Transition Form Factors in the H Atom*

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Transition form factors between arbitrary excited states of the H atom have been evaluated in closed highly symmetric form within the framework of the noncompact dynamical group $O(4,2)$.

THE purpose of this paper is to present the exact form of the charge form factors in the H atom for transitions between any two excited states. The motivations for this work are: (1) These arbitrary transition form factors have, to our knowledge, not been given in the literature; (2) to show the power of the new simple algebraic methods using the representations of noncompact groups; and (3) for possible adaptation of the results to the dynamics of strongly interacting particles.

The form factors in question, denoted by $\mathfrak{F}_{n'l'm',nlm}{}^{\mu}\times(q^2)$, are the vertex amplitudes shown in Fig. 1 as a function of the momentum transfer $q^2=t=k^2+k'^2-2kk'\cos\theta$. They govern the inelastic scattering of the H atom by other charged particles or atoms if single photon exchange is dominant, and are measured by such scattering experiments. The transition form factors from the ground state $|100\rangle$ to an arbitrary state $|nlm\rangle$ were first calculated by Massey and Mohr[1] by Schrödinger theory. To our knowledge, these are the only form factors known explicitly. We present here an evaluation of arbitrary form factors solely within the conformal group $O(4,2)$. The method does not make any reference to spatial wave functions.

It has been shown recently that the dipole transitions in the H atom can be described in a simple manner by using the dynamical group $O(4,2)$, the conformal group.[2] Nambu[3] has investigated relativistic infinite-component wave equations for H-like systems and has indicated the calculation of form factors. Later, further properties of the H atom within the group $O(4,2)$ were investigated by Fronsdal[4] and the present authors.[5] Fronsdal gave also the form of the Galilei booster transformations on the group $O(4,2)$, and evaluated the form factor of the ground state.

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[1] H. S. W. Massey and C. B. O. Mohr, Proc. Roy. Soc. (London) A132, 605 (1931). See also H. S. W. Massey, in *Handbuch der Physik*, edited by S. Flügge (Springer Verlag, Berlin, 1956), Vol. 34.

[2] A. O. Barut and H. Kleinert, Phys. Rev. **156**, 1541 (1967). The relevance of the group $O(4,2)\sim SU(2,2)$ to the H atom, beyond the minimal dynamical group $O(4,1)$, was also noticed by I. A. Malkin and V. I. Man'ko, JETP Pis'ma v Redaktsiyn **2**, 230 (1966) [English transl.: JETP Letters **3**, 146 (1966)], but these authors did not consider dipole operators.

[3] Y. Nambu, Progr. Theoret. Phys. (Kyoto) Suppl. **37**, 368 (1966).

[4] C. Fronsdal, Phys. Rev. **156**, 1665 (1967).

[5] A. O. Barut and H. Kleinert, Phys. Rev. **157**, 1180 (1967).

We summarize briefly the $O(4,2)$ description of the H atom. Let $L_{ab}=-L_{ba}$; $a, b=1, 2, \cdots 6$, be the 16 generators of $O(4,2)$. The subgroup $O(4)$ generated by L_{ab}; $a, b=1, 2, 3, 4$, describes the degeneracy of the states of a given energy; the subgroup $O(4,1)$—dynamical group in the rest frame—describes *all* bound states $|nlm\rangle$, and, finally, the remaining generators L_{i6} are associated with dipole transitions, and L_{56} with the quantum number n.

The vector form factors are given by

$$\mathfrak{F}_{n'l'm',nlm}=\langle n'l'm'|nlm,k\rangle\,, \qquad (1)$$

where $|nlm,k\rangle$ is the Galilei-boosted state of momentum

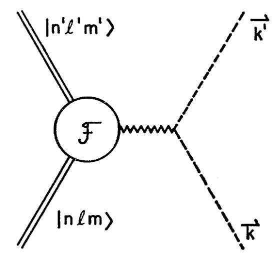

FIG. 1. The transition form factor $\mathfrak{F}$ is measured in an inelastic collision with another system of momentum k when one-photon exchange is dominant.

k, i.e., $e^{ikx}\psi_{nlm}(x)$. The generators M_i of the Galilei transformations $\exp(i\mathbf{k}\cdot\mathbf{M})$ are given by[4]

$$M_i=(L_{i5}-L_{i4})\,, \qquad (2)$$

provided we introduce the new states[5]

$$|\bar{n}lm\rangle=\frac{1}{n}e^{-i\theta_nL_{45}}|nlm\rangle,\quad \theta_n=\ln n\,, \qquad (3)$$

and a current operator $\Gamma_\mu=(L_{56}-L_{46}, L_{i6})$. Then the charge form factors can be written as (for a booster in the 3 direction)

$$\mathfrak{F}_{n',n}{}^{lm}=\langle\bar{n}'lm|\Gamma_0e^{-ikM_3}|\bar{n}lm\rangle\,. \qquad (4)$$

Using Eqs. (2) and (3) and the commutation relations of $O(4,2)$ we can bring Eq. (4) to the form

$$\mathfrak{F}_{n',n}{}^{lm}=\frac{1}{n}\langle n'lm|\Gamma_0|n''lm\rangle \times\langle n''lm|e^{-i\vartheta_{n'n}L_{45}}e^{-ink(L_{34}-L_{35})}|nlm\rangle\,, \qquad (5)$$

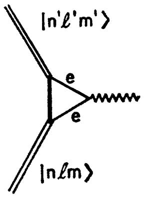

FIG. 2. The triangular diagram which gives the anomalous threshold obtained in Eq. (17).

where

$$\tanh\tfrac{1}{2}\vartheta_{n'n}=\frac{n-n'}{n+n'};\quad \vartheta_{n'n}=\ln(n/n').$$

In the evaluation of Eq. (5) we notice that $L_{34}=K_3$, $L_{35}=-K_2$, $L_{45}=K_1$ generate an $O(2,1)$ subalgebra (transition group[5] K). The second matrix element in Eq. (5) is that of a finite transformation of $O(2,1)$ that we express in terms of the Euler angles φ, ψ, χ:

$$e^{-i\vartheta_{n'n}K_1}e^{-ikn(K_3+K_2)}=e^{-i\varphi K_3/2}e^{-i\chi K_2/2}e^{-i\psi K_3/2},\quad (6)$$

and obtain

$$\sinh\tfrac{1}{2}\chi=\frac{1}{(4n'n)^{1/2}}[(n-n')^2+tn^2n'^2]^{1/2},$$
$$\sin\varphi=\frac{(n'^2-n^2)+tn'^2n^2}{\{[(n'-n)^2+tn'^2n^2][(n'+n)^2+tn'^2n^2]\}^{1/2}};\quad (7)$$

$\sin\psi$ is exactly like $\sin\varphi$ with n and n' interchanged. Next we express the operators K in terms of the generators corresponding to parabolic coordinates[2]

$$[N_i^+,N_i^-]=-2N_i^3,\quad [N_i^3,N_i^\pm]=\pm N_i^\pm,\quad i=1,2\quad (8)$$

where

$$N_i^3|n_1n_2m\rangle=[n_i+(m+1)/2]|n_1n_2m\rangle;$$
$$N_1^\pm|n_1n_2m\rangle=-[(n_1+\tfrac{1}{2}\pm\tfrac{1}{2})(n_1+m+\tfrac{1}{2}\pm\tfrac{1}{2})]^{1/2}|n_1\pm1,n_2m\rangle\quad (9)$$

as follows:

$$K_3=N_1^3-N_2^3;\quad K_1=\frac{1}{2i}(N_1^++N_2^+-N_1^--N_2^-);$$
$$K_2=-\tfrac{1}{2}(N_1^+-N_2^++N_1^--N_2^-).\quad (10)$$

It is therefore easy to evaluate the matrix elements of Eq. (6) in parabolic coordinates:

$$\langle n_1''n_2''m|e^{-i\varphi K_3}e^{-i\chi K_2}e^{-i\psi K_3}|n_1n_2m\rangle$$
$$=e^{-i(n_1''-n_2'')\varphi}e^{-i(n_1-n_2)\psi}\langle n_1''n_2''m|$$
$$\times e^{-i(\chi/2)(N_1^++N_1^-)}e^{-i(\chi/2)(N_2^++N_2^-)}|n_1n_2m\rangle,\quad (11)$$

where the last matrix element is the product of two finite $O(2,1)$ transformations:

$$G_{n_1''n_2''n_1n_2}{}^m\equiv V_{n_1''+(m+1)/2,n_1+(m+1)/2}{}^{(m+1)/2}(\chi)\times V_{n_2''+(m+1)/2,n_2+(m+1)/2}{}^{(m+1)/2}(\chi).\quad (12)$$

The V function for $n_1'>n_1$ is given by

$$V_{n_1'+(m+1)/2,n_1+(m+1)/2}{}^{(m+1)/2}(\chi)$$
$$=\theta_{n_1'n_1}(\cosh\tfrac{1}{2}\chi)^{-(n_1'+n_1+m+1)}(-i\sinh\tfrac{1}{2}\chi)^{n_1'-n_1}$$
$$\times F(-n_1,-n_1-m,1+n_1'-n_1,-\sinh^2(\tfrac{1}{2}\chi)),\quad (13)$$
$$\theta_{n_1'n_1}=\frac{1}{(n_1'-n_1)!}\left[\frac{n_1'!(n_1'+m)!}{n_1!(n_1+m)!}\right]^{1/2}$$

[for $n_1'<n_1$, use $V_{n_1+(m+1)/2,n_1'+(m+1)/2}{}^{(m+1)/2}(\chi)$], and occurs universally in all form-factor calculations (scalar or vector) and in the approximate evaluation of scattering amplitudes.[5–7]

Similarly, the first matrix element in Eq. (5) is easily calculated in parabolic coordinates, because

$$L_{46}=\tfrac{1}{2}(N_1^++N_1^-+N_2^++N_2^-);\quad L_{56}=N.\quad (14)$$

It remains then to change the basis $|n_1n_2m\rangle$ into $|nlm\rangle$. Because this change of basis is connected with the reduction of $O(4)$ into $O(3)\times O(3)$, we have immediately in terms of the $3j$ symbols

$$\langle nlm|n_1n_2m\rangle=(-1)^m(2l+1)^{1/2}\times\begin{pmatrix}(n-1)/2 & (n-1)/2 & l\\ (m-n_1+n_2)/2 & (m+n_1-n_2)/2 & -m\end{pmatrix}.\quad (15)$$

Consequently, collecting all the terms the final result is

$$\mathfrak{F}_{n'n}{}^{lm}=\frac{(2l+1)}{n}\sum_{n_1'n_2',n_1n_2}\begin{pmatrix}(n'-1)/2 & (n'-1)/2 & l\\ (m-n_1'+n_2')/2 & (m+n_1'-n_2')/2 & -m\end{pmatrix}\begin{pmatrix}(n-1)/2 & (n-1)/2 & l\\ (m-n_1+n_2)/2 & (m+n_1-n_2)/2 & -m\end{pmatrix}$$
$$\times\{n'e^{-i[(n_1'-n_2')\varphi+(n_1-n_2)\psi]}G_{n_1'n_2'n_1n_2}{}^m+[(n_1'+1)(n_1'+m+1)]^{1/2}h^+{}_{n_1'-n_2',n_1-n_2}{}^{n',n}G_{n_1'+1,n_2'n_1n_2}{}^m$$
$$+[n_1'(n_1'+m)]^{1/2}h^-{}_{n_1'-n_2',n_1-n_2}{}^{n',n}G_{n_1'-1,n_2'n_1n_2}{}^m\},\quad (16)$$

where

$$h^\pm{}_{s',s}{}^{n',n}=\cos[(s'\pm1)\varphi+s\psi]\quad\text{for}\quad(-1)^{n'-n}=-1$$
$$=-i\sin[(s'\pm1)\varphi+s\psi]\quad\text{for}\quad(-1)^{n'-n}=+1.$$

[6] A. O. Barut and H. Kleinert, Phys. Rev. **156**, 1546 (1967).
[7] A. O. Barut and H. Kleinert, Phys. Rev. Letters **18**, 754(1967).

The form factor $\mathfrak{F}$ has a singularity where the $\cosh\frac{1}{2}\chi$ term in Eq. (11) vanishes. From Eq. (7), this occurs at

$$t=t_1=-\frac{(n'+n)^2}{n^2n'^2}=2(\sqrt{B'}+\sqrt{B})^2 \tag{17}$$

and coincides exactly with the anomalous threshold singularity obtained from the triangular diagram shown in Fig. 2 (B=binding energy).

The final result, Eq. (16), reduces in the special case to the Massey and Mohr result[1] which now has been written in a highly symmetric form.

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Theory of Sequential Decays*

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The theory of sequential decays of an unstable system is studied. Examples include the sequential emission of two or more photons by an excited atom which reaches its ground state via one or more intermediate levels, and the decay of an unstable particle into other unstable particles. To describe these phenomena, a factorization of the Green's function is introduced. This leads to a simple, and intuitively obvious, description of sequential decays. It also makes possible an assessment of the accuracy of this description.

I. INTRODUCTION

We consider the quantum-mechanical description of a system undergoing a sequence of decays. An example of this is provided by the de-excitation of an atom radiating two or more photons in sequence. Other examples include the study of angular correlations in successive nuclear decays and the decay of an unstable daughter in particle physics.

Previous treatments of these phenomena have tended to be heuristic or have introduced approximations at the outset which have obscured many of the subtle features of sequential decays. In this paper we shall apply the Green's-function method used by Goldberger and Watson[1] for single-step decays to a general description of multistep decay processes. Somewhat related techniques have been used by Reff,[2] by Kroll,[3] and by Goldberger and Watson[4] for specific cases of two-step decays. An alternative formulation of the decay problem has been given recently by Mower.[5] His method treats as "closely coupled" all the states involved in a decay. This leads to the algebraic problem of inversion of a matrix whose dimensionality is the number of states considered. Our method takes account from the outset of the time ordering of sequential decays.

The value of the Green's-function approach lies in the fact that it gives a rigorous formulation of multistep decays in which the usual description by a product of Breit-Wigner resonance factors is a natural first approximation. This is not true of ordinary (e.g., Rayleigh-Schrödinger) perturbation methods. Correction terms depending on the ratio of level widths to level spacing may be estimated in a straightforward way. Qualitative statements about the time dependence of the decay may be obtained from the analytic behavior of the Green's function.[4]

We begin with a collection of some relevant results of the Goldberger-Watson[1] formulation of decay processes. A physical system is assumed to be described by a Hamiltonian H. This is written as $H=K+V$, where V is responsible for transitions between eigenstates of K. These eigenstates are written as g_a, g_b, $\cdots$ and satisfy the respective Schrödinger equations

$$Kg_a=\epsilon_a g_a\,,\quad Kg_b=\epsilon_b g_b\,,\cdots. \tag{1.1}$$

* This work was supported in part by the U. S. Atomic Energy Commission and in part by a grant from the Office of Scientific Research of the U. S. Air Force.

[1] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964), Chap. 8.

[2] The earliest use of these techniques seems to be that of I. Reff, Phys. Rev. **91**, 150 (1953).

[3] N. M. Kroll, in *Quantum Optics and Electronics*, edited by C. DeWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach Science Publishers, Inc., New York, 1964).

[4] Reference 1, p. 454.

[5] L. Mower, Phys. Rev. **142**, 799 (1966).