expansion in the u channel:

$$C^I(u,z(s))=\sum_{l=0}^{\infty} f_{l+}(u)P_{l+1}'(\cos\theta_u) - \sum_{l=2}^{\infty} f_{l-}(u)P_{l-1}'(\cos\theta_u)\,, \quad (24)$$

$$D^I(u,z(s))=\sum_{l=0}^{\infty} (f_{l-}-f_{l+})P_l(\cos\theta_u)\,, \quad (25)$$

where $f=(\eta_{l\pm}e^{2i\delta_{l\pm}}-1)/q_u$, $\eta_{l\pm}$ is the inelasticity factor with $0\leq\eta_{l\pm}\leq 1$, and

$$\operatorname{Im} f_{l\pm}(u)=\frac{1}{2}\int_{-1}^{1} dz[C_u{}^I(z)P_l(z)+D_u{}^I(z)P_{l\pm1}(z)]. \quad (26)$$

From this we see that $\operatorname{Im} f_{l\pm}$ vanishes along with $C_u{}^I$ and $D_u{}^I$,

$$\operatorname{Im} f_{l\pm}(u)=0 \quad \text{for all } l. \quad (27)$$

However, $\operatorname{Im} f_{l\pm}(u)$ are related to the modulus of $f_{l\pm}(u)$ by the partial-wave unitarity condition,

$$\operatorname{Im} f_{l\pm}(u)=q_u|f_{l\pm}(u)|^2+q_u(1-\eta_{l\pm}/4)\,, \quad (28)$$

where $q_u((1-\eta_{l\pm})/4)$ is the inelastic contribution to $\operatorname{Im} f_{l\pm}(u)$ in the u channel and is non-negative, so that

$$f_{l\pm}(u)=0 \quad \text{for all } l. \quad (29)$$

Putting this back into Eqs. (24) and (25), we see that C^I and D^I, and hence A^I and B^I, vanish in the physical region of the u channel. By analytic continuation, they must be identically zero and we reach the final result that there can be no scattering at all if there are no production processes in one channel.

It is worth remarking that the result we have obtained remains true even if there is a finite number of subtractions in the Mandelstam representations for $A^\pm$ and $B^\pm$ in Eq. (15). This is so because the result is based only on the crossing relations Eq. (2), the structure of the Landau curves on the real s-u plane, and the elastic-unitarity integrals Eq. (19), and none of these is changed by the presence of a finite number of subtractions in Eq. (15).

PHYSICAL REVIEW VOLUME 160, NUMBER 5 25 AUGUST 1967

Quantum Theory of Parametric Amplification. I*

B. R. Mollow and R. J. Glauber
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts
(Received 29 December 1966)

The amplification of electromagnetic fields is analyzed in a quantum-mechanical context by discussing the behavior of a simple theoretical model of the parametric amplifier. The statistical properties of the amplifier fields are described by means of the time-dependent density operator for the system. In doing this, extensive use is made of the coherent states and the P representation of the density operator, which provide a quantum-mechanical description of the fields closely resembling their classical description. Explicit solutions are obtained for the density operator for either of the two field modes for a variety of initial states of the modes. Initial states considered include combinations of coherent states, chaotic mixtures, and n-quantum states. Particular attention is given the behavior of the amplifier fields in the limit of large amplification. The conditions are established under which the amplification process leads in this limit to the existence of a non-negative P representation for the density operator for a single mode of oscillation.

I. INTRODUCTION

THE fundamental process which has become known as parametric amplification in electronic contexts plays a central role in several physical phenomena of interest. These include the coherent Raman and Brillouin effects and the frequency splitting of light beams in nonlinear media. The most familiar form of the parametric amplifier is designed to amplify an oscillating signal by means of a particular coupling of the mode in which it appears to a second mode of oscillation, the idler mode. The coupling parameter is made to oscillate with time in a way which gives rise to a steady increase of the energy in both the signal and idler modes.

The physical processes we have indicated as depending upon parametric amplification may be described in parallel terms. In the coherent Raman effect, for example, the presence of a monochromatic light wave in a Raman active medium gives rise to parametric coupling between an optical vibrational mode and a mode of the radiation field which represents the scattered (Stokes) wave. In the case of Brillouin scattering a similar form of coupling holds, with the vibrational mode oscillating at an acoustic rather than an optical frequency. The frequency splitting of light beams is an example of parametric amplification in which both of the coupled modes are electromagnetic. An intense light wave in a nonlinear dielectric medium couples pairs of electromagnetic field modes whose frequencies sum

* Supported in part by the U. S. Air Force Office of Scientific Research.

to the frequency of the original wave. It is worth noting that in this example and in the case of the Raman effect, the modes of oscillation in which the fields are amplified are initially free of excitation or very nearly so. The amplification process, in other words, not only intensifies pre-existing fields, but creates fields as well.

It is this property which indicates most clearly that the theory of the amplification process must be constructed in quantum-mechanical terms. An initially unexcited field mode can receive quanta only by means of spontaneous emission processes, which bring about what may be described as amplification of the zero-point oscillations of the mode. The correct treatment of such processes obviously lies outside the scope of classical theory. Classical analysis can only be applied to the amplification of fields which already contain many quanta, and that condition is met in the optical frequency range only by fields of high intensity.

In the radio-frequency region of the spectrum, signals containing few quanta are quite weak in comparison to the noise levels of the most sensitive detectors. The recent development of extremely low-noise amplifiers, however, has raised the possibility that it may not be long before such weak signals are indeed detectable at the higher microwave frequencies. This possibility has already inspired a lively discussion of the quantum-mechanical theory of amplification.[1–16] A quantum-mechanical model of the parametric amplifier has been proposed within this context by Louisell, Yariv, and Siegman.[6] Their model is an elementary one which may be applied as well to the optical phenomena we have noted. In this paper and the one that follows we shall adopt it as the basis of our analysis.

Previous discussions of the quantum-mechanical parametric amplifier have been based on the equations of motion for the field variables. The solutions to these equations have been used to find various time-dependent expectation values and moments of the field strengths. These data are only a part of the information implicitly contained in the time-dependent density operator for the fields. Since the density operator provides the most complete statistical description available for the system we shall devote most of our analysis to the ways in which it can be found and represented.

In a sense the amplifier we discuss is a device for transforming quantum fields into classical ones. The mathematical methods we use to discuss the amplification process must be well adapted to the treatment of both extremes. We will find it particularly convenient, in this connection, to make use of the set of coherent states in describing the quantum state of the system. In many of the examples we consider, the density operator for the system may be expressed as a species of statistical mixture of pure coherent states which we have called the P representation.[17] This representation, when it is available, can be the source of a good deal of insight, since it describes quantum states in terms not unlike those of classical probability theory. Part of our interest in the present work has therefore been directed toward determining, in a dynamical context, the usefulness and the limitations of this way of representing the density operator.

The kinds of observations which one may imagine making upon the parametric amplifier fall naturally into two classes, those which measure the field of only one mode, either the signal mode or the idler, and those which measure the fields in both. In the present paper, which is the first of two on the amplification process, we shall restrict ourselves to describing the time-dependent behavior of just one of the two interacting modes. This restriction allows us to base our analysis on a reduced form of the density operator, which is simpler in its structure than the full density operator and is somewhat more easily found. In the paper which follows we discuss the complete form of the density operator and correlations in the behavior of the two modes which it describes.

In the next two sections of this paper we outline some of the basic properties of the coherent states and the P representation, and then describe the theoretical model of the parametric amplifier. The expression for the reduced density operator for a single mode of the amplifier system is formulated in Sec. IV, and calculations of its value for a variety of initial states of the system are presented in Secs. V–VIII. The case in which both modes are initially in coherent states represents an ideally precise specification of the initial fields and therefore is discussed in some detail in Secs. V and VI. By superposing solutions of this form we then discuss a broader variety of initial states. A number of examples of the important case in which one of the two modes is initially in a chaotic state, e.g., a thermal state, are discussed in Secs. VII and VIII. The remaining three sections of the paper are devoted to more general questions concerning the properties of the P representa-

[1] M. W. Muller, Phys. Rev. **106**, 8 (1957).
[2] K. Shimoda, H. Takahasi, and C. H. Townes, J. Phys. Soc. Japan **12**, 686 (1957).
[3] R. V. Pound, Ann. Phys. (N. Y.) **1**, 24 (1957).
[4] M. W. P. Strandberg, Phys. Rev. **106**, 617 (1957).
[5] R. Serber and C. H. Townes, *Quantum Electronics—A Symposium*, edited by C. H. Townes (Columbia University Press, New York, 1960), p. 233.
[6] W. H. Louisell, A. Yariv, and A. E. Siegman, Phys. Rev. **124**, 1646 (1961).
[7] J. Schwinger, J. Math. Phys. **2**, 407 (1961).
[8] W. H. Wells, Ann. Phys. (N. Y.) **12**, 1 (1961).
[9] A. E. Siegman, Proc. Inst. Radio Engrs. **49**, 633, (1961).
[10] H. A. Haus and J. A. Mullen, Phys. Rev. **128**, 2407 (1962).
[11] I. R. Senitzky, Phys. Rev. **128**, 2864 (1962).
[12] J. P. Gordon, W. H. Louisell, and L. R. Walker, Phys. Rev. **129**, 481 (1963).
[13] J. P. Gordon, L. R. Walker, and W. H. Louisell, Phys. Rev. **130**, 806 (1963).
[14] R. P. Feynman and F. L. Vernon, Jr., Ann. Phys. (N. Y.) **24**, 118 (1963).
[15] W. H. Louisell and L. R. Walker, Phys. Rev. **137**, B204 (1965).
[16] D. Holliday and A. E. Glassgold, Phys. Rev. **139**, A1717 (1965).
[17] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

tion. In Sec. IX we discuss cases in which the mode of interest is in an arbitrary initial state, while the other mode is initially in a chaotic state. The arbitrary initial state need not possess a P representation. We show that as the field is altered by the amplification process, a critical time is reached after which the P representation necessarily exists for the mode of interest. We show that the weight function for the representation must be non-negative beginning at somewhat later times. We thus follow the evolution of the function which, in the classical limit of prolonged amplification, becomes the probability distribution for the mode amplitude. An illustration of this evolution is discussed in Sec. X, and the arguments we have presented are extended to the treatment of somewhat more general initial states in Sec. XI.

II. THE COHERENT STATES AND THE P REPRESENTATION

The dynamical behavior of a single mode of the electromagnetic field may be described in terms analogous to those used for a harmonic oscillator. Each field mode is characterized by an annihilation operator a and its adjoint $a^\dagger$; these operators obey the commutation relation

$$[a,a^\dagger]=1. \tag{2.1}$$

In the case of the free field, the different modes are dynamically independent. The Hamiltonian for a freely oscillating mode of frequency ω may be written in the form

$$H_0=\hbar\omega a^\dagger a. \tag{2.2}$$

The stationary states of this Hamiltonian are the n-quantum states

$$|n\rangle=(n!)^{-1/2}(a^\dagger)^n|0\rangle, \tag{2.3}$$

where $n=0, 1, 2, \cdots$ and $|0\rangle$ is the ground state of the mode, which is defined to satisfy

$$a|0\rangle=0. \tag{2.4}$$

The n-quantum states form a complete set, but one which is not particularly convenient to use when the quantum numbers of the field are large or quite uncertain, as they are in cases in which we have some knowledge of the phase of oscillation of the field. For such cases an alternative set of states, the coherent states, has been found especially useful.[17–22] The coherent state with complex amplitude α is defined to satisfy the eigenvalue equation

$$a|\alpha\rangle=\alpha|\alpha\rangle. \tag{2.5}$$

An explicit expression for $|\alpha\rangle$, which is determined apart from a phase factor by Eq. (2.5), is

$$|\alpha\rangle=e^{-(1/2)|\alpha|^2}\sum_{n=0}^{\infty}\frac{(\alpha a^\dagger)^n}{n!}|0\rangle. \tag{2.6}$$

It should be emphasized that $|\alpha\rangle$ is not an eigenstate of $a^\dagger$, or of the real and imaginary parts of a considered individually. Such properties are excluded by the commutation relation (2.1).

The configuration-space wave function for a coherent state has the form of a minimum uncertainty wave packet; it is simply a displaced[23] form of the wave function for the ground state of the oscillator, which is a coherent state with complex eigenvalue equal to zero. The properties of the vacuum fluctuations present in a coherent state are such that two coherent states with different complex eigenvalues are not orthogonal; it is easily shown that

$$\langle\alpha|\beta\rangle=e^{-(1/2)|\alpha|^2-(1/2)|\beta|^2+\alpha^*\beta}, \tag{2.7}$$

and therefore

$$|\langle\alpha|\beta\rangle|^2=e^{-|\alpha-\beta|^2}. \tag{2.8}$$

Although the coherent states lack orthogonality, they do form a complete set. They may be shown to satisfy the completeness relation[24,17]

$$\frac{1}{\pi}\int|\alpha\rangle\langle\alpha|d^2\alpha=1, \tag{2.9}$$

where $d^2\alpha\equiv d(\mathrm{Re}\alpha)d(\mathrm{Im}\alpha)$. This relation may be used to expand arbitrary state vectors and operators in terms of the coherent state vectors.[17]

An operator we shall be particularly interested in expanding in terms of the coherent states is the density operator. This operator provides the most general statistical description of the state of a system; indeed no simpler description is useful, in general, for individual systems which interact with others. A density operator ρ must be Hermitian, must have non-negative eigenvalues, and must satisfy the trace relation

$$\mathrm{tr}\rho=1. \tag{2.10}$$

The mean value of a dynamical operator F for a system described by the density operator ρ is $\mathrm{tr}\{\rho F\}$.

There exists a considerable variety of ways of representing the density operator for oscillator systems. The most common, perhaps, is the use of the n-quantum state matrix elements $\langle n|\rho|m\rangle$, which are the basis of the expansion

$$\rho=\sum_{n,m=0}^{\infty}|n\rangle\langle n|\rho|m\rangle\langle m|. \tag{2.11}$$

[18] R. J. Glauber, Phys. Rev. Letters **10**, 84 (1963).
[19] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[20] R. J. Glauber, in *Quantum Electronics*, Proceedings of the Third International Congress, Paris, 1963, edited by N. Bloembergen and P. Grivet (Columbia University Press, New York 1964), Vol. I, p. 111.
[21] R. J. Glauber, *Quantum Optics and Electronics*, edited by C. de Witt *et al.* (Gordon and Breach Science Publishers, Inc., New York, 1965).
[22] P. Carruthers and M. M. Nieto, Phys. Rev. Letters **14**, 387 (1965).

[23] The ground state of the oscillator is displaced in both coordinate and momentum space in general. See, for example, Ref. 17, p. 2771.
[24] J. R. Klauder, Ann. Phys. (N. Y.) **11**, 123 (1960).

This expansion is simplest when few quanta are present, and when the off-diagonal matrix elements of ρ vanish, as they do for many stationary fields. For nonstationary fields, on the other hand, e.g., fields for which we have some information about the phase of the complex field strength, ρ is expressed more conveniently, as a rule, by means of the expansion

$$\rho=\frac{1}{\pi^2}\int|\alpha\rangle\langle\alpha|\rho|\beta\rangle\langle\beta|d^2\alpha d^2\beta\,, \tag{2.12}$$

which holds for an arbitrary density operator, according to the completeness relation (2.9).

We may also note that ρ is uniquely determined by its characteristic function[25] $\chi(\eta)$, which is defined for arbitrary complex η by the expression[26]

$$\chi(\eta)\equiv\mathrm{tr}\{\rho e^{\eta a^\dagger-\eta^* a}\}\,. \tag{2.13}$$

A particularly simple way of representing the density operator, the P representation,[17,18,21,27,28] corresponds to writing it as a statistical mixture of pure coherent states:

$$\rho=\int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha\,. \tag{2.14}$$

The P representation, when it is available, leads to great simplifications in the calculation of statistical averages of quantum-mechanical operators. It is especially suitable for making comparisons between quantum and classical theory, and for exhibiting the classical limit of quantum mechanics. The principal drawback of the P representation is that it cannot be used to represent all varieties of quantum states.[27] We shall presently discuss this limitation in greater detail.

The Hermiticity of ρ implies that the function P which appears in the expansion (2.14) must be real. The relation $\mathrm{tr}\rho=1$ leads to the requirement

$$\int P(\alpha)d^2\alpha=1\,. \tag{2.15}$$

For any real non-negative function $P(\alpha)$ satisfying this integral condition, the operator ρ defined by Eq. (2.14) is necessarily a Hermitian positive-definite operator with unit trace, and hence is a permissible density operator. The physical constraints imposed on the density operator do not, however, exclude negative values $P(\alpha)$ for or require that it be a very well-behaved function.

[25] That χ determines ρ uniquely follows from an expansion theorem due to H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover Publications, Inc., New York, 1950), p. 272. See also Ref. 16, Sec. III.

[26] For a discussion of the characteristic function with complex argument, see R. J. Glauber, (Ref. 21); Lecture XIII.

[27] R. J. Glauber, in *Physics of Quantum Electronics*, edited by P. L. Kelley *et al.* (McGraw-Hill Book Company Inc., New York, 1966), p. 788.

[28] The same kind of representation is discussed from a different standpoint by E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).

The nonorthogonality of the projection operators $|\alpha\rangle\langle\alpha|$, which is a reflection of the noncommutation of the real and imaginary parts of a, makes it impossible in general to interpret the function $P(\alpha)$ as a probability density. In the classical limit, however, which corresponds to fields with $|\alpha|\gg 1$, we often deal with density operators for which the function $P(\alpha)$ varies slowly over unit distances in the complex α plane. In such cases the lack of orthogonality of the states $|\alpha\rangle$ becomes unimportant and the function P may be identified in an asymptotic sense with the classical probability distribution for finding the oscillator with the complex coordinate α.

The normally ordered form of an operator expression is obtained by writing the creation operators to the left of the annihilation operators wherever they appear. The mean value of a normally ordered operator is evaluated in the P representation by an integral which has the form of a classical statistical average, with the function P playing the role of the classical distribution function. By using Eqs. (2.5) and (2.14) we find, for example,

$$\mathrm{tr}\{\rho a^{\dagger n}a^m\}=\int\langle\alpha|a^{\dagger n}a^m|\alpha\rangle P(\alpha)d^2\alpha$$
$$=\int\alpha^{*n}\alpha^m P(\alpha)d^2\alpha\,. \tag{2.16}$$

We have noted that the characteristic function uniquely determines the density operator. In order to evaluate the function P in terms of the characteristic function, we first introduce the normally ordered characteristic function[27] $\chi_N(\eta)$. This function is defined by an expression analogous to the definition (2.13) of the ordinary characteristic function $\chi(\eta)$, but with the exponential written in normally ordered form:

$$\chi_N(\eta)\equiv\mathrm{tr}\{\rho e^{\eta a^\dagger}e^{-\eta^* a}\}\,. \tag{2.17}$$

By making use of the well-known operator identity[29]

$$e^Ae^B=e^{A+B+(1/2)[A,B]}\,, \tag{2.18}$$

which holds for any two operators A and B satisfying

$$[[A,B],A]=[[A,B],B]=0\,, \tag{2.19}$$

we find that $\chi_N(\eta)$ and $\chi(\eta)$ are related by

$$\chi_N(\eta)=e^{\frac{1}{2}|\eta|^2}\chi(\eta)\,. \tag{2.20}$$

If the density operator ρ has a P representation, then $\chi_N(\eta)$ is given by

$$\chi_N(\eta)=\int\langle\alpha|e^{\eta a^\dagger}e^{-\eta^* a}|\alpha\rangle P(\alpha)d^2\alpha$$
$$=\int e^{\eta\alpha^*-\eta^*\alpha}P(\alpha)d^2\alpha\,. \tag{2.21}$$

[29] See, for example, A. Messiah, *Quantum Mechanics* (North-Holland Publishing Company, Amsterdam, 1961), Vol. I, p. 442.

If we write η and α in terms of their real and imaginary parts, we find that Eq. (2.21) expresses $\chi_N(\eta)$ as the two-dimensional Fourier transform of $P(\alpha)$. The solution to Eq. (2.21) for $P(\alpha)$, which follows simply from the Fourier inversion theorem, is[27]

$$P(\alpha)=\frac{1}{\pi^2}\int e^{\alpha\eta^*-\alpha^*\eta}\chi_N(\eta)d^2\eta. \tag{2.22}$$

We have shown, then, that if the P representation exists, the function $\chi_N(\eta)$ has the Fourier transform $P(\alpha)$. The converse, which follows simply from the uniqueness of the correspondence between a density operator and its characteristic function, is also true: If we *define* $P(\alpha)$ by Eq. (2.22), where $\chi_N(\eta)$ is defined for an arbitrary density operator ρ by Eq. (2.17), then we may construct ρ by substituting $P(\alpha)$ into Eq. (2.14). Our criterion for the existence of the P representation will therefore be simply the existence of a Fourier transform for the normally ordered characteristic function $\chi_N(\eta)$.

Although the class of states for which the Fourier transform of $\chi_N(\eta)$ exists is a broad one, there exist many well-behaved quantum states for which the function $\chi_N(\eta)$ increases so rapidly as $|\eta|\rightarrow\infty$ that no meaning can be attached to the integral (2.22), even as a tempered distribution.[30,31] In such unmanageably singular cases it seems most reasonable to say that the P representation does not exist.[32,33] The density operator may always be expressed in terms of the coherent states, on the other hand, by using the more general expansion of Eq. (2.12).

III. MODEL OF THE PARAMETRIC AMPLIFIER

The dynamical elements of the amplifier model proposed by Louisell, Yariv, and Siegman[6] are two modes of oscillation of the electromagnetic field. These play a symmetrical role in the amplification process; it will be somewhat briefer to refer to them as the A and B modes in the work which follows, than to designate one arbitrarily as the signal mode and the other as the idler. It is assumed that the uncoupled A and B modes have the dynamical behavior of harmonic oscillators, which are described by the annihilation operators $a(t)$ and $b(t)$, respectively. These operators and their adjoints satisfy the canonical commutation relations

$$\begin{aligned}[][a(t),b(t)]=[a(t),b^\dagger(t)]=0,\\ [a(t),a^\dagger(t)]=[b(t),b^\dagger(t)]=1.\end{aligned} \tag{3.1}$$

The A and B modes are assumed to be coupled by a parameter which oscillates harmonically at a frequency ω equal to the sum of the natural frequencies ω_a and ω_b of the unperturbed oscillators:

$$\omega=\omega_a+\omega_b. \tag{3.2}$$

The Hamiltonian for the two coupled modes is taken to have the form[6]

$$\begin{aligned}H(t)=\hbar\omega_a a^\dagger(t)a(t)+\hbar\omega_b b^\dagger(t)b(t)\\ -\hbar\kappa[a^\dagger(t)b^\dagger(t)e^{-i\omega t}+a(t)b(t)e^{i\omega t}],\end{aligned} \tag{3.3}$$

in which κ is a coupling constant, and the phase of the externally imposed oscillation of the coupling is chosen to be zero[34] at $t=0$. This Hamiltonian contains all of the terms which are most essential to the description of a number of physical realizations of the parametric amplifier. It is possible to construct such an amplifier, for example, by using a sample of a lossless nonlinear dielectric substance to couple the modes of a resonant cavity with reflecting walls.[6] By imposing on the dielectric an external field, the "pump" field, which oscillates at a frequency equal to the sum of the frequencies of two particular modes, those modes are made to undergo a closely coupled forced oscillation. The Hamiltonian (3.3) is intended only to describe the behavior of the two modes which are resonantly coupled in this way; the nonresonant couplings to other modes have been omitted, and the pump field has been assumed strong enough to be represented in classical terms.

The frequency-splitting of light beams may be described by essentially the same Hamiltonian as is given in Eq. (3.3). The pump field in this case is the incident light beam, which excites, by means of a nonlinear dielectric, the emission of light in each of two modes which meet appropriate resonance conditions. Coherent Raman and Brillouin scattering may be described in a similar way. In these cases one of the two harmonic oscillators described by the Hamiltonian (3.3) represents a vibrational mode of the medium (an optical mode for the Raman effect and an acoustic mode for Brillouin scattering). The coupling of the vibrational mode with the scattered light wave is provided by the presence of the intense incident light wave of frequency ω. While these familiar phenomena are all described by the Hamiltonian (3.3), we should emphasize that we are discussing a model of an amplifier rather than an oscillator. Raman and Brillouin scattering have typically been observed under conditions in which the leakage of scattered light from the medium has tended to quench

[30] L. Schwartz, *Theorie des Distributions* (Hermann et Cie, Paris, 1957), Vol. II, Chap. VII.

[31] K. E. Cahill, Phys. Rev. **138**, B1566 (1965).

[32] An alternative approach is suggested by J. R. Klauder, J. McKenna, and D. G. Currie, J. Math. Phys. **6**, 734 (1965), C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. **138**, B274 (1965), and J. R. Klauder, Phys. Rev. Letters **16**, 534 (1966), who represent the density operator as the limit of an infinite sequence of P representations. Statistical averages, they show, may be evaluated for any density operator by carrying out an appropriate limiting procedure in each instance. The usefulness of this approach in physical contexts is not yet clear.

[33] Restriction of the functions $P(\alpha)$, on the other hand, to a narrower class than that of tempered distributions has been suggested by R. Bonifacio, L. M. Narducci, and E. Montaldi, Phys. Rev. Letters **16**, 1125 (1966).

[34] Any other initial phase for the coupling may be treated either by redefining the initial time or by performing a canonical transformation which appropriately readjusts the phases of the operators a and b.

the amplification process. The scattering medium has functioned as an amplifier only for a brief interval after the appearance of the incident field, and then has continued to function as an oscillator. The effects of leakage and dissipation, on the other hand, are omitted from the Hamiltonian (3.3), and the amplification process it describes therefore continues indefinitely without quenching.

The Heisenberg equations of motion which follow from the Hamiltonian $H(t)$ are

$$i\hbar\frac{d}{dt}a(t)=[a(t),H(t)],\qquad i\hbar\frac{d}{dt}b(t)=[b(t),H(t)]. \tag{3.4}$$

Before discussing these equations further, let us note that the Hamiltonian (3.3) possesses a simple invariance property. It remains unchanged under the transformation $a(t)\to a(t)e^{i\theta}$, $b(t)\to b(t)e^{-i\theta}$ (θ real), which is generated by the unitary operator

$$V(\theta,t)=e^{i\theta[a^\dagger(t)a(t)-b^\dagger(t)b(t)]} \tag{3.5}$$

via the relations

$$V^{-1}(\theta,t)a(t)V(\theta,t)=a(t)e^{i\theta},\qquad V^{-1}(\theta,t)b(t)V(\theta,t)=b(t)e^{-i\theta}. \tag{3.6}$$

Since $H(t)$ is invariant under the group of transformations defined by $V(\theta,t)$, i.e.,

$$V^{-1}(\theta,t)H(t)V(\theta,t)=H(t), \tag{3.7}$$

it follows that $H(t)$ commutes with the generator of the group:

$$[(a^\dagger(t)a(t)-b^\dagger(t)b(t)),H(t)]=0. \tag{3.8}$$

This equation may also be deduced directly from the explicit form for $H(t)$ and the commutation relations (3.1). It implies, according to the equations of motion (3.4), that the generator is a constant of the motion:

$$a^\dagger(t)a(t)-b^\dagger(t)b(t)=a^\dagger(0)a(0)-b^\dagger(0)b(0). \tag{3.9}$$

This relation expresses a conservation law for the difference in the number of quanta in the A and B modes; the law follows directly from the form of the coupling between the modes.

The foregoing equations have been formulated in the Heisenberg picture, which is characterized by the time-dependent operators $a(t)$ and $b(t)$, and by a time-independent state vector $|\ \rangle$ for the system. The Schrödinger picture, on the other hand, is characterized by a time-dependent state vector $|t\rangle$, and by the time-independent operators a and b. We take the two representations to coincide at $t=0$ by writing

$$a(0)\equiv a,\qquad b(0)\equiv b, \tag{3.10}$$

and

$$|\ \rangle\equiv|t=0\rangle. \tag{3.11}$$

The time dependence of the Schrödinger state vector is given by

$$i\hbar\frac{d}{dt}|t\rangle=H_S(t)|t\rangle, \tag{3.12}$$

in which the Schrödinger Hamiltonian $H_S(t)$ may be obtained from the Heisenberg Hamiltonian $H(t)$ by making the substitutions $a(t)\to a$, $b(t)\to b$ in Eq. (3.3). The Hamiltonian remains time-dependent in the Schrödinger picture because of the explicit time dependence of the coupling.

The significance of the conservation law (3.9) in the Schrödinger picture may be expressed as follows: If a Schrödinger state vector $|t\rangle$ is initially an eigenstate of $a^\dagger a-b^\dagger b$ with eigenvalue m, then it remains so at all times; if $|t\rangle$ is expanded in the n-quantum representation, only those terms appear for which the difference in the number of quanta in the A and B modes is equal to m.

The Heisenberg equations of motion for the operators $a(t)$ and $b(t)$ are obtained by substituting Eq. (3.3) for $H(t)$ into Eq. (3.4), and making use of the commutation relations (3.1); we find

$$i\frac{d}{dt}a(t)=\omega_a a(t)-\kappa b^\dagger(t)e^{-i\omega t},\qquad i\frac{d}{dt}b(t)=\omega_b b(t)-\kappa a^\dagger(t)e^{-i\omega t}. \tag{3.13}$$

Although these are operator equations their linear character means that they are no more difficult to solve than the corresponding linear equations for c numbers. The solutions to the coupled equations (3.13) may be written in the form

$$a(t)=ac_a(t)+b^\dagger s_a(t), \tag{3.14a}$$

$$b(t)=bc_b(t)+a^\dagger s_b(t), \tag{3.14b}$$

in which the abbreviations $c_a(t)$, $s_a(t)$, $c_b(t)$, and $s_b(t)$ have been introduced for the c-number functions defined by

$$c_a(t)\equiv e^{-i\omega_a t}\cosh\kappa t,\quad s_a(t)\equiv ie^{-i\omega_a t}\sinh\kappa t,\quad c_b(t)\equiv e^{-i\omega_b t}\cosh\kappa t,\quad s_b(t)\equiv ie^{-i\omega_b t}\sinh\kappa t. \tag{3.15}$$

A direct insight into the way the number of quanta present changes with time may be gained by noting that the equations of motion (3.13) lead to a second-order rate equation involving only the occupation numbers of the A and B modes.[35] By using Eqs. (3.13)

[35] T. Von Foerster (private communication).

to evaluate the second derivative of $a^\dagger(t)a(t)$ we find

$$\frac{d^2}{dt^2}\{a^\dagger(t)a(t)\}=2\kappa^2\{a^\dagger(t)a(t)+b(t)b^\dagger(t)\}. \quad (3.16)$$

If we now introduce the notation

$$\begin{aligned}N_a(t)&\equiv a^\dagger(t)a(t),\\ N_b(t)&\equiv b^\dagger(t)b(t)\end{aligned} \quad (3.17)$$

for the occupation number operators for the two modes, and use the commutation relations (3.1) for the b operators, we may write Eq. (3.16) as

$$\frac{d^2}{dt^2}N_a(t)=2\kappa^2[N_a(t)+N_b(t)+1]. \quad (3.18)$$

We now make use of the conservation law (3.9) to write the difference $N_a(t)-N_b(t)$ as a time-independent operator:

$$N_a(t)-N_b(t)\equiv M. \quad (3.19)$$

By using this equation to eliminate $N_b(t)$ from Eq. (3.18) we obtain a rate equation for the operator $N_a(t)$:

$$\frac{d^2}{dt^2}N_a(t)=2\kappa^2[2N_a(t)+1-M]. \quad (3.20)$$

The solution of this equation for $N_a(t)$ in terms of its initial value $N_a(0)$ and the initial value of its time derivative $\dot{N}_a(0)$ is

$$N_a(t)=\frac{\dot{N}_a(0)}{2\kappa}\sinh 2\kappa t+[N_a(0)+\tfrac{1}{2}(1-M)]\cosh 2\kappa t-\tfrac{1}{2}(1-M). \quad (3.21)$$

A similar result, with the sign of M reversed, holds for the operator $N_b(t)$. It is clear, therefore, that the number of quanta in both modes tends, at large enough times, to increase exponentially with time.

The operator solutions (3.14) to the equations of motion may be used to calculate the mean values of products of arbitrary numbers of creation and annihilation operators as a function of time. As the simplest example, we consider first the mean values of $a(t)$ and $b(t)$, which are defined in terms of the Heisenberg density operator ρ by

$$\begin{aligned}\bar{\alpha}(t)&\equiv \mathrm{tr}\{\rho a(t)\},\\ \bar{\beta}(t)&\equiv \mathrm{tr}\{\rho b(t)\}.\end{aligned} \quad (3.22)$$

The solutions for $\bar{\alpha}(t)$ and $\bar{\beta}(t)$ as a function of time take the same form as those for the complex mode amplitudes in the corresponding classical problem:

$$\bar{\alpha}(t)=\alpha_0 c_a(t)+\beta_0^* s_a(t), \quad (3.23a)$$

$$\bar{\beta}(t)=\beta_0 c_b(t)+\alpha_0^* s_b(t), \quad (3.23b)$$

where we have written

$$\begin{aligned}\bar{\alpha}(0)&\equiv\alpha_0\\ \bar{\beta}(0)&\equiv\beta_0.\end{aligned} \quad (3.24)$$

It is possible to calculate the expectation values and variances of numbers of quanta and complex field strengths for a variety of initial states in a straightforward manner with the aid of the solutions (3.14) and the commutation relations (3.1). Let us assume for example that both the A and B modes are initially in pure coherent states. We may write this initial state of the system as $|\alpha_0,\beta_0\rangle$, where α_0 and β_0 are the complex amplitudes of the A and B modes, respectively. For certain types of couplings of harmonic oscillator systems it has been shown that an initially coherent Schrödinger state retains its coherent character at later times.[36] The state vectors for the parametric amplifier modes behave quite differently, however. A simple indication of their behavior may be obtained by evaluating the variance of the complex field strength for the A mode. We easily deduce with the aid of Eq. (2.5), the commutation relations (3.1), and Eq. (3.14a), that

$$\langle\alpha_0,\beta_0|(a^\dagger(t)-\bar{\alpha}^*(t))(a(t)-\bar{\alpha}(t))|\alpha_0,\beta_0\rangle=|s_a(t)|^2=\sinh^2\kappa t. \quad (3.25)$$

In the Schrödinger picture this equation takes the form

$$\langle t|(a^\dagger-\bar{\alpha}^*(t))(a-\bar{\alpha}(t))|t\rangle=\sinh^2\kappa t, \quad (3.26)$$

in which the Schrödinger state vector $|t\rangle$ is initially coherent:

$$|t=0\rangle=|\alpha_0,\beta_0\rangle, \quad (3.27)$$

and satisfies the Schrödinger Eq. (3.12).

In either of the two pictures it is evident that the variance of the complex field amplitude grows exponentially with time, much as the average occupation number of the field does. It is clear from this behavior of the variance that the state which evolves from an initially coherent state does not retain its coherent character. The uncertainty of the field, which is initially minimal, grows rapidly with time.[12] The amplification process amplifies the vacuum fluctuations along with the expectation value of the field strength, and the minimum uncertainty character of the initial state is quickly lost.

IV. REDUCED DENSITY OPERATOR FOR THE A MODE

To evaluate statistical averages of time-dependent operators in the Heisenberg picture we must make explicit use of the solutions to the operator equations of motion for the system. The operators $a(t)$ and $b(t)$, for example, must be expressed in terms of their initial values before the statistical average of a function which

[36] See, for example, R. J. Glauber, Phys. Letters **21**, 650 (1966).

depends on them can be evaluated in the initial state of the system. The Schrödinger picture, on the other hand, offers a more compact way of evaluating such averages; it combines the dynamical part of the calculation with the statistical part by describing the system in terms of a time-dependent density operator $\rho(t)$. We shall confine our attention in the present paper to discussing the statistical behavior of either one of the two modes of the amplifier system. For this purpose we are led to consider a reduced form of the Schrödinger density operator which is defined in terms of the variables for the mode of interest.

The Heisenberg and Schrödinger pictures of the motion of the system are related by the unitary time translation operator $U(t)$, which is defined by the equations

$$i\hbar\frac{d}{dt}U(t)=H_S(t)U(t) \tag{4.1}$$

and

$$U(0)=1. \tag{4.2}$$

The Heisenberg operators $a(t)$ and $b(t)$, for example, are given formally in terms of their initial values a and b by

$$a(t)=U^{-1}(t)aU(t), \tag{4.3a}$$

$$b(t)=U^{-1}(t)bU(t). \tag{4.3b}$$

In the Schrödinger picture the density operator, like the state vector, is a time-dependent quantity. The Schrödinger density operator $\rho(t)$ is given in terms of the time-independent Heisenberg density operator ρ by the relation

$$\rho(t)=U(t)\rho U^{-1}(t). \tag{4.4}$$

The reduced Schrödinger density operator for the A mode is defined by

$$\rho_A(t)=\mathrm{tr}_B\rho(t), \tag{4.5}$$

where tr_B means trace with respect to the (initial or time-independent) states of the B mode. The mean value of an operator T_A which refers to the variables of the A mode only is given by

$$\begin{aligned}\mathrm{tr}\{\rho(t)T_A\}&=\mathrm{tr}_A\mathrm{tr}_B\{\rho(t)T_A\}\\&=\mathrm{tr}_A\{\rho_A(t)T_A\}.\end{aligned} \tag{4.6}$$

The time-dependent form of the normally ordered characteristic function $\chi_N(\eta,t)$ for the A mode is given by

$$\chi_N(\eta,t)=\mathrm{tr}\{\rho(t)e^{\eta a^\dagger}e^{-\eta^* a}\} \tag{4.7}$$

$$=\mathrm{tr}_A\{\rho_A(t)e^{\eta a^\dagger}e^{-\eta^* a}\}. \tag{4.8}$$

The time-dependent function $\chi_N(\eta,t)$, according to Eq. (4.8), is defined in terms of the reduced density operator $\rho_A(t)$ by an expression identical to the definition (2.17) for the normally ordered characteristic function corresponding to an arbitrary (single mode) density operator.

We shall say that a P representation for the A mode exists at time t if the reduced Schrödinger density operator $\rho_A(t)$ can be written in the form

$$\rho_A(t)=\int P(\alpha,t)|\alpha\rangle\langle\alpha|d^2\alpha. \tag{4.9}$$

From the discussion of Sec. II it is evident that $\rho_A(t)$ has a P representation if $\chi_N(\eta,t)$ possesses a Fourier transform,

$$P(\alpha,t)=\frac{1}{\pi^2}\int e^{\alpha\eta^*-\alpha^*\eta}\chi_N(\eta,t)d^2\eta. \tag{4.10}$$

The function $\chi_N(\eta,t)$, which has been defined as the trace of a Schrödinger operator, is invariant under transformation to the Heisenberg picture. By substituting Eq. (4.4) for $\rho(t)$ into Eq. (4.7), and making use of the cyclical symmetry of the traces of products, we obtain

$$\chi_N(\eta,t)=\mathrm{tr}\{\rho U^{-1}(t)e^{\eta a^\dagger}e^{-\eta^* a}U(t)\}. \tag{4.11}$$

By making use of Eq. (4.3a) and its adjoint we then find

$$\chi_N(\eta,t)=\mathrm{tr}\{\rho e^{\eta a^\dagger(t)}e^{-\eta^* a(t)}\}. \tag{4.12}$$

This equation expresses $\chi_N(\eta,t)$ in terms of the initial density operator ρ for the joint system of A and B modes, and the time-dependent operator $a(t)$ and its adjoint. A formal solution for $\chi_N(\eta,t)$ may thus be constructed by substituting the solution (3.14a) for $a(t)$ into Eq. (4.12). If the function $\chi_N(\eta,t)$ obtained in this way has the Fourier transform $P(\alpha,t)$, then the reduced density operator $\rho_A(t)$ is given by Eq. (4.9).

V. INITIALLY COHERENT STATE: *P* REPRESENTATION FOR THE *A* MODE

In Sec. III we considered the case in which the joint system of A and B modes is initially described by the pure coherent state vector $|\alpha_0,\beta_0\rangle$. We showed, by evaluating the variance of the complex field strength for the A mode, that such a state does not remain coherent at later times. In this section we shall consider the case of an initially coherent state in greater detail, and solve for the function $P(\alpha,t)$, which provides a full description of the behavior of the A mode.

Let us assume, then, that the density operator for the joint system of A and B modes is initially given by

$$\rho=|\alpha_0,\beta_0\rangle\langle\alpha_0,\beta_0|. \tag{5.1}$$

To evaluate the time-dependent normally ordered characteristic function for the A mode, we first use the operator identity (2.18) to write

$$e^{\eta a^\dagger(t)}e^{-\eta^* a(t)}=e^{\frac{1}{2}|\eta|^2}e^{\eta a^\dagger(t)-\eta^* a(t)}. \tag{5.2}$$

By substituting this relation into Eq. (4.12), using and

Eq. (5.1) for ρ and Eq. (3.14a) for $a(t)$, we obtain

$$\chi_N(\eta,t)=e^{\frac{1}{2}|\eta|^2}\langle\alpha_0,\beta_0|\exp[\eta(a^\dagger c_a^*(t)+bs_a^*(t)) -\eta^*(ac_a(t)+b^\dagger s_a(t))]|\alpha_0,\beta_0\rangle. \quad (5.3)$$

We denote by $s(t)$ and $c(t)$ the moduli of the complex functions $s_{a,b}(t)$ and $c_{a,b}(t)$, respectively:

$$s(t)\equiv\sinh\kappa t,\quad c(t)\equiv\cosh\kappa t. \quad (5.4)$$

By using Eq. (2.18) to write the exponential operator in Eq. (5.3) in normally ordered form, we find

$$\begin{aligned}\chi_N(\eta,t)&=\exp[|\eta|^2(\tfrac{1}{2}-\tfrac{1}{2}c^2(t)-\tfrac{1}{2}s^2(t))]\\ &\quad\times\langle\alpha_0,\beta_0|\exp[a^\dagger\eta c_a^*(t)-b^\dagger\eta^*s_a(t)]\\ &\quad\times\exp[-a\eta^*c_a(t)+b\eta s_a^*(t)]|\alpha_0,\beta_0\rangle\\ &=\exp[-|\eta|^2s^2(t)+\alpha_0^*\eta c_a^*(t)-\beta_0^*\eta^*s_a(t)\\ &\quad-\alpha_0\eta^*c_a(t)+\beta_0\eta s_a^*(t)] \quad (5.5)\\ &=\exp[-|\eta|^2s^2(t)+\eta\bar\alpha^*(t)-\eta^*\bar\alpha(t)], \quad (5.6)\end{aligned}$$

in which $\bar\alpha(t)$, the mean value of $a(t)$, is defined by Eq. (3.23a).

Substituting Eq. (5.6) for $\chi_N(\eta,t)$ into Eq. (4.10), we find that $P(\alpha,t)$ is given by the complex Fourier integral

$$P(\alpha,t)=\frac{1}{\pi^2}\int\exp[-|\eta|^2s^2(t)+\eta(\bar\alpha^*(t)-\alpha^*) -\eta^*(\bar\alpha(t)-\alpha)]d^2\eta. \quad (5.7)$$

This integral is easily evaluated with the aid of the useful identity

$$\frac{1}{\pi}\int d^2\eta\, e^{-\mu|\eta|^2+\lambda\eta+\nu\eta^*}=\frac{1}{\mu}\exp\left(\frac{\lambda\nu}{\mu}\right), \quad (5.8)$$

which holds for $\mathrm{Re}\mu>0$, and for arbitrary complex numbers λ and ν. By making the appropriate identifications of these parameters we find from Eq. (5.7) that $P(\alpha,t)$ takes the form

$$P(\alpha,t)=\frac{1}{\pi s^2(t)}\exp\left[-\frac{|\alpha-\bar\alpha(t)|^2}{s^2(t)}\right]. \quad (5.9)$$

The function $P(\alpha,t)$ for the A mode at time t is thus a Gaussian function about the complex mean value $\bar\alpha(t)$. The variance of the distribution is $s^2(t)=\sinh^2\kappa t$, a result which was obtained in Sec. III from the solution to the equations of motion.

In Ref. 17 it is shown that the P representation provides a natural means of extending to the quantum-mechanical domain the classical concept of the superposition of two independent fields: The P function for the superposition of two fields is the convolution of the P functions for each field considered individually. Since the function $P(\alpha)$ for a coherent state with complex amplitude $\bar\alpha$ is the delta function $\delta^{(2)}(\alpha-\bar\alpha)\equiv\delta(\mathrm{Re}(\alpha-\bar\alpha))\delta(\mathrm{Im}(\alpha-\bar\alpha))$, we may say that Eq. (5.9) describes the superposition of a coherent state with complex amplitude $\bar\alpha(t)$, and a chaotic or Gaussian mixture with variance $\sinh^2\kappa t$.

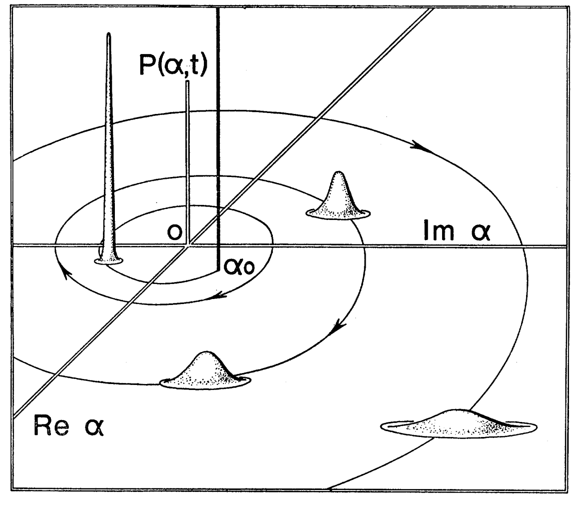

FIG. 1. Schematic picture of the way in which the function $P(\alpha,t)$ varies with time when the system is initially in a pure coherent state. At the initial time $t=0$ the function $P(\alpha,t)$ is the delta function $\delta^{(2)}(\alpha-\alpha_0)$. The amplification process leads to a function $P(\alpha,t)$ which is Gaussian in form at later times and has a variance which increases monotonically with time. The mean value of the complex amplitude α describes a spiral trajectory in the complex α plane.

In Fig. 1 we have plotted the function $P(\alpha,t)$ at several points along the curve $\bar\alpha(t)$, for the case $\beta_0=0$, and therefore $\bar\alpha(t)=e^{-i\omega_a t}\alpha_0\cosh\kappa t$. The variance of the distribution, which is initially zero, grows rapidly with time; for large amplification ($\kappa t\gg1$), both the variance and the square of the mean field strength are proportional to $e^{2\kappa t}$.

The result (5.9) has been derived for the initially coherent state $|\alpha_0,\beta_0\rangle$. If the joint system of A and B modes is initially described by a P representation,

$$\rho=\int P(\alpha_0,\beta_0)|\alpha_0,\beta_0\rangle\langle\alpha_0,\beta_0|d^2\alpha_0d^2\beta_0, \quad (5.10)$$

then the function P for the A mode at time t is obtained by averaging the right-hand side of Eq. (5.9) with respect to the weight function $P(\alpha_0,\beta_0)$:

$$P(\alpha,t)=\frac{1}{\pi s^2(t)}\int P(\alpha_0,\beta_0) \times\exp\left[-\frac{|\alpha-\bar\alpha(\alpha_0,\beta_0,t)|^2}{s^2(t)}\right]d^2\alpha_0d^2\beta_0, \quad (5.11)$$

in which the function $\bar\alpha(\alpha_0,\beta_0,t)$ is defined by Eq. (3.23a).

The initial state specified by $\alpha_0=\beta_0=0$ in Eq. (5.1) corresponds to the absence of any initial excitation in the system. A field is generated nonetheless; it may be thought of as an amplified form of the zero-point fluctuations of the vacuum. The function $P(\alpha,t)$ for this case is obtained by setting $\bar{\alpha}(t)=0$ in Eq. (5.9):

$$P(\alpha,t)=\frac{1}{\pi s^2(t)}\exp\left[-\frac{|\alpha|^2}{s^2(t)}\right]. \quad (5.12)$$

This function describes a chaotic mixture with variance $s^2(t)$; since $\bar{\alpha}(t)=0$ the variance is equal to the mean number of quanta present in the mode,

$$\langle n(t)\rangle = s^2(t)=\sinh^2\kappa t. \quad (5.13)$$

In Ref. 17 it is shown that a density operator with a Gaussian P function may be written in the n-quantum representation in the form characteristic of thermal equilibrium. The density operator $\rho_A(t)$ is then given by

$$\rho_A(t)=\sum_{n=0}^{\infty}\frac{[\langle n(t)\rangle]^n}{[1+\langle n(t)\rangle]^{n+1}}|n\rangle\langle n|$$

$$=\sum_{n=0}^{\infty}\frac{[s^2(t)]^n}{[c^2(t)]^{n+1}}|n\rangle\langle n|, \quad (5.14)$$

where $|n\rangle$ represents the n-quantum state of the A mode. The probability of finding n quanta in the A mode after time t is given by the coefficient of $|n\rangle\langle n|$ in this expansion,

$$p_n(t)=\frac{[s^2(t)]^n}{[c^2(t)]^{n+1}}. \quad (5.15)$$

VI. INITIALLY COHERENT STATE; MOMENTS, MATRIX ELEMENTS, AND EXPLICIT REPRESENTATION FOR $\rho_A(t)$

We have shown that the reduced density operator for the A mode at time t, corresponding to an initially coherent state for the joint system, has the P representation

$$\rho_A(t)=\frac{1}{\pi s^2(t)}\int\exp\left[-\frac{|\alpha-\bar{\alpha}(t)|^2}{s^2(t)}\right]|\alpha\rangle\langle\alpha|d^2\alpha. \quad (6.1)$$

In this section we shall derive a number of statistical properties of the mode which follow from this description, and exhibit an explicit form for the density operator in terms of the operators a and $a^\dagger$.

Let us find the mean value of the normally ordered product of an arbitrary number of factors of $a^\dagger$ and a. We may express this average in terms of the function $\chi_N(\eta,t)$, by means of the formula

$$\mathrm{tr}\{\rho_A(t)a^{\dagger n}a^m\}=\left(\frac{\partial}{\partial\eta}\right)^n\left(-\frac{\partial}{\partial\eta^*}\right)^m\chi_N(\eta,t)\bigg|_{\eta=\eta^*=0}, \quad (6.2)$$

which may be deduced by differentiating Eq. (4.8) directly. If we substitute Eq. (5.6) for $\chi_N(\eta,t)$ into Eq. (6.2), we find

$$\mathrm{tr}\{\rho_A(t)a^{\dagger n}a^m\}=\left(\frac{\partial}{\partial\eta}\right)^n\left(-\frac{\partial}{\partial\eta^*}\right)^m$$

$$\exp[\eta\bar{\alpha}^*(t)-\eta^*\bar{\alpha}(t)-\eta\eta^*s^2(t)]\bigg|_{\eta=\eta^*=0}. \quad (6.3)$$

We show in the Appendix that an expression of this form may be reduced, apart from some simple factors, to an associated Laguerre polynomial. By making use of Eqs. (A6) and (A7) of the Appendix we obtain for the average in Eq. (6.3) the two equivalent expressions

$$\mathrm{tr}\{\rho_A(t)a^{\dagger n}a^m\}$$

$$=n![s^2(t)]^n[\bar{\alpha}(t)]^{m-n}L_n^{(m-n)}\left(-\frac{|\bar{\alpha}(t)|^2}{s^2(t)}\right) \quad (6.4)$$

$$=m![s^2(t)]^m[\bar{\alpha}^*(t)]^{n-m}L_m^{(n-m)}\left(-\frac{|\bar{\alpha}(t)|^2}{s^2(t)}\right). \quad (6.5)$$

It follows from the easily proved identity

$$a^{\dagger n}a^n=a^\dagger a(a^\dagger a-1)\cdots(a^\dagger a-n+1)$$

$$\equiv\frac{(a^\dagger a)!}{(a^\dagger a-n)!}$$

that the factorial moments of the quantum distribution in the mode, which are obtained by evaluating Eq. (6.4) or Eq. (6.5) at $n=m$, are given by

$$\mathrm{tr}\left\{\rho_A(t)\frac{(a^\dagger a)!}{(a^\dagger a-n)!}\right\}=n![s^2(t)]^nL_n\left(-\frac{|\bar{\alpha}(t)|^2}{s^2(t)}\right). \quad (6.6)$$

For the case of vacuum amplification, i.e., when no quanta are initially present in either mode, the quantity $\mathrm{tr}\{\rho_A(t)a^{\dagger n}a^m\}$ is evaluated by letting $\bar{\alpha}(t)\to 0$, in either of the relations (6.4) or (6.5). By making use of the identity $L_n(0)=1$ we find

$$\mathrm{tr}\{\rho_A(t)a^{\dagger n}a^m\}=\delta_{nm}n![s^2(t)]^n. \quad (6.7)$$

These expectation values vanish for $n\neq m$ since the phase of the field is completely uncertain.

The matrix elements of $\rho_A(t)$ in the n-quantum representation are easily evaluated. By multiplying Eq. (4.9) on the left by $\langle m|$ and on the right by $|n\rangle$, we obtain the relation

$$\langle m|\rho_A(t)|n\rangle=\int\langle m|\alpha\rangle\langle\alpha|n\rangle P(\alpha,t)d^2\alpha$$

$$=\int\frac{\alpha^m\alpha^{*n}}{(m!n!)^{1/2}}P(\alpha,t)e^{-|\alpha|^2}d^2\alpha. \quad (6.8)$$

If we define a generating function $R(w,z,t)$ by the equation[17]

$$R(w,z,t)=\int P(\alpha,t)e^{-|\alpha|^2+w\alpha+z\alpha^*}d^2\alpha\,, \tag{6.9}$$

we find, on differentiating this expression and comparing the result to Eq. (6.8), that the matrix elements of $\rho_A(t)$ are given by

$$\langle m|\rho_A(t)|n\rangle=(m!n!)^{-1/2}\times\left(\frac{\partial}{\partial w}\right)^m\left(\frac{\partial}{\partial z}\right)^n R(w,z,t)\Big|_{w=z=0}. \tag{6.10}$$

If we substitute Eq. (5.9) for $P(\alpha,t)$ into Eq. (6.9), we find that the desired generating function is given by

$$R(w,z,t)=\frac{1}{\pi s^2(t)}\int \exp\left\{-\frac{|\alpha-\bar{\alpha}(t)|^2}{s^2(t)}-|\alpha|^2+w\alpha+z\alpha^*\right\}d^2\alpha. \tag{6.11}$$

This integral may be evaluated straightforwardly with the aid of the identity (5.8). We find then

$$R(w,z,t)=\frac{1}{c^2(t)}\exp\left\{-\frac{|\bar{\alpha}(t)|^2}{c^2(t)}+w\left(\frac{\bar{\alpha}(t)}{c^2(t)}\right)+z\left(\frac{\bar{\alpha}^*(t)}{c^2(t)}\right)+wz\left(\frac{s^2(t)}{c^2(t)}\right)\right\}. \tag{6.12}$$

The derivatives of this expression, which according to Eq. (6.10) correspond to the matrix elements of $\rho_A(t)$, may be expressed in terms of associated Laguerre polynomials much as in the case of the derivatives we discussed in Eq. (6.3). By using Eqs. (A6) and (A7) we find for the matrix elements of $\rho_A(t)$ the two equivalent expressions

$$\langle m|\rho_A(t)|n\rangle=\exp\left[-\frac{|\bar{\alpha}(t)|^2}{c^2(t)}\right]\frac{[s^2(t)]^m}{[c^2(t)]^{n+1}}[\bar{\alpha}^*(t)]^{n-m}\times\left(\frac{m!}{n!}\right)^{1/2}L_m^{(n-m)}\left(-\frac{|\bar{\alpha}(t)|^2}{s^2(t)c^2(t)}\right) \tag{6.13}$$

$$=\exp\left[-\frac{|\bar{\alpha}(t)|^2}{c^2(t)}\right]\frac{[s^2(t)]^n}{[c^2(t)]^{m+1}}[\bar{\alpha}(t)]^{m-n}\times\left(\frac{n!}{m!}\right)^{1/2}L_n^{(m-n)}\left(-\frac{|\bar{\alpha}(t)|^2}{s^2(t)c^2(t)}\right). \tag{6.14}$$

The right-hand side of Eq. (6.14) may be obtained from the right-hand side of Eq. (6.13) by taking the complex conjugate of the latter and interchanging n and m; this relation is a reflection of the Hermiticity requirement

$$\langle m|\rho_A(t)|n\rangle=(\langle n|\rho_A(t)|m\rangle)^*. \tag{6.15}$$

The probability of finding n quanta in the A mode at time t, which is obtained by evaluating Eq. (6.13) or Eq. (6.14) for $n=m$, is

$$\langle n|\rho_A(t)|n\rangle=\exp\left[-\frac{|\bar{\alpha}(t)|^2}{c^2(t)}\right]\frac{[s^2(t)]^n}{[c^2(t)]^{n+1}}\times L_n\left(-\frac{|\bar{\alpha}(t)|^2}{s^2(t)c^2(t)}\right). \tag{6.16}$$

For the case of vacuum amplification, the matrix elements of $\rho_A(t)$ in the n-quantum representation are given by Eq. (6.13) or Eq. (6.14) evaluated at $\bar{\alpha}(t)=0$:

$$\langle m|\rho_A(t)|n\rangle=\delta_{nm}\frac{[s^2(t)]^n}{[c^2(t)]^{n+1}}, \tag{6.17}$$

which is equivalent to the result already noted in Eq. (5.14).

We need not confine our discussion of the density operator $\rho_A(t)$ to the form it takes in the P representation or to its matrix elements in terms of n-quantum states. It is a simple matter to give an explicit construction of the operator itself. For the case of vacuum amplification, for example, $\rho_A(t)$ may be obtained by expressing Eq. (5.14) formally as follows:

$$\rho_A(t)=\frac{1}{c^2(t)}\left[\frac{s^2(t)}{c^2(t)}\right]^{a^\dagger a} \tag{6.18}$$

$$=\frac{1}{c^2(t)}\exp\left[a^\dagger a\ln\left(\frac{s^2(t)}{c^2(t)}\right)\right]. \tag{6.19}$$

If the initial state of the system is an arbitrary coherent state, it is easy to show that $\rho_A(t)$ is given by Eq. (6.19), with a replaced by $a-\bar{\alpha}(t)$ and $a^\dagger$ by $a^\dagger-\bar{\alpha}^*(t)$:

$$\rho_A(t)=\frac{1}{c^2(t)}\exp\left[[a^\dagger-\bar{\alpha}^*(t)][a-\bar{\alpha}(t)]\times\ln\left(\frac{s^2(t)}{c^2(t)}\right)\right]. \tag{6.20}$$

This result applies to the initial density operator (5.1). If the initial density operator for the system has the P representation (5.10), then the linearity of the dependence of $\rho_A(t)$ on ρ enables us to see that $\rho_A(t)$ is given more generally by

$$\rho_A(t)=\frac{1}{c^2(t)}\int\exp\left[[a^\dagger-\bar{\alpha}^*(\alpha_0,\beta_0,t)]\times[a-\bar{\alpha}(\alpha_0,\beta_0,t)]\ln\left(\frac{s^2(t)}{c^2(t)}\right)\right]P(\alpha_0,\beta_0)d^2\alpha_0d^2\beta_0, \tag{6.21}$$

where $\bar{\alpha}(\alpha_0,\beta_0,t)$ is defined by Eq. (3.23a).

VII. SOLUTIONS FOR AN INITIALLY CHAOTIC B MODE

Cases in which one of the modes is in a chaotically mixed state (e.g., a thermal equilibrium distribution) are important from a practical standpoint and relatively simple to treat. In this section we shall consider the behavior of the A mode of the amplifier when the initial state of the B mode is chaotic in character. The simplest such case is the one in which both the A and B modes are initially in chaotic states. Other cases we shall discuss are those in which the A mode is initially in the vacuum state or a coherent state, and cases in which its initial state is specified more generally by a P representation.

If the initial states of the A and B modes are independent chaotic mixtures with mean quantum numbers $\langle n\rangle$ and $\langle m\rangle$, respectively, we may write the initial density operator in the form

$$\rho_c=\int P_c(\alpha_0,\beta_0)\,|\alpha_0,\beta_0\rangle\langle\alpha_0,\beta_0|\,d^2\alpha_0 d^2\beta_0\,, \tag{7.1}$$

where

$$P_c(\alpha_0,\beta_0)=\frac{1}{\pi^2\langle n\rangle\langle m\rangle}\exp\left[-\frac{|\alpha_0|^2}{\langle n\rangle}-\frac{|\beta_0|^2}{\langle m\rangle}\right]. \tag{7.2}$$

One way of finding the function $P(\alpha,t)$ is to substitute this Gaussian expression into the formula given by Eq. (5.11) and evaluate the resulting integral. An equivalent and somewhat simpler procedure is to find $P(\alpha,t)$ by beginning once again with the normally ordered characteristic function. That function, it is clear, may be obtained by averaging the result in Eq. (5.5) with respect to the weight function $P_c(\alpha_0,\beta_0)$. We have then

$$\chi_{Nc}(\eta,t)=\int \exp[-|\eta|^2s^2(t)+\alpha_0{}^*\eta c_a{}^*(t)-\beta_0{}^*\eta^*s_a(t)$$
$$-\alpha_0\eta^*c_a(t)+\beta_0\eta s_a{}^*(t)]P_c(\alpha_0,\beta_0)d^2\alpha_0d^2\beta_0\,. \tag{7.3}$$

By substituting the Gaussian form for $P_c(\alpha_0,\beta_0)$ given by Eq. (7.2) into this expression and performing the integration with the aid of Eq. (5.8), we obtain

$$\chi_{Nc}(\eta,t)=e^{-|\eta|^2N(t)}\,, \tag{7.4}$$

where

$$N(t)=\langle n\rangle c^2(t)+(1+\langle m\rangle)s^2(t)\,. \tag{7.5}$$

The function P for the A mode at time t is evaluated as the Fourier transform of $\chi_{Nc}(\eta,t)$:

$$P_c(\alpha,t)=\pi^{-2}\int e^{-|\eta|^2N(t)+\alpha\eta^*-\alpha^*\eta}\,d^2\eta$$
$$=\frac{1}{\pi N(t)}\exp\left[-\frac{|\alpha|^2}{N(t)}\right]. \tag{7.6}$$

The reduced Schrödinger density operator $\rho_{Ac}(t)$ for the A mode thus corresponds to a chaotic mixture with mean quantum number $N(t)$. For $\langle n\rangle=\langle m\rangle=0$ the joint system is initially in the vacuum state; for this case $N(t)=s^2(t)$, and Eq. (7.6) becomes identical to the result (5.12) found earlier for the case of vacuum amplification. The effect of the chaotic fields initially present in both the A and B modes is to increase the fluctuations in the field strength of the A mode at time t, from $s^2(t)$ to $N(t)$.

Holliday and Glassgold[16] have discussed an approximate model of laser amplification described by equations which may be cast in a form similar to those for the parametric amplifier. The B mode of oscillation in their model represents formally the effect of the pumping molecules; its state is taken to be a chaotic mixture. They show that if the initial state of the field mode in their model is also taken to be chaotic, then the function $P(\alpha,t)$ which describes it takes the form of Eq. (7.6).

An explicit expression for the density operator $\rho_{Ac}(t)$ may be obtained by replacing $s^2(t)$ in Eq. (6.19) by $N(t)$ (and $c^2(t)$ by $N(t)+1$). Formulas for the moments and n-quantum state matrix elements of $\rho_{Ac}(t)$ are obtained by making the same substitutions in Eqs. (6.7) and (6.17), respectively. The probability of finding l quanta in the A mode at time t, for the initially chaotic mixture (7.1), is

$$p(l,t|\langle n\rangle,\langle m\rangle)=\frac{[N(t)]^l}{[1+N(t)]^{l+1}}. \tag{7.7}$$

The choice $\langle n\rangle=0$ in Eq. (7.2) implies that the initial state of the A mode is the ground state $|0\rangle_A$. The initial density operator for the joint system in this case may be written in the form

$$\rho=|0\rangle_A\,{}_A\langle 0|\quad \rho_{B,\langle m\rangle}\,, \tag{7.8}$$

in which the chaotic density operator $\rho_{B,\langle m\rangle}$ for the B mode is defined by

$$\rho_{B,\langle m\rangle}=\frac{1}{\pi\langle m\rangle}\int \exp\left[-\frac{|\beta_0|^2}{\langle m\rangle}\right]|\beta_0\rangle\langle\beta_0|\,d^2\beta_0\,. \tag{7.9}$$

To obtain the function $P(\alpha,t)$ which corresponds to the initial state (7.8), we substitute $\langle n\rangle=0$ in Eqs. (7.5) and (7.6). We thereby find

$$P(\alpha,t)=\frac{1}{\pi(1+\langle m\rangle)s^2(t)}\exp\left[-\frac{|\alpha|^2}{(1+\langle m\rangle)s^2(t)}\right]. \tag{7.10}$$

If we begin with the A mode in the coherent state $|\alpha'\rangle$ and the B mode in the chaotic mixture $\rho_{B,\langle m\rangle}$, then the initial density operator for the system is

$$\rho=|\alpha'\rangle\langle\alpha'|\,\rho_{B,\langle m\rangle}\,, \tag{7.11}$$

which has the P representation

$$P(\alpha_0,\beta_0)=\delta^{(2)}(\alpha_0-\alpha')\frac{1}{\pi\langle m\rangle}\exp\left[-\frac{|\beta_0|^2}{\langle m\rangle}\right]. \quad (7.12)$$

The normally ordered characteristic function for the A mode at time t may be obtained by evaluating the integral in Eq. (7.3), with $P_c(\alpha_0,\beta_0)$ replaced by $P(\alpha_0,\beta_0)$ as defined by Eq. (7.12). We find

$$\chi_N(\eta,t)=\exp[\eta\alpha'^*c_a^*(t)-\eta^*\alpha' c_a(t) - |\eta|^2(1+\langle m\rangle)s^2(t)]. \quad (7.13)$$

By substituting this relation into Eq. (4.10) and performing the integration with the aid of the integral identity (5.8) we find that the function P for the A mode at time t is given by[16]

$$P(\alpha,t)=\frac{1}{\pi(1+\langle m\rangle)s^2(t)}\exp\left\{-\frac{|\alpha-\alpha' c_a(t)|^2}{(1+\langle m\rangle)s^2(t)}\right\}. \quad (7.14)$$

Thus the effect of choosing the initial state of the A mode to be the coherent state $|\alpha'\rangle$ rather than the ground-state $|0\rangle_A$ is to shift the mean complex amplitude of the distribution from zero to $\alpha' c_a(t)$; the variance of the distribution remains unchanged.

The result (7.14) may be immediately generalized to the case in which the A mode initially has an arbitrary P representation. If we assume that the initial state of the B mode is the chaotic mixture $\rho_{B,<m>}$, then the initial density operator for the system takes the form

$$\rho=\int P(\alpha',0)|\alpha'\rangle\langle\alpha'|d^2\alpha'\ \rho_{B,<m>}, \quad (7.15)$$

where $P(\alpha',0)$ is the initial P function for the A mode. The value of this function at time t may be obtained by averaging Eq. (7.14) with respect to the weight function $P(\alpha',0)$. We thus have

$$P(\alpha,t)=\frac{1}{\pi(1+\langle m\rangle)s^2(t)} \times\int\exp\left[-\frac{|\alpha-\alpha' c_a(t)|^2}{(1+\langle m\rangle)s^2(t)}\right]P(\alpha',0)d^2\alpha'. \quad (7.16)$$

The effect of amplification upon the function $P(\alpha,t)$ may thus be expressed quite generally by means of a convolution transform with a Gaussian kernel.

It is interesting to compare the asymptotic form of $P(\alpha,t)$ for large times, $\kappa t\gg 1$, with the probability distribution for the amplitude of the field which would emerge from a purely classical linear amplifier taken to have the same amplitude gain factor, $c_a(t)\sim\frac{1}{2}e^{\kappa t-i\omega_a t}$. If the probability distribution for the input amplitude α in the classical linear amplifier were $p(\alpha)$, then the probability distribution for the output amplitude at large times would be

$$P(\alpha,t)\approx 4e^{-2\kappa t}p(2\alpha e^{i\omega_a t-\kappa t}), \quad (7.17)$$

if the amplification process were noiseless. By comparing this equation with Eq. (7.16) for $\kappa t\gg 1$ we see that the output of the quantum-mechanical parametric amplifier may be represented by thinking of the classical linear amplifier as having an input amplitude distribution

$$p(\alpha)=\frac{1}{\pi(1+\langle m\rangle)}\int\exp\left[-\frac{|\alpha-\alpha'|^2}{1+\langle m\rangle}\right]P(\alpha',0)d^2\alpha'. \quad (7.18)$$

The distribution represented by this convolution integral corresponds to the superposition of two fields. One of these fields is the true input or signal field represented by $P(\alpha',0)$, and the other is a chaotic field with mean quantum number $1+\langle m\rangle$. The effect of noise in the quantum amplifier is thus equivalent to the addition at the input of the classical amplifier of an intensity of Gaussian noise corresponding to $1+\langle m\rangle$ quanta. The $\langle m\rangle$ quanta of noise are contributed by the chaotic nature of the excitation of the B mode. The single quantum of input noise which remains when $\langle m\rangle=0$ represents the unavoidable quantum noise[6] intrinsic to the amplification process.

VIII. SOLUTION FOR INITIAL n-QUANTUM STATE OF A MODE; B MODE CHAOTIC

The density operator for the case in which both modes are initially in chaotic mixtures can be used to generate the density operators for cases in which the initial quantum states of the modes are specified more closely. Let us assume, for example, that the A mode is initially in the n-quantum state $|n\rangle_A$. Then if we again assume that the B mode is initially in a chaotic mixture with mean quantum number $\langle m\rangle$, the system is described at $t=0$ by the density operator

$$\rho_{n,c}=|n\rangle_A\,{}_A\langle n|\rho_{B,<m>}, \quad (8.1)$$

where $\rho_{B,<m>}$ is defined by Eq. (7.9).

Let us now recall that the initial density operator ρ_c is the product of density operators representing individual chaotic mixtures for each of the two modes:

$$\rho_c=\rho_{A,<n>}\rho_{B,<m>}. \quad (8.2)$$

The operator $\rho_{A,<n>}$ is defined by an expression similar to Eq. (7.9) but involving the variables of the A mode. In the n-quantum representation $\rho_{A,<n>}$ takes the form

$$\rho_{A,<n>}=\frac{1}{1+\langle n\rangle}\sum_{n=0}^{\infty}\left[\frac{\langle n\rangle}{1+\langle n\rangle}\right]^n|n\rangle_A\,{}_A\langle n|. \quad (8.3)$$

It is convenient at this point to introduce a parameter x defined as

$$x=\frac{\langle n\rangle}{1+\langle n\rangle}, \quad (8.4)$$

which permits us to write the density operator $\rho_{A,\langle n\rangle}$ in the more compact form

$$\rho_{A,\langle n\rangle}=(1-x)\sum_{n=0}^{\infty}x^n|n\rangle_A\,{}_A\langle n|\,. \tag{8.5}$$

If we now construct the operator ρ_c by substituting the form for $\rho_{A,\langle n\rangle}$ given by Eq. (8.5) into Eq. (8.2), and make use of the definition stated in Eq. (8.1), we find the identity

$$\rho_c=(1-x)\sum_{n=0}^{\infty}x^n\rho_{n,c}. \tag{8.6}$$

This equation expresses the relationship between the initial density operators ρ_c and $\rho_{n,c}$, in terms of the parameter x. Since the time-dependent density operator is linearly related to its initial value, it follows that the density operators at time t corresponding to the initial values ρ_c and $\rho_{n,c}$, respectively, satisfy a relation identical to Eq. (8.6); the symbol x is just a constant parameter. Quantities linearly related to the density operator, such as the weight function P or the mean values of dynamical operators, obey similar identities. If we know these quantities for the initially chaotic state, we can find them for the state in which the A mode begins with a specified number of quanta by substituting

$$\langle n\rangle=x/(1-x) \tag{8.7}$$

into the known solutions, and identifying the coefficients in the expansion of these solutions in powers of x.

Let us consider, as an example of some interest, the P function for the A mode at time t, corresponding to the initial density operator $\rho_{n,c}$; we shall denote this function by $P_{nc}(\alpha,t)$. It follows from Eqs. (7.6) and (8.6) that $P_{nc}(\alpha,t)$ satisfies the relation

$$\frac{1}{\pi N(x,t)}\exp\left[-\frac{|\alpha|^2}{N(x,t)}\right]=(1-x)\sum_{n=0}^{\infty}x^nP_{nc}(\alpha,t)\,, \tag{8.8}$$

in which $N(x,t)$ is obtained by substituting Eq. (8.7) for $\langle n\rangle$ into Eq. (7.5):

$$N(x,t)=[x/(1-x)]c^2(t)+(1+\langle m\rangle)s^2(t)\,. \tag{8.9}$$

If we insert this expression in Eq. (8.8) and divide by $1-x$, we obtain

$$\pi^{-1}\{x[1-\langle m\rangle s^2(t)]+(1+\langle m\rangle)s^2(t)\}^{-1}\cdot\exp\left[-\frac{|\alpha|^2(1-x)}{x[1-\langle m\rangle s^2(t)]+(1+\langle m\rangle)s^2(t)}\right]=\sum_{n=0}^{\infty}x^nP_{nc}(\alpha,t)\,. \tag{8.10}$$

The function $P_{nc}(\alpha,t)$ is thus the coefficient of x^n in the power series expansion of the left-hand side of Eq. (8.10). This expression is in fact very closely related to the familiar generating function for the Laguerre polynomials. We may transform it into this generating function by separating from it the factor

$$\mathfrak{M}(\alpha,t)=[\pi(1+\langle m\rangle)s^2(t)]^{-1}\times\exp\{-|\alpha|^2/(1+\langle m\rangle)s^2(t)\}\,, \tag{8.11}$$

and introducing the parameter z through the scale transformation

$$z=\mu(t)x\,, \tag{8.12}$$

where

$$\mu(t)=\frac{\langle m\rangle s^2(t)-1}{(1+\langle m\rangle)s^2(t)}. \tag{8.13}$$

We may then make use of the generating expansion[37]

$$\frac{1}{1-z}e^{-yz/(1-z)}=\sum_{n=0}^{\infty}z^nL_n(y) \tag{8.14}$$

by letting the parameter y be

$$y(\alpha,t)=\frac{|\alpha|^2c^2(t)}{(1+\langle m\rangle)s^2(t)[1-\langle m\rangle s^2(t)]}. \tag{8.15}$$

If we express Eq. (8.10) in terms of the variable z rather than x, then on equating coefficients of z^n in Eqs. (8.10) and (8.14) we find[16]

$$P_{nc}(\alpha,t)=\mathfrak{M}(\alpha,t)[\mu(t)]^nL_n[y(\alpha,t)]\,, \tag{8.16}$$

where the functions $\mathfrak{M}(\alpha,t)$, $\mu(t)$, and $y(\alpha,t)$ are defined by Eqs. (8.11), (8.13), and (8.15), respectively.

One of the senses in which the function $P_{nc}(\alpha,t)$ differs from a probability density is immediately evident. The Laguerre polynomial $L_n(y)$ has n real roots which are simple and positive. As long as the function $y(\alpha,t)$ is positive, the function $P_{nc}(\alpha,t)$ must take on negative values in concentric rings of the α plane which lie between pairs of zeros. As t approaches zero, $P_{nc}(\alpha,t)$ becomes a highly singular, rapidly oscillating function which vanishes everywhere except within an infinitesimal neighborhood of $\alpha=0$.

Although the function $P_{nc}(\alpha,t)$ is not a probability density, we may expect it, in general, to approximate one when the fields we are dealing with are strong enough to be described in classical terms. The linear amplification process we are considering leads by its very nature to fields of arbitrarily great strength when it continues for a sufficiently long time. It is reasonable, therefore, to expect the functions P to approach classical probability densities in the limit of large times.

As far as the sign of $P_{nc}(\alpha,t)$ is concerned this expectation is easily shown to be justified. For $\langle m\rangle\neq 0$ the function $y(\alpha,t)$ defined by Eq. (8.15) is positive for

[37] *Handbook of Mathematical Functions*, edited by M. Abramovitz and Irene A. Stegun (National Bureau of Standards Applied Mathematics Series 55, U. S. Government Printing and Publishing Office, Washington, D. C., 1964), p. 784.

times smaller than the root of the equation $s(t) = (\langle m\rangle)^{-1/2}$. For all larger times $y(\alpha,t)$ is negative-valued and $\mu(t)$ is positive-valued; since $L_n(y)\geq 0$ for $y\leq 0$, it follows from Eq. (8.16) that $P_{nc}(\alpha,t)\geq 0$. Thus for $\langle m\rangle\neq 0$ the amplification process does lead before too long a time to a non-negative function P.

In the particular case $\langle m\rangle=0$, which corresponds to the B mode beginning in its ground state, the behavior of the function P is somewhat different. In this case Eq. (8.16) reduces to

$$P_{nc}(\alpha,t)=\frac{1}{\pi s^2(t)}\exp\left[-\frac{|\alpha|^2}{s^2(t)}\right]\times\left(-\frac{1}{s^2(t)}\right)^n L_n\left(|\alpha|^2\frac{c^2(t)}{s^2(t)}\right). \quad (8.17)$$

As the time t increases the annular regions of the α plane in which P is negative do not shrink to zero radius as they do for $\langle m\rangle\neq 0$. Instead they become asymptotically fixed in radius and the function $P_{nc}(\alpha,t)$ always takes on negative values as well as positive ones. It is clear, however, from the nature of the α dependence of the factor multiplying the Laguerre polynomial in Eq. (8.17), that for large times, i.e., when $s^2(t)$ greatly exceeds the largest root of $L_n(y)$, the negative values of P will be small in magnitude compared with the positive values which occur for $|\alpha|\sim s(t)$.

The answer to the question whether the function P approaches a classical probability density in the limit of large times depends in this case on the physical nature of the quantities we are investigating. If we wish to find the mean value of an operator F for which the expectation value $\langle\alpha|F|\alpha\rangle$ assumes its most significant values for $|\alpha|^2$ smaller than the largest root of L_n, then the nonclassical character of the function P given by Eq. (8.17) will in general be quite significant. If on the other hand we seek the average of an operator F for which $\langle\alpha|F|\alpha\rangle$ assumes its most significant values for large $|\alpha|$, e.g., $F=a^\dagger a$, and hence $\langle\alpha|F|\alpha\rangle=|\alpha|^2$, then $P_{nc}(\alpha,t)$ can be accurately approximated for large t by the positive function

$$P_{nc}(\alpha,t)\approx\frac{1}{n!\pi s^2(t)}\left[\frac{|\alpha|^2}{s^2(t)}\right]^n\exp\left[-\frac{|\alpha|^2}{s^2(t)}\right], \quad (8.18)$$

which is obtained by approximating the Laguerre polynomial by its dominant term.

As a further example of the use of the generating function technique let us now calculate the probability of finding l quanta in the A mode at time t, given that the system is initially described by the density operator $\rho_{n,c}$ defined by Eq. (8.1). If we write this probability as $p(l,t|n,\langle m\rangle)$, and recall that $p(l,t|\langle n\rangle,\langle m\rangle)$ represents the probability of finding l quanta in the A mode at time t when the two modes are initially described by ρ_c, then these probabilities must by virtue of Eq. (8.6) satisfy the identity

$$p(l,t|\langle n\rangle,\langle m\rangle)=(1-x)\sum_{n=0}^{\infty}x^n p(l,t|n,\langle m\rangle). \quad (8.19)$$

The probability $p(l,t|\langle n\rangle,\langle m\rangle)$ may be expressed as a function of x by replacing the quantity $N(t)$ in Eq. (7.7) by the function $N(x,t)$ defined by Eq. (8.9), so that we have

$$p(l,t|\langle n\rangle,\langle m\rangle)=\frac{[N(x,t)]^l}{[1+N(x,t)]^{l+1}}. \quad (8.20)$$

We next substitute the expression (8.9) for $N(x,t)$ into Eq. (8.20), and equate the result to the summation in Eq. (8.19). To express the identity derived in this way more compactly it is convenient to introduce the functions

$$\mathcal{E}(t)=1-\langle m\rangle s^2(t), \quad (8.21)$$

$$\mathfrak{F}(t)=(1+\langle m\rangle)s^2(t), \quad (8.22)$$

$$\mathcal{S}(t)=-\langle m\rangle s^2(t), \quad (8.23)$$

and

$$\mathcal{T}(t)=c^2(t)+\langle m\rangle s^2(t). \quad (8.24)$$

Then the identity found by equating the expressions in Eqs. (8.19) and (8.20) may be written in the form

$$\frac{[\mathcal{E}(t)x+\mathfrak{F}(t)]^l}{[\mathcal{S}(t)x+\mathcal{T}(t)]^{l+1}}=\sum_{n=0}^{\infty}x^n p(l,t|n,\langle m\rangle). \quad (8.25)$$

The probability $p(l,t|n,\langle m\rangle)$ may be solved for by evaluating the coefficient of x^n in the power series expansion of the rational fraction on the left side of Eq. (8.25).

As a simple illustration, let us consider the case $\langle m\rangle=0$, for which Eq. (8.25) takes the form

$$\frac{[x+s^2(t)]^l}{[c^2(t)]^{l+1}}=\sum_{n=0}^{\infty}x^n p(l,t|n,0). \quad (8.26)$$

By equating coefficients of x^n, we find

$$p(l,t|n,0)=\binom{l}{n}\frac{[s^2(t)]^{l-n}}{[c^2(t)]^{l+1}} \quad \text{for } l\geq n \quad (8.27)$$

$$=0 \text{ for } l<n. \quad (8.28)$$

It is interesting to note that for this case the A mode can never have fewer than n quanta. The reason for this behavior is indicated by the conservation law stated in Eq. (3.9). For the case $\langle m\rangle=0$ the modes are initially in quantum-number eigenstates, and the operator $a^\dagger a-b^\dagger b$ has the eigenvalue n. The state of the system must retain this eigenvalue for the quantum-number difference at all later times, even though the two modes are no longer in states with well determined quantum numbers. The A mode cannot have fewer than n

quanta because in that case the B mode would have to have fewer than zero.

In the general case ($\langle m\rangle\neq 0$) the left-hand side of Eq. (8.25) can be expanded in powers of x without great difficulty. We find that the coefficient of x^n in this expansion may be expressed in terms of the associated Jacobi[38] polynomials $P_i^{(j,k)}(x)$ as follows:

$$p(l,t|n,\langle m\rangle)=\frac{[\mathcal{E}(t)]^l[-\mathcal{S}(t)]^{n-l}}{[\mathcal{T}(t)]^{n+1}}\times P_l^{(n-l,0)}\left[1-2\frac{\mathcal{F}(t)\mathcal{S}(t)}{\mathcal{E}(t)\mathcal{T}(t)}\right]\text{ for } n\geq l \quad (8.29)$$

$$=\frac{[\mathcal{E}(t)]^n[\mathcal{F}(t)]^{l-n}}{[\mathcal{T}(t)]^{l+1}}\times P_n^{(l-n,0)}\left[1-2\frac{\mathcal{F}(t)\mathcal{S}(t)}{\mathcal{E}(t)\mathcal{T}(t)}\right]\text{ for } n\leq l. \quad (8.30)$$

IX. GENERAL DISCUSSION OF AMPLIFICATION WITH B MODE INITIALLY CHAOTIC

Let us now generalize some of the considerations of the preceding sections. We continue to assume that the B mode is initially in the mixed state $\rho_{B,\langle m\rangle}$ defined by Eq. (7.9); the A mode, on the other hand, we take to be described initially by an arbitrary density operator ρ_A. Since the two modes are assumed to be initially independent, the Heisenberg density operator for the joint system is

$$\rho=\rho_A\rho_{B,\langle m\rangle}. \quad (9.1)$$

We have remarked that the amplification process leads to arbitrarily strong fields, and that it is therefore reasonable to expect a classical description of the fields to be valid in the limit of large times. As far as the behavior of the A mode is concerned, then, we may expect that the reduced density operator $\rho_A(t)$ will eventually be described by a P representation, and that this P representation will eventually become non-negative. We shall show that for the initial density operator defined by Eq. (9.1), a P representation for the A mode does indeed exist after a certain characteristic time, and does eventually become non-negative, unless $\langle m\rangle=0$.

We begin by considering the time-dependent form of the ordinary characteristic function for the A mode, which is defined by

$$\chi(\eta,t)=\mathrm{tr}_A\{\rho_A(t)e^{\eta a^\dagger-\eta^* a}\}. \quad (9.2)$$

The function $\chi(\eta,t)$ like $\chi_N(\eta,t)$, may be expressed in terms of the Heisenberg density operator ρ and the Heisenberg operators $a(t)$ and $a^\dagger(t)$. By steps precisely analogous to those leading to Eq. (4.12) for $\chi_N(\eta,t)$, we find

$$\chi(\eta,t)=\mathrm{tr}\{\rho e^{\eta a^\dagger(t)-\eta^* a(t)}\}. \quad (9.3)$$

If we substitute Eq. (3.14a) for $a(t)$ into Eq. (9.3), we find

$$\chi(\eta,t)=\mathrm{tr}\{\rho\exp[\eta(a^\dagger c_a^*(t)+bs_a^*(t))-\eta^*(ac_a(t)+b^\dagger s_a(t))]\}, \quad (9.4)$$

and if we then substitute Eq. (9.1) for ρ into this relation, we obtain the separated expression for the characteristic function

$$\begin{aligned}\chi(\eta,t)&=\mathrm{tr}_A\{\rho_A\exp[\eta c_a^*(t)a^\dagger-\eta^* c_a(t)a]\}\\&\quad\cdot\mathrm{tr}_B\{\rho_{B,\langle m\rangle}\exp[-\eta^* s_a(t)b^\dagger+\eta s_a^*(t)b]\}\\&=\chi(\eta c_a^*(t),0)\\&\quad\cdot\mathrm{tr}_B\{\rho_{B,\langle m\rangle}\exp[-\eta^* s_a(t)b^\dagger+\eta s_a^*(t)b]\}. \quad (9.5)\end{aligned}$$

The factor multiplying $\chi(\eta c_a^*(t),0)$ in the latter equation is just the ordinary characteristic function, evaluated at the complex argument $-\eta^* s_a(t)$, for the chaotic density operator $\rho_{B,\langle m\rangle}$. This function may be evaluated by using the expression (7.9) for $\rho_{B,\langle m\rangle}$ and employing the integral identity (5.8); we then find

$$\mathrm{tr}_B\{\rho_{B,\langle m\rangle}\exp[-\eta^* s_a(t)b^\dagger+\eta s_a^*(t)b]\}=\exp[-|\eta|^2 s^2(t)(\langle m\rangle+\tfrac{1}{2})]. \quad (9.6)$$

By substituting this relation into Eq. (9.5) and making use of Eq. (2.20), we find that the normally ordered characteristic function for the A mode at time t is

$$\chi_N(\eta,t)=\exp\{-|\eta|^2[s^2(t)(\langle m\rangle+\tfrac{1}{2})-\tfrac{1}{2}]\}\cdot\chi(\eta c_a^*(t),0). \quad (9.7)$$

In Sec. IV it was shown that a P representation for the A mode exists at time t if $\chi_N(\eta,t)$ has a Fourier transform. It can be shown, on the other hand, that the ordinary characteristic function corresponding to an arbitrary density operator necessarily has a well-defined Fourier transform. That transform is in fact the Wigner function[39] written with complex argument. It follows that when the coefficient in square brackets in the exponent of Eq. (9.7) is non-negative, a P representation for the A mode exists. We may define a characteristic time t_0 by the relation

$$s^2(t_0)(\langle m\rangle+\tfrac{1}{2})-\tfrac{1}{2}=0,$$

so that

$$\sinh^2\kappa t_0=\frac{1}{2\langle m\rangle+1}. \quad (9.8)$$

We have shown, then, that a P representation for the A mode must exist for $t\geq t_0$. For many initial states, of course, it will begin to exist at a time prior to $t=t_0$, and for some it will exist at all times $t\geq 0$.

In order to discuss the sign of the function P, we

[38] See Ref. 37, p. 775.

[39] See, for example, J. E. Moyal, Proc. Cambridge Phil. Soc. **45**, 99 (1948), or Ref. 21, Lecture XIII.

introduce the antinormally ordered characteristic function, which is defined by

$$\chi_{\text{ant.}}(\eta,t)=\text{tr}_A\{\rho_A(t)e^{-\eta^*a}e^{\eta a\dagger}\} \quad (9.9)$$

$$=e^{-\frac{1}{2}|\eta|^2}\chi(\eta,t). \quad (9.10)$$

If we insert the expression for the unit operator given by the completeness relation (2.9) between the exponentials in Eq. (9.9) and make use of Eq. (2.5), we find

$$\chi_{\text{ant.}}(\eta,t)=\int e^{\eta\alpha^*-\eta^*\alpha}[\pi^{-1}\langle\alpha|\rho_A(t)|\alpha\rangle]d^2\alpha. \quad (9.11)$$

The function $\chi_{\text{ant.}}(\eta,t)$ is thus the Fourier transform of the non-negative function $\pi^{-1}\langle\alpha|\rho_A(t)|\alpha\rangle$. The inverse of Eq. (9.11) is

$$\pi^{-1}\langle\alpha|\rho_A(t)|\alpha\rangle=\pi^{-2}\int e^{\alpha\eta^*-\alpha^*\eta}\chi_{\text{ant.}}(\eta,t)d^2\eta. \quad (9.12)$$

By evaluating Eq. (9.10) at $t=0$ and with η replaced by $\eta c_a^*(t)$, we find

$$\chi(\eta c_a^*(t),0)=e^{\frac{1}{2}|\eta|^2c^2(t)}\chi_{\text{ant.}}(\eta c_a^*(t),0), \quad (9.13)$$

and if we substitute this relation into Eq. (9.7), we obtain

$$\chi_N(\eta,t)=\exp\{-|\eta|^2[\langle m\rangle s^2(t)-1]\}\cdot\chi_{\text{ant.}}(\eta c_a^*(t),0). \quad (9.14)$$

We now define a second characteristic time t_1 by the relation

$$\sinh^2\kappa t_1=\frac{1}{\langle m\rangle} \quad (9.15)$$

which implies that

$$t_1>t_0,$$

and hence that a P representation for the A mode exists at time t_1. By evaluating Eq. (9.14) at $t=t_1$, we find

$$\chi_N(\eta,t_1)=\chi_{\text{ant.}}(\eta c_a^*(t_1),0). \quad (9.16)$$

If we substitute this expression for $\chi_N(\eta,t_1)$ into Eq. (4.10), we obtain

$$P(\alpha,t_1)=\pi^{-2}\int e^{\alpha\eta^*-\alpha^*\eta}\chi_{\text{ant.}}(\eta c_a^*(t_1),0)d^2\eta. \quad (9.17)$$

It is convenient, before performing the integration in this equation, to change the variable of integration from η to $\eta c_a^*(t_1)$, so that we have

$$P(\alpha,t_1)=\frac{1}{\pi^2c^2(t_1)}\times\int\exp\left[\frac{\alpha\eta^*}{c_a(t_1)}-\frac{\alpha^*\eta}{c_a^*(t_1)}\right]\chi_{\text{ant.}}(\eta,0)d^2\eta. \quad (9.18)$$

By evaluating Eq. (9.12) at $t=0$ and with α replaced by $\alpha/c_a(t_1)$, and then comparing the result to Eq. (9.18) we deduce that

$$P(\alpha,t_1)=[1/\pi c^2(t_1)]\times\langle\alpha[c_a(t_1)]^{-1}|\rho_A(0)|\alpha[c_a(t_1)]^{-1}\rangle. \quad (9.19)$$

Since the function $\langle\alpha|\rho|\alpha\rangle$ is intrinsically positive for an arbitrary density operator, it follows that the P function for the A mode is positive-definite at $t=t_1$.

It is not difficult to see that the P function for the A mode remains positive-definite for $t>t_1$. The proof may be outlined briefly as follows: For $t>t_1$ the coefficient in square brackets in the exponent in Eq. (9.14) is positive, and hence the exponential function in which it appears is the Fourier transform of a Gaussian function which is always real and positive. The Fourier transform of the right-hand side of Eq. (9.14) is therefore the convolution of this Gaussian function with the Fourier transform of $\chi_{\text{ant.}}(\eta c_a^*(t),0)$. We have shown that Fourier transform to be a diagonal matrix element of the density operator and hence to be positive-definite. It follows that for $t>t_1$, or $s^2(t)>(\langle m\rangle)^{-1}$, the function $P(\alpha,t)$ is positive-definite. It is worth noting that if the initial chaotic distribution in the B mode corresponds to thermal equilibrium, then in the high-temperature limit (or equivalently in the classical limit) the existence and positive-definiteness of the P representation for the A mode are guaranteed after an infinitesimally small time interval.

For the case $\langle m\rangle=0$, no conclusion about the sign of $P(\alpha,t)$ can be drawn. In Sec. VIII it was shown that for $\rho_A=|n\rangle_A\,{}_A\langle n|$ and $\langle m\rangle=0$, the function P for the A mode continues to take on negative values at all times.

X. DISCUSSION OF P REPRESENTATION: CHARACTERISTIC FUNCTIONS INITIALLY GAUSSIAN

The cases we examined in Sec. VII in which the characteristic functions were initially Gaussian in form were ones in which the P representation for the A mode exists at $t=0$ and at all later times. It is worth noting, therefore, that when the amplifier system begins in states which are described by somewhat more general Gaussian forms for the characteristic functions, the P representation only comes into existence at times $t>0$. These cases provide simple illustrations of some of the results derived in the preceding section.

Let us begin by discussing some of the properties of a single mode which is described by a Gaussian characteristic function. We define the real variables x and y which are proportional to the real and imaginary parts of η as

$$x=2^{-1/2}(\eta^*+\eta),\quad y=i2^{-1/2}(\eta^*-\eta). \quad (10.1)$$

A simple example of a characteristic function which takes a more general Gaussian form than those considered earlier is

$$\chi(\eta)=\exp[-\tfrac{1}{2}(y^2\mathcal{Q}^2+x^2\mathcal{P}^2)], \qquad (10.2)$$

where $\mathcal{Q}$ and $\mathcal{P}$ are a pair of real numbers which may be taken to be positive.

If we define the pair of Hermitian operators q and p via the relations

$$\begin{aligned} q&=2^{-1/2}(a^\dagger+a), \\ p&=i2^{-1/2}(a^\dagger-a), \end{aligned} \qquad (10.3)$$

then according to Eq. (2.13), the general expression for the characteristic function $\chi(\eta)$ may be written as

$$\chi(\eta)=\mathrm{tr}\{\rho e^{i(yq-xp)}\}. \qquad (10.4)$$

It is evident that when this function takes the form given in Eq. (10.2), the mean values of q^2 and p^2 are given by

$$\begin{aligned} \mathrm{tr}\{\rho q^2\}&=-\frac{\partial^2}{\partial y^2}\chi(\eta)\Big|_{\eta=0}=\mathcal{Q}^2, \\ \mathrm{tr}\{\rho p^2\}&=-\frac{\partial^2}{\partial x^2}\chi(\eta)\Big|_{\eta=0}=\mathcal{P}^2. \end{aligned} \qquad (10.5)$$

Since q and p satisfy the canonical commutation relation $[q,p]=i$, the second moments of these operators must satisfy the inequality

$$\mathcal{Q}^2\mathcal{P}^2\geq\tfrac{1}{4}, \qquad (10.6)$$

which corresponds to the Heisenberg uncertainty relation.

The normally ordered characteristic function, when it is expressed in terms of the real variables x and y, takes the form

$$\begin{aligned} \chi_N(\eta)&=e^{\frac{1}{2}|\eta|^2}\chi(\eta) \\ &=\exp\{-\tfrac{1}{2}[y^2(\mathcal{Q}^2-\tfrac{1}{2})+x^2(\mathcal{P}^2-\tfrac{1}{2})]\}. \end{aligned} \qquad (10.7)$$

This function becomes infinite with great rapidity as $|\eta|\to\infty$ unless the coefficients of x^2 and y^2 in the exponential are both negative, i.e., unless

$$\begin{aligned} \mathcal{Q}^2&\geq\tfrac{1}{2}, \\ \mathcal{P}^2&\geq\tfrac{1}{2}. \end{aligned} \qquad (10.8)$$

If these inequalities are satisfied, the function $\chi_N(\eta)$ possesses a two-dimensional Fourier transform which is the function $P(\alpha)$. If we introduce the real variables q' and p' via the relations

$$\begin{aligned} \alpha&=2^{-1/2}(q'+ip'), \\ \alpha^*&=2^{-1/2}(q'-ip'), \end{aligned} \qquad (10.9)$$

then we may write the function $P(\alpha)$ which corresponds to the Fourier integral (2.22) as

$$P(\alpha)=\pi^{-1}(\mathcal{Q}^2-\tfrac{1}{2})^{-1/2}(\mathcal{P}^2-\tfrac{1}{2})^{-1/2} \times\exp\left\{-\tfrac{1}{2}\left[\frac{q'^2}{\mathcal{Q}^2-\frac{1}{2}}+\frac{p'^2}{\mathcal{P}^2-\frac{1}{2}}\right]\right\}. \qquad (10.10)$$

In the limit in which $\mathcal{Q}^2\to\frac{1}{2}$ this function reduces to the one-dimensional δ function

$$P(\alpha)=(2/\pi)^{1/2}(\mathcal{P}^2-\tfrac{1}{2})^{-1/2}\delta(q') \times\exp\left\{-\tfrac{1}{2}\left[\frac{p'^2}{\mathcal{P}^2-\frac{1}{2}}\right]\right\}, \qquad (10.11)$$

and a corresponding result holds for $\mathcal{P}^2\to\frac{1}{2}$. When both $\mathcal{Q}^2$ and $\mathcal{P}^2$ approach $\frac{1}{2}$, the function $P(\alpha)$ reduces to

$$P(\alpha)=2\delta(q')\delta(p')=\delta^{(2)}(\alpha), \qquad (10.12)$$

which represents the ground state of the mode.

If either of the inequalities (10.8) fails to be satisfied, on the other hand, the function $\chi_N(\eta)$ increases so rapidly as $|\eta|\to\infty$ that no P representation exists. It is not difficult, of course, to find states for which either $\mathcal{Q}^2=\mathrm{tr}\{\rho q^2\}$ or $\mathcal{P}^2=\mathrm{tr}\{\rho p^2\}$ is less than $\frac{1}{2}$, and there is nothing unphysical about them. Indeed the condition $\mathcal{Q}^2\mathcal{P}^2=\frac{1}{4}$ leads uniquely to a family of minimum uncertainty states, one member of which corresponds to any positive value for $\mathcal{Q}^2$. The only case among these for which the P representation exists corresponds to $\mathcal{Q}^2=\mathcal{P}^2=\frac{1}{2}$, which specifies the ground state of the mode. In all other cases either $\mathcal{Q}^2$ or $\mathcal{P}^2$ is smaller than $\frac{1}{2}$, and the normally ordered characteristic function has no Fourier transform.

The arguments we have given regarding the existence of the P representation are not materially altered if the characteristic function specified by Eq. (10.2) is replaced by more general types of Gaussian functions. If, for example, we have

$$\chi(\eta)=\exp[-\tfrac{1}{2}(y^2\mathcal{Q}^2+x^2\mathcal{P}^2)+i(y\bar{q}-x\bar{p})] \qquad (10.13)$$

for some pair of real numbers $\bar{q}$ and $\bar{p}$, then we see from Eq. (10.4) that the mean values of q and p are just

$$\begin{aligned} \mathrm{tr}\{\rho q\}&=-i\frac{\partial}{\partial y}\chi(\eta)\Big|_{\eta=0}=\bar{q}, \\ \mathrm{tr}\{\rho p\}&=i\frac{\partial}{\partial x}\chi(\eta)\Big|_{\eta=0}=\bar{p}. \end{aligned} \qquad (10.14)$$

The variances of q and p are given by

$$\begin{aligned} \Delta q^2&=\mathrm{tr}\{\rho(q^2-\bar{q}^2)\} \\ &=-\frac{\partial^2}{\partial y^2}\ln\chi(\eta)\Big|_{\eta=0}=\mathcal{Q}^2, \end{aligned} \qquad (10.15a)$$

$$\begin{aligned} \Delta p^2&=\mathrm{tr}\{\rho(p^2-\bar{p}^2)\} \\ &=-\frac{\partial^2}{\partial x^2}\ln\chi(\eta)\Big|_{\eta=0}=\mathcal{P}^2. \end{aligned} \qquad (10.15b)$$

The condition that a Fourier transform exist for the normally ordered characteristic function is once again that both the inequalities (10.8) hold. When they do hold the function $P(\alpha)$ is given by a displaced form of the Gaussian function in Eq. (10.10) obtained by letting $q'\to q'-\bar{q}$ and $p'\to p'-\bar{p}$. Let us define the complex number $\bar{\alpha}$ as

$$\bar{\alpha}=\mathrm{tr}\{\rho a\}=2^{-1/2}(\bar{q}+i\bar{p}). \tag{10.16}$$

Then in the limit $\Delta q^2\to\frac{1}{2}$, $\Delta p^2\to\frac{1}{2}$, the function $P(\alpha)$ reduces to

$$P(\alpha)=2\delta(q'-\bar{q})\delta(p'-\bar{p})=\delta^{(2)}(\alpha-\bar{\alpha}), \tag{10.17}$$

which represents the pure coherent state $|\bar{\alpha}\rangle$. This state is but one member of an infinite family of minimum uncertainty states for which $\Delta q^2\Delta p^2=\mathcal{Q}^2\mathcal{P}^2=\frac{1}{4}$ and $\mathrm{tr}\{\rho a\}=\bar{\alpha}$. The fact that the other members of the set, for which $\mathcal{Q}^2\neq\mathcal{P}^2$, have no P representations in the basis we are using, simply expresses the fact that there is no way of mixing coherent states, which have $\Delta q^2=\Delta p^2=\frac{1}{2}$, to form states with smaller values of either Δq^2 or Δp^2 and minimum uncertainty $\Delta q^2\Delta p^2=\frac{1}{4}$.

A theorem which holds for all quantum states is perhaps worth noting at this point: A non-negative P representation can only exist for states for which the variances Δq^2 and Δp^2 obey the inequalities

$$\Delta q^2\geq\tfrac{1}{2},\quad \Delta p^2\geq\tfrac{1}{2}. \tag{10.18}$$

Indeed, if we calculate the variance Δq^2 in the P representation we find

$$\begin{aligned}\Delta q^2&=\mathrm{tr}\{\rho(q-\bar{q})^2\}\\&=\tfrac{1}{2}\,\mathrm{tr}\{\rho(a^\dagger+a-\bar{\alpha}^*-\bar{\alpha})^2\}\\&=\tfrac{1}{2}+\tfrac{1}{2}\int P(\alpha)|\alpha^*+\alpha-\bar{\alpha}^*-\bar{\alpha}|^2d^2\alpha,\end{aligned} \tag{10.19}$$

where the commutation relation has been used to reach the latter expression. We similarly find

$$\Delta p^2=\tfrac{1}{2}+\tfrac{1}{2}\int P(\alpha)|(\alpha^*-\alpha-\bar{\alpha}^*+\bar{\alpha})|^2d^2\alpha. \tag{10.20}$$

It is clear from these expressions that the variances obey the inequalities (10.18) as long as the function $P(\alpha)$ exists and takes on no negative values.

Quantum states of the field exist for all values of Δq^2 and Δp^2 for which $\Delta q^2\Delta p^2\geq\frac{1}{4}$. These states are indicated by the points in the $\Delta q^2-\Delta p^2$ plane lying within the hyperbola shown in Fig. 2. The states which correspond to points lying within the shaded regions between the hyperbola and the bounds $\Delta q^2=\frac{1}{2}$ and $\Delta p^2=\frac{1}{2}$ are the ones for which no positive-valued function $P(\alpha)$ can exist. In the examples we have considered, which correspond to Gaussian characteristic functions, no function $P(\alpha)$ exists at all in the sense discussed in Sec. II for the states lying within the shaded regions.

To illustrate the foregoing arguments in a dynamical context let us assume that the A mode of the parametric amplifier is initially in the state specified by the characteristic function (10.2) and that the B mode is

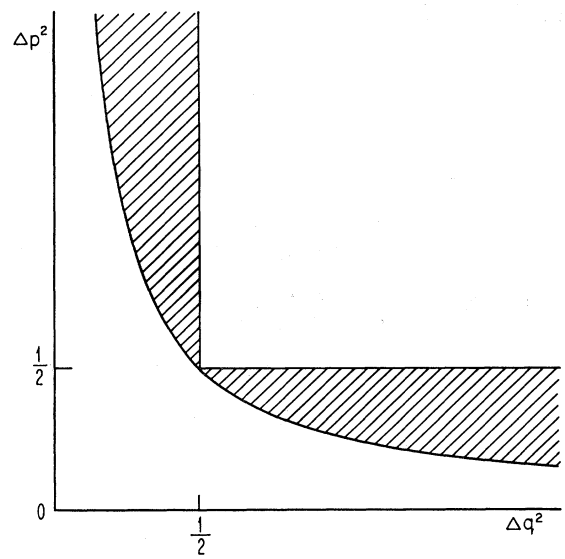

FIG. 2. Characterization of quantum states according to their coordinate and momentum uncertainties. The variables Δq^2 and Δp^2 represent the variances of the coordinate and the momentum, respectively, for an arbitrary state of a one-mode system. The hyperbola is defined by the equation $\Delta q^2\Delta p^2=\frac{1}{4}$. Points within the shaded region represent states allowed by the uncertainty relation, but for which no function P can exist which takes on only positive values.

initially in the chaotic mixture $\rho_{B,\langle m\rangle}$ defined by Eq. (7.9). Let us also assume that $\mathcal{Q}^2<\frac{1}{2}$ (and therefore $\mathcal{P}^2>\frac{1}{2}$), so that no P representation exists initially for the A mode. We may use Eq. (9.7) to evaluate the normally ordered characteristic function at times $t>0$. When the characteristic function is not circularly symmetric in the complex η plane, its behavior is most simply described in a frame of reference which rotates uniformly with angular velocity $-\omega_a$. If we therefore write the argument of χ_N as $\eta e^{-i\omega_a t}$, we find

$$\chi_N(\eta e^{-i\omega_a t},t)=\exp\{-\tfrac{1}{2}[y^2(\mathcal{Q}'^2(t)-\tfrac{1}{2})+x^2(\mathcal{P}'^2(t)-\tfrac{1}{2})]\}, \tag{10.21}$$

where

$$\begin{aligned}\mathcal{Q}'^2(t)&=\mathcal{Q}^2c^2(t)+(\langle m\rangle+\tfrac{1}{2})s^2(t),\\\mathcal{P}'^2(t)&=\mathcal{P}^2c^2(t)+(\langle m\rangle+\tfrac{1}{2})s^2(t),\end{aligned} \tag{10.22}$$

and x and y are defined by Eqs. (10.1).

Let us define the Hermitian operators $q(t)$ and $p(t)$ appropriate to a rotating coordinate system by the relation

$$a=2^{-1/2}[q(t)+ip(t)]e^{-i\omega_a t}, \tag{10.23}$$

so that we have

$$\begin{aligned}q(t)&=2^{-1/2}[a^\dagger e^{-i\omega_a t}+ae^{i\omega_a t}],\\p(t)&=i2^{-1/2}[a^\dagger e^{-i\omega_a t}-ae^{i\omega_a t}].\end{aligned} \tag{10.24}$$

Then it is easily shown that the functions $\mathcal{Q}'^2(t)$ and $\mathcal{P}'^2(t)$ are just the mean-squared values of $q(t)$ and $p(t)$, i.e., we have

$$\operatorname{tr}\{\rho(t)q^2(t)\}=\tfrac{1}{2}\operatorname{tr}\{\rho[a^\dagger(t)e^{-i\omega_a t}+a(t)e^{i\omega_a t}]^2\}$$
$$=\mathcal{Q}'^2(t) \tag{10.25}$$

and

$$\operatorname{tr}\{\rho(t)p^2(t)\}=\tfrac{1}{2}\operatorname{tr}\{-\rho[a^\dagger(t)e^{-i\omega_a t}-a(t)e^{i\omega_a t}]^2\}.$$
$$=\mathcal{P}'^2(t). \tag{10.26}$$

It is clear from Eq. (10.22) that these mean-squared values increase monotonically with time.

The function χ_N given by Eq. (10.21) will possess a two-dimensional Fourier transform only if $\mathcal{Q}'^2(t)\geq\frac{1}{2}$ and $\mathcal{P}'^2(t)\geq\frac{1}{2}$. The latter condition is satisfied at all times, since we have assumed that the initial value $\mathcal{P}^2>\frac{1}{2}$. The condition on $\mathcal{Q}'(t)$ implies that a P representation for the A mode only exists for times t satisfying the inequality

$$\sinh^2\kappa t\geq\frac{1-2\mathcal{Q}^2}{1+2\langle m\rangle+2\mathcal{Q}^2}. \tag{10.27}$$

For $\mathcal{Q}^2\neq 0$, then, a P representation comes into existence for the A mode at a time prior to the time t_0 defined by Eq. (9.8).

For times which satisfy the condition (10.27) the function P is given by

$$P(\alpha e^{-i\omega_a t},t)=\pi^{-1}[\mathcal{Q}'^2(t)-\tfrac{1}{2}]^{-1/2}[\mathcal{P}'^2(t)-\tfrac{1}{2}]^{-1/2}$$
$$\times\exp\left\{-\tfrac{1}{2}\left[\frac{q'^2}{\mathcal{Q}'^2(t)-\frac{1}{2}}+\frac{p'^2}{\mathcal{P}'^2(t)-\frac{1}{2}}\right]\right\}, \tag{10.28}$$

where q' and p' are the variables defined by the Eqs. (10.9). At the instant at which the inequality (10.27) is initially satisfied the function P is a one-dimensional δ function similar to that in Eq. (10.11).

XI. SOME GENERAL PROPERTIES OF $P(\alpha,t)$

The results we have derived in the previous sections have corresponded to the choice of particular initial states for the B mode. We shall now derive a general expression for the function $P(\alpha,t)$ which corresponds to the choice of an arbitrary initial density operator for the two-mode system. We illustrate the use of this expression by proving a simple theorem about the existence of the P representation for the case in which the two modes are statistically independent of each other in the initial state.

To treat arbitrary initial states we first introduce the ordinary characteristic function for the joint system of A and B modes, which is defined at $t=0$ by

$$\chi(\eta,\zeta,0)=\operatorname{tr}\{\rho e^{\eta a^\dagger+\zeta b^\dagger-\eta^* a-\zeta^* b}\}, \tag{11.1}$$

where η and ζ are complex variables. By comparing this expression with Eq. (9.4) we find that the time-dependent characteristic function for the A mode, which we now designate by $\chi_A(\eta,t)$, is related to the initial value of the characteristic function for the joint system by the equation

$$\chi_A(\eta,t)=\chi(\eta c_a^*(t),\,-\eta^* s_a(t),\,0). \tag{11.2}$$

The normally ordered characteristic function $\chi_{N,A}(\eta,t)$ is therefore given by

$$\chi_{N,A}(\eta,t)=e^{\frac{1}{2}|\eta|^2}\chi(\eta c_a^*(t),\,-\eta^* s_a(t),\,0). \tag{11.3}$$

If a P representation for the A mode exists at time t, the function $P(\alpha,t)$ is the Fourier transform of $\chi_{N,A}(\eta,t)$, so that we have

$$P(\alpha,t)=\pi^{-2}\int e^{\alpha\eta^*-\alpha^*\eta+\frac{1}{2}|\eta|^2}$$
$$\times\chi(\eta c_a^*(t),\,-\eta^* s_a(t),\,0)d^2\eta. \tag{11.4}$$

The forms of this function derived in the earlier sections correspond to appropriately specialized forms of the characteristic function χ.

Let us now assume that the initial state of the two-mode system is separable, i.e., that the Heisenberg density operator factors into a direct product of density operators for each of the two modes,

$$\rho=\rho_A\rho_B. \tag{11.5}$$

We shall allow ρ_B to represent an arbitrary initial state of the B mode. The density operator ρ_A, on the other hand, is assumed to have the P representation

$$\rho_A=\int P(\alpha,0)|\alpha\rangle\langle\alpha|d^2\alpha. \tag{11.6}$$

It follows from Eq. (11.5) that the ordinary characteristic function for the joint system of A and B modes is given at time $t=0$ by

$$\chi(\eta,\zeta,0)=\operatorname{tr}_A\{\rho_A e^{\eta a^\dagger-\eta^* a}\}\times\operatorname{tr}_B\{\rho_B e^{\zeta b^\dagger-\zeta^* b}\}$$
$$=\chi_A(\eta,0)\chi_B(\zeta,0)$$
$$=e^{-\frac{1}{2}|\eta|^2}\chi_{N,A}(\eta,0)\chi_B(\zeta,0). \tag{11.7}$$

Since ρ_A possesses the P representation (11.6), the function $\chi_{N,A}(\eta,0)$ has the Fourier transform

$$P(\alpha,0)=\pi^{-2}\int e^{\alpha\eta^*-\alpha^*\eta}\chi_{N,A}(\eta,0)d^2\eta. \tag{11.8}$$

If we use the relation (11.7) for $\chi(\eta,\zeta,0)$ in Eq. (11.3), we obtain

$$\chi_{N,A}(\eta,t)=e^{-\frac{1}{2}|\eta|^2 s^2(t)}\chi_{N,A}(\eta c_a^*(t),0)$$
$$\times\chi_B[-\eta^* s_a(t),0]. \tag{11.9}$$

The absolute magnitude of an ordinary characteristic function such as $\chi_B(\zeta,0)$ possesses a simple upper bound. Since the operator $\exp(\zeta b^\dagger-\zeta^* b)$ is unitary, its expectation value can not exceed unity in modulus. Hence we have

$$|\chi_B(\zeta)|\leq 1. \tag{11.10}$$

It follows then from Eq. (11.9) that

$$|\chi_{N,A}(\eta,t)|^2 \leq e^{-|\eta|^2 s^2(t)} |\chi_{N,A}(\eta c_a^*(t),0)|^2. \quad (11.11)$$

It is easily seen that this inequality implies the continued existence of the P representation for the A mode.

Let us suppose, for example, that the function $P(\alpha,0)$ which describes the initial state of the A mode is quadratically integrable. Then its Fourier transform $\chi_{N,A}(\eta,0)$ must be quadratically integrable, and it follows from the inequality (11.11) that $\chi_{N,A}(\eta,t)$ is quadratically integrable at all times t. The quadratic integrability of $\chi_{N,A}(\eta,t)$ implies that its Fourier transform $P(\alpha,t)$ must exist and remain quadratically integrable at all times.

If, for the sake of greater generality, we take $P(\alpha,0)$ to be a tempered distribution,[30] then its Fourier transform $\chi_{N,A}(\eta,0)$ must also be a tempered distribution and so too is the expression for $\chi_{N,A}(\eta,t)$ given by Eq. (11.9). It follows then that $P(\alpha,t)$ must remain a tempered distribution at all times t.

The general formula (11.4) enables us to discuss the asymptotic form of $P(\alpha,t)$ at large times for a much wider class of initial states than we assumed in deriving the asymptotic relation (7.17). By changing the variable of integration in Eq. (11.4) from η to $\eta c_a^*(t)$, we find

$$P(\alpha,t)=\frac{1}{\pi^2 c^2(t)}\int \exp\left[\frac{\alpha\eta^*}{c_a(t)}-\frac{\alpha^*\eta}{c_a^*(t)}\right] \times\left\{\exp\left[\tfrac{1}{2}\frac{|\eta|^2}{c^2(t)}\right]\chi(\eta,-i\eta^*\tanh\kappa t,0)\right\}d^2\eta. \quad (11.12)$$

The function $P(\alpha,t)$ is thus the Fourier transform, evaluated at the complex argument $\alpha/c_a(t)$, of the time-dependent function in curly brackets in Eq. (11.12). As long as the asymptotic value of the Fourier transform is equal to the Fourier transform of the asymptotic value of the function in curly brackets, the function $P(\alpha,t)$ takes the form given by Eq. (7.17) in the limit of large times. The effective input amplitude distribution $p(\alpha)$ is then given by the integral

$$p(\alpha)=\pi^{-2}\int e^{\alpha\eta^*-\alpha^*\eta}\chi(\eta,-i\eta^*,0)d^2\eta. \quad (11.13)$$

The function given by Eq. (7.18) corresponds to the special case in which the B mode is initially in a chaotic state and the A mode is described by means of a P representation.

APPENDIX

In Sec. VI use was made of a theorem involving the associated Laguerre polynomials, which we shall now prove. Let us define the set of functions $D_{nm}(\lambda,\mu,\nu)$ as the coefficients in the double power-series expansion

$$e^{\lambda w+\mu z+\nu wz}\equiv\sum_{n,m=0}^{\infty}\frac{w^n z^m}{n!m!}D_{nm}(\lambda,\mu,\nu). \quad (A1)$$

Alternatively, we may write

$$D_{nm}(\lambda,\mu,\nu)\equiv\left(\frac{\partial}{\partial w}\right)^n\left(\frac{\partial}{\partial z}\right)^m e^{\lambda w+\mu z+\nu wz}\bigg|_{w,z=0}. \quad (A2)$$

The function $D_{nm}(\lambda,\mu,\nu)$ clearly obeys the symmetry relation

$$D_{nm}(\lambda,\mu,\nu)=D_{mn}(\mu,\lambda,\nu). \quad (A3)$$

By expanding the exponential first in powers of z, we find

$$e^{\lambda w+\mu z+\nu wz}=\sum_{m=0}^{\infty}\frac{z^m}{m!}(\mu+\nu w)^m e^{\lambda w} =\sum_{m=0}^{\infty}\frac{z^m}{m!}\mu^m\left(1+\frac{\nu w}{\mu}\right)^m e^{\lambda w}. \quad (A4)$$

We now make use of the identity[40]

$$\sum_{n=0}^{\infty}L_n^{(m-n)}(x)y^n=(1+y)^m e^{-xy}. \quad (A5)$$

If we write $y=\nu w/\mu$ and $x=-\lambda\mu/\nu$, and substitute the resulting form of Eq. (A5) into Eq. (A4), we deduce, by referring to the definition (A1),

$$D_{nm}(\lambda,\mu,\nu)=n!\nu^n\mu^{m-n}L_n^{(m-n)}(-\lambda\mu/\nu). \quad (A6)$$

By making use of the symmetry relation (A3) we obtain the alternative expression

$$D_{nm}(\lambda,\mu,\nu)=m!\nu^m\lambda^{n-m}L_m^{(n-m)}(-\lambda\mu/\nu). \quad (A7)$$

If we equate the expressions (A6) and (A7), we obtain

$$\frac{t^n}{n!}L_m^{(n-m)}(-t)=\frac{t^m}{m!}L_n^{(m-n)}(-t), \quad (A8)$$

where $t=\lambda\mu/\nu$. This identity also follows for arbitrary t directly from the explicit expression for the associated Laguerre polynomials.

[40] W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1949), p. 85.