-model histories are altered by the requirement that the model both begin and end in very improbable states.

In Sec. III we develop the mathematical machinery for applying two-time statistical boundary conditions to systems undergoing transitions between states which can be described via Markov matrices. The scheme is truly time-symmetric only for a certain class of Markov chains. Necessary and sufficient conditions for time symmetry are exhibited which have an intuitively pleasing interpretation. The application of these ideas to thermodynamic systems, quantum dynamics, and the quantum measurement process is carried out. Complete quantum-mechanical time symmetry under very weak assumptions is demonstrated.

[4] Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, Phys. Rev. **134**, B1410 (1964).

[5] R. H. Penfield, Am. J. Phys. **34**, 422 (1966).

Conclusions about the behavior of statistical processes in a time-symmetric oscillating cosmology are then discussed in Sec. IV. We indicate that the two-time boundary-condition formalism provides a self-consistent way of treating the coexistence of different statistical directions of time. It also alleviates the "irretrodictability" of the usual probability concepts in physics, and thus casts further light on the paradox presented by Boltzmann's H theorem.

II. TWO-TIME BOUNDARY CONDITIONS AND THE EHRENFEST URN MODEL

We now study a particular statistical system which will give us some insight into the thermodynamic time development of physical systems, namely, the famous Ehrenfest urn model for temperature equalization.[6]

The Ehrenfest urn model consists of two urns, A and B, together with a collection of $2R$ balls, numbered from 1 to $2R$. Distribute all the balls in some arbitrary fashion in the two urns, and then choose at random an integer from 1 to $2R$ and move the corresponding ball from the urn in which it is found to the other one. Repeat this drawing process some desired number of times. We can expect that if there is initially a large excess in one of the urns, repeated drawings will result in the number of balls in the two urns becoming more or less equal.

We define the "states" of our system as follows: Let n_e be the *excess* number of balls in urn A over the *equilibrium* value R, i.e., $n_e \equiv$ (number of balls in A)$-R$. Of course, n_e might be negative. Then if $P_{n,n\pm1}$ is the conditional probability that $n_e=n\pm1$ after the $(m+1)$th draw, given that $n_e=n$ after the mth draw, we obviously have

$$P_{n,n\pm1}=\tfrac{1}{2}(1\mp n/R). \tag{1}$$

We call these conditional probabilities the transition probabilities for the transition $n\rightarrow n\pm1$. Of course, the probabilities for other single-step transitions are zero.

An interesting physical application of the Ehrenfest model is given by the one-dimensional Brownian motion of an elastically bound particle.[7] Our excess n_e is then interpreted as the displacement x of the particle away from the center of force.

Let us call a particular sequence of transitions $a\rightarrow b\rightarrow\cdots\rightarrow f$ a "history" of the urn model. The usual probability of such a history is then given by the expression $P(a)P_{ab}P_{bc}\cdots P_{ef}$, where $P(a)$ is the *a priori* probability that the initial state was $n_e=a$. We might call such a compound probability a probability with a "single-time" boundary condition.

We may generalize to two-time boundary conditions as follows: Consider all urn-model histories

[6] M. Kac, *Probability and Related Topics in Physical Sciences* (Interscience Publishers, Inc., New York, 1959), pp. 72–99.

[7] M. Kac, Am. Math. Monthly **43**, 369 (1947). Reprinted in *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover Publications, Inc., New York, 1954), pp. 295–317.

$j_0 \to j_1 \to \cdots \to j_s$ consisting of s transitions (s is considered fixed throughout the discussion), and construct an ensemble $E(a,f)$ consisting of all possible histories such that $j_0=a$ and $j_s=f$. The two-time statistical boundary conditions are then applied by assigning *arbitrary* probabilities $P(a,f)$ to each ensemble $E(a,f)$, with the restriction that $P(a,f)=0$ if the ensemble contains no possible histories; i.e., if $(P^s)_{af}=0$, where $(P^s)_{af}$ is the a, f element of the matrix P^s. Thus for our urn model we must have $P(a,f)=0$ if $|a-f|>s$ or if $a-f$ and s have different parity. Of course, we require $\sum_{i,j} P(i,j)=1$.

Having allocated the probabilities $P(a,f)$, we determine the absolute probabilities $P(a,b,\cdots,f)$ of the elementary histories by setting

$$P(a,b,\cdots,f)=P(a,f)P_{ab}P_{bc}\cdots P_{ef}/(P^s)_{af}, \quad (2)$$

if $(P^s)_{af}\neq 0$, or $P(a,b,\cdots,f)=0$ otherwise. In the usual theory, $P_{ab}P_{bc}\cdots P_{ef}/(P^s)_{af}$ would be the conditional probability that the history $ab\cdots f$ occurred, given that the initial state was a and the final state was f; i.e., given that the history was chosen to begin with from $E(a,f)$.

In general, the use of two-time boundary conditions changes the entire probability metric on the transition histories. In particular, $P_{n,n\pm1}$ is no longer the conditional probability that $n_e=n\pm1$ after the $(m+1)$th draw, given that $n_e=n$ after the mth draw. This probability is easily seen to be

$$\sum P(j_0,\cdots,j_{m-1},n,n\pm1,j_{m+2},\cdots,j_s)/\sum P(j_0,\cdots,j_{m-1},n,j_{m+1},\cdots,j_s)$$

by definition of conditional probability, where the summands are given by Eq. (2), and the sums are taken over all the variables except n and $n\pm1$.

Note that if we set $P(a,f)=P(a)(P^s)_{af}$, where $P(a)$ is an arbitrary initial probability, we recover the usual theory back again.

The formalism manifested by Eq. (2) appears time-symmetric in that initial and final states are equally represented. The question of detailed statistical time symmetry for general Markov matrices P_{kj} is discussed in the next section, where we show under what circumstances the probabilities $P(a,b,\cdots,f)$ given by Eq. (2) are invariant under the reflection $ab\cdots f\to f\cdots ba$, provided that $P(a,f)\to P(f,a)$ simultaneously. We will show that the Ehrenfest model is time-symmetric in this sense.

To illustrate the way in which the two-time boundary conditions influence the time development of statistical systems, we now apply the concept to our model. A simple sort of statistical boundary condition would be to let the initial excess $n_e=a$ be some value fairly close to R (i.e., the initial state is far removed from equilibrium) and to require that after s transitions the system be back in the same state $n_e=a$. Thus, $P(i,j)=\delta_{ai}\delta_{aj}$.

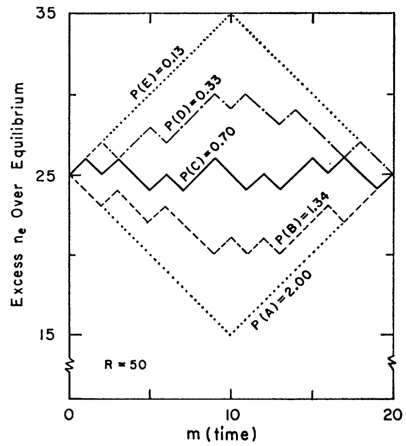

FIG. 1. A plot of the urn-model histories A, B, C, D, and E, labeled with their relative probabilities. The abscissa m is the number of the draw (the time), and the ordinate n_e is the excess in one of the urns over the equilibrium occupation number $R=50$.

We then consider all possible histories for which these boundary conditions are satisfied, i.e., all integer sequences $j_0, j_1, \cdots, j_s$ such that $j_0=j_s=a$, $j_k=j_{k+1}\pm1$ and $|j_k|\leqslant R$. One might then proceed to find the probabilities $P(j_0,j_1,\cdots,j_s)$ as defined by Eq. (2) for the various histories, so that some idea might be formed of the most probable type of history. However, a general analysis of the problem of finding normalized probabilities even for this simple boundary condition is very complicated, and we cannot go into it here.

Nevertheless, we have computed a few relative probabilities [omitting the normalizing denominator in Eq. (2)] for the case $R=50$, $a=25$, $s=20$. With reference to the histories A, B, C, D, and E plotted on Fig. 1, we have obtained $P(A)=2.00$, $P(B)=1.34$, $P(C)=0.70$, $P(D)=0.33$, and $P(E)=0.13$.

Obviously, A is the most probable trajectory. Still, there are many more paths that look somewhat like type B. Thus, the most probable general type or "macropath" (thermodynamic path) is something like type B. The indication is that instead of proceeding more or less directly downward toward equilibrium, the probable behavior of the system during the first $s/2$ transitions is to remain generally in less probable states than would ordinarily be the case. One might say that the system exhibits a statistical "pre-effect" because of the final boundary condition. This observation will allow us to make plausible inferences about the behavior of statistical processes in time-symmetric oscillating cosmologies, which we discuss in Sec. V.

III. STATISTICAL TIME SYMMETRY FOR THERMODYNAMIC AND QUANTUM SYSTEMS

The main point of this paper is to develop a formalism for describing physical statistical processes in a time-symmetric fashion. Hence, we wish to show under what conditions the history probabilities defined by Eq. (2) for general statistical process are invariant under time inversion. Of course, Eq. (2) may be applied to any Markov chain, but it will be seen that only certain types of Markov chains are time-symmetric.

The analogy between a Markov chain and a classical thermodynamic system is fairly evident. If the system is quasiclosed, its phase point will perform a random walk in phase space. The number of states that such a system can assume, characterized, for example, by the instantaneous value of its total energy, is uncountably infinite, and the transitions occur continuously instead of in discrete jumps. Still, the motion of the phase point is Markovian since, if we know the position $P(t_0)$ of the phase point at a time t_0, the probability distribution of P at times $t>t_0$ depends only on $P(t_0)$ and not on P at times previous to t_0. The statistical–boundary-condition formalism could easily be extended to continuous distribution functions.

It is easy to construct artificially the two-time ensembles in terms of the ensembles of statistical mechanics. The collections $E(a,f)$ are then defined by observing a very large number of dynamically identical thermodynamic systems and picking out certain proportions $[\propto P(a,f)]$ of those systems according to initial state a and final state f, without regard to the histories in between.

In the usual theory, it can be objected that this is a rather high-handed way to construct an ensemble. It is more in the spirit of probability theory simply to specify an initial probability distribution and then work out the final distribution in the usual way. In our case the contrary course of action is motivated by the fact that the equations of motion of physics are all invariant under time inversion, so that the distinction between initial and final becomes purely conventional from the microscopic standpoint. We now prove that also for a rather broad class of Markov chains, including the Ehrenfest urn model, there is no distinction between initial and final states (past and future) if the two-time boundary condition concept is used.

Consider a general statistical system characterized by denumberable states j, such that the base probability for the transition $j\rightarrow k$ is P_{jk}. Of course, $\sum_k P_{jk}=1$. We treat only the case where P_{jk} does not depend explicitly on the time. Then having assigned a number s and boundary condition pair probabilities $P(i,j)$, the probability for a history $ab\cdots f$ is given by Eq. (2).

Time symmetry is investigated by requiring that the probability of the history $ab\cdots f$ be the same as the *reversed* history $f\cdots ba$ calculated for ensembles in which the boundary condition pair probabilities are also reversed. Let $\bar{P}(f,\cdots,b,a)$ be this reversed probability and $\bar{P}(i,j)=P(j,i)$ be the reverse boundary condition probabilities. Time symmetry then requires

$$P(a,b,\cdots,f)=\bar{P}(f,\cdots,b,a),$$

or

$$\begin{aligned}P(a,f)P_{ab}P_{bc}\cdots P_{ef}/(P^s)_{af}&=\bar{P}(f,a)P_{fe}\cdots P_{cb}P_{ba}/(P^s)_{fa}\\&=P(a,f)P_{fe}\cdots P_{cb}P_{ba}/(P^s)_{fa},\end{aligned}$$

or

$$P_{ab}P_{bc}\cdots P_{ef}(P^s)_{fa}=P_{fe}\cdots P_{cb}P_{ba}(P^s)_{af}. \quad (3)$$

In general, we may require this for all values of s. Obviously, this will not hold for an arbitrary transition matrix P_{kj}, and thus not all Markov chains are time-symmetric in this sense.

An interesting necessary and sufficient condition for Eq. (3) to hold for all s is that

$$P_{ab}P_{bc}\cdots P_{de}P_{ea}=P_{ae}P_{ed}\cdots P_{cb}P_{ba} \quad (4)$$

for all histories $ab\cdots ea$ that begin and end in the same state. This relation may be obtained from Eq. (3) by setting $f=a$; and, conversely, one may take Eq. (4) for $2s$ factors and then sum over $s-1$ neighboring pairs of indices to obtain Eq. (3).

A rather general sufficient condition for Eqs. (3) and (4) to hold is that there exist a set of numbers v_a, none of which are zero, such that

$$v_aP_{ab}=v_bP_{ba}. \quad (5)$$

That this is *necessary* for time symmetry is not clear.

Equation (5) has a satisfying interpretation. Summing over a, we obtain $\sum_a v_aP_{ab}=v_b$. This relation means that if v_a is taken as an initial probability distribution, the absolute probabilities remain independent of the number of transitions made, i.e., v_a is "stationary" under P_{ab}. With this interpretation of v_a, Eq. (5) means that the *absolute* probability of the occurrence of state a followed by a transition to state b is the same as for the occurrence of b followed by a. This is the usual definition of reversibility in Markov chain theory,[8] and thus any reversible chain is also time-symmetric in our sense.

The Ehrenfest urn model may easily be shown to be time-symmetric. In fact, its matrix, given by Eq. (1), satisfies Eq. (5) with

$$v_a=\binom{2R}{R+a},$$

where

$$\binom{m}{n}$$

is the binominal coefficient.

To show the reversibility of thermodynamic systems within the scheme of classical and quantum-statistical mechanics is now trivial. We have only to point out that for both the microcanonical and the Gibbs distribution the probabilities of the different states a are independent of any prior measurement at all, so that we

[8] J. G. Kemeny and J. L. Snell, *Finite Markov Chains* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1960), p. 105.

have $P_{ab}=w_b$. Equation (5) is then satisfied with $v_a=w_a$.

In exact classical and quantum mechanics, reversibility also depends on changing the signs of all the momenta. For thermodynamic states we do not have to deal with this since the quantities total energy, volume, etc., are invariant under momentum reflection. However, in discussing quantum-mechanical time symmetry, we will have to consider the reverse of the history $ab\cdots f$ to be $\bar{f}\cdots\bar{b}\bar{a}$, where $\bar{a}$ is the time-reversed analog of the quantum state a. This will be but a slight complication in our formalism.

It is well known that the quantum-mechanical equations of motion are time-symmetric.[1,2] However, the role of the quantum measurement process has long been a subject of debate, and many authors have maintained that the discontinuous change of the wave function upon measurement is a source of time asymmetry in the real world. It is not our purpose here to discuss all the ramifications of this question, and the reader is referred to the articles of Aharonov, Bergmann, and Lebowitz[4] and Penfield[5] cited before, as well as the recent review article by Bohm and Bub.[9]

We now demonstrate that all aspects of quantum mechanics fit naturally into the two-time boundary-condition formalism, provided that we perform the quantum-mechanical analog of momentum reversal when requiring time symmetry. Aharonov, Bergmann, and Lebowitz[4] have treated this same topic, but their discussion was limited to measurement of observables which were constants of the motion, i.e., to stationary states. We present the generalization of their result to include transitions between nonstationary states.

We represent time inversion by the usual antiunitary operator Θ on the quantum-state Hilbert space, so that $\Theta|a\rangle\equiv|\bar{a}\rangle$ is the time-reversed analog of $|a\rangle$.[10] The antiunitary property means that

$$\langle\Theta b|\Theta a\rangle\equiv\langle\bar{b}|\bar{a}\rangle=\langle b|a\rangle^*=\langle a|b\rangle \tag{6}$$

for any two states $|a\rangle$ and $|b\rangle$. We may complete the definition by requiring that Θ anticommute with the linear and angular-momentum operators for each particle in the state:

$$\Theta\mathbf{p}+\mathbf{p}\Theta=\Theta\mathbf{J}+\mathbf{J}\Theta=0. \tag{7}$$

Equations (6) and (7) imply that the expection values of $\mathbf{p}$ and $\mathbf{J}$ reverse on time inversion: $\langle\bar{a}|\mathbf{p}|\bar{a}\rangle=\langle\Theta a|\mathbf{p}\Theta|a\rangle$ $=-\langle\Theta a|\Theta\mathbf{p}|a\rangle=-\langle a|\mathbf{p}^\dagger|a\rangle=-\langle a|\mathbf{p}|a\rangle$, etc. For an ordinary nonrelativistic one-component Schrödinger wave function $\psi(\mathbf{r})$ one can show that time inversion amounts to taking the complex conjugate.[11]

All Hamiltonian operators used in quantum mechanics, except for those containing an external magnetic field, commute with Θ: $H\Theta=\Theta H$. With this assumption, it is very simple to demonstrate the general symmetry of the quantum-measurement transition probabilities.[12] Suppose that a complete measurement is made on a quantum system and its state determined to be $|a\rangle$. Then at a time Δt later its quantum state is given by the unitary time-development operator[13] $T(\Delta t)$ $=\exp(iH\Delta t/\hbar)$ to be $|a,\Delta t\rangle=T(\Delta t)|a\rangle$. Since H commutes with Θ, and since Eq. (6) implies that $i\Theta=-\Theta i$, we have

$$T(\Delta t)\Theta=\Theta T(-\Delta t)=\Theta T(\Delta t)^\dagger. \tag{8}$$

The probability for determining the system to be in the state $|b\rangle$ after time Δt is then $P_{ab}=|\langle b|T(\Delta t)|a\rangle|^2$. The time-reversed analog of $a\rightarrow b$ is $\bar{b}\rightarrow\bar{a}$, and the probability for this transition over the same time interval $\Delta t\geqslant 0$ is given by $P_{\bar{b}\bar{a}}=|\langle\bar{a}|T(\Delta t)|\bar{b}\rangle|^2$. But Eqs. (6) and (8) imply that $\langle\bar{a}|T(\Delta t)|\bar{b}\rangle=\langle\Theta a|\Theta T(\Delta t)^\dagger|b\rangle$ $=\langle a|T(\Delta t)^\dagger|b\rangle^*=\langle b|T(\Delta t)|a\rangle$.

Therefore, we have the elementary result $P_{ab}=P_{\bar{b}\bar{a}}$ for the quantum-measurement transition probabilities. In order to fit this into the two-time boundary-condition formalism we have only to make the obvious generalization that the inverse of a history $ab\cdots f$ is not $f\cdots ba$, but rather $\bar{f}\cdots\bar{b}\bar{a}$. Time symmetry is then investigated by asking whether or not $P(a,b,\cdots,f)=\bar{P}(\bar{f},\cdots,\bar{b},\bar{a})$, with the definition of Eq. (2). But our relation $P_{ab}=P_{\bar{b}\bar{a}}$ obviously guarantees that this symmetry condition is fulfilled, and hence the quantum measurement process is completely time symmetric.

IV. STATISTICAL PROCESSES IN OSCILLATING COSMOLOGIES AND A COMMENT ON BOLTZMANN'S *H* THEOREM

Now that we have established the time symmetry of statistical and quantum mechanics, we can discuss in more detail the behavior of general statistical processes in time-symmetric oscillating cosmologies.

Let us choose, for example, a closed Friedmann model with zero cosmological constant, so that the model starts from a singular state, expands out to a certain maximum radius of curvature, and then contracts down to a singular state again. To avoid discussing the behavior at the singularities, we choose the following symmetrical statistical boundary conditions: ΔT seconds after the initial singularity all the matter energy of the universe is to be in the form of uniformly distributed primordial matter of some appropriate composition. Similarly, ΔT sec before the final singularity the *same* state, with the same composition, density, and temperature, is to occur. We may, of course, take ΔT as close to zero as we wish.

Our study of the Ehrenfest model in Sec. II indicates that ordinary galactic and stellar evolution processes

[9] D. Bohm and J. Bub, Rev. Mod. Phys. 38, 453 (1966).

[10] F. A. Kaempffer, *Concepts in Quantum Mechanics* (Academic Press Inc., New York, 1965), Sec. 15.

[11] R. C. Tolman, *The Principles of Statistical Mechanics* (Oxford University Press, London, 1959), pp. 395–398.

[12] For a version of this proof in *S*-matrix theory, see Ref. 10, pp. 255–258.

[13] E. Merzbacher, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1962), pp. 330–332.

will operate during most of the expanding phase. But we must also conclude that in the contracting phase the same processes will also operate, but in reverse. Observers living in the contracting phase would have their time-senses opposite to those in the expanding phase and would conclude that they live in the expanding phase themselves. And each would be correct. There exists no criterion by which one can decide which half is really the expanding phase. Thus, complete time-symmetry is attained. Furthermore, if the calculations for the Ehrenfest model are any indication, we would expect the galactic histories to be predominantly of type B, and thus the statistical "pre-effect" of the final boundary condition would slow down the rates of entropy-producing processes in both phases.

One might also use the two-time formalism to analyze specific problems involving the appearance of observers in regions having a time sense opposite from their own, and the like, as examined, for example, by Wiener.[14] Unfortunately, this would be exceedingly complicated even for the simplest situations, but we may at least claim to have found a self-consistent theoretical framework for such problems. The time symmetry demonstrated in Sec. III for quantum processes shows that the calculated history probabilities for such problems do not depend on the choice of the direction of the time axis.

Watanabe[15] has discussed the fact that the very notion of probability in statistical and quantum physics introduces a past-future asymmetry ("irretrodictability") into the conceptual framework. This asymmetry is indeed a characteristic of ordinary probability theory, but the use of the two-time boundary-condition concept removes this defect for systems whose base transitions probabilities satisfy the symmetry condition of Eqs. (3) and (4).

We can now comment intelligently on the famous controversy which Boltzmann's H theorem raised when it appeared long ago: Boltzmann showed that the collision term in the kinetic equation had the effect that[16,17]

$$dH/dt \equiv (d/dt) \int d\mathbf{v}\, f(\mathbf{v},t) \ln f(\mathbf{v},t) \leq 0\,,$$

with equality only for a Maxwellian distribution. Thus the H theorem singles out a time direction and contradicts microscopic reversibility. The reason for this is, of course, that the collision term is constructed by means of a scattering cross section, which is a type of transition probability and no longer a purely dynamical concept. Thus the H theorem is in line with Watanabe's results. It is noteworthy that the kinetic equation without the collision term is time-symmetric.

The two-time boundary condition concept could be used to construct a time-symmetric replacement for the kinetic equation including the collision term. This could also be done for other statistical transport equations, such as the heat conduction equation and the viscosity terms in the Navier-Stokes equation.

We here conclude our discussion of statistical boundary conditions. Many points still remain obscure, including the interesting question of the rates of statistical processes in oscillating cosmologies.

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[14] N. Wiener, *Cybernetics* (M.I.T. Press, New York, 1961), 2nd ed., pp. 30–36.
[15] S. Watanabe, Progr. Theoret. Phys. (Kyoto) Suppl., Extra Number, p. 135 (1965).
[16] C. L. Longmire, *Elementary Plasma Physics* (Interscience Publishers, Inc., New York, 1963), pp. 184–192.
[17] M. Kac, Ref. 6, pp. 59–62.