Necessity of Production Amplitudes in Quantum Field Theory*

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A nonvanishing production amplitude in the pion-nucleon scattering is proved to be a necessary consequence of relativistic invariance, causality, crossing, and unitarity relations in quantum field theory.

I. INTRODUCTION

IN potential models, elastic scattering can occur without the existence of production processes. However, in quantum field theory,[1] relativistic invariance, causality, crossing symmetry, and unitarity lead to inevitable connections between scattering and production. Previous proofs[2,3] of this relationship have involved various special assumptions. In this article we will give a general proof that in pion-nucleon scattering, consistency between the unitarity and crossing relations requires the existence of a nonvanishing production amplitude. This example is chosen because of its physical importance, but the method can be generalized to other cases involving spins and isotopic spins, so long as single-dispersion relations for these processes can be proved.

II. s-CHANNEL ELASTIC UNITARITY CONDITION AND THE MANDELSTAM REPRESENTATION

The T matrix for the πN scattering has the following spin and isospin decomposition:

$$T=\delta_{\beta\alpha}[-A^{+}(s,t,u)+\tfrac{1}{2}i\gamma(q+q')B^{+}(s,t,u)]+\tfrac{1}{2}[\tau_\beta,\tau_\alpha]\times[-A^{+}(s,t,u)+\tfrac{1}{2}i\gamma(q+q')B^{-}(s,t,u)], \quad (1)$$

where α, β are charge indices of the incoming and outgoing pions (Fig. 1), $s=(p+q)^2$, $t=(p+p')^2$, and $u=(p+q')^2$. The scattering amplitudes $A^{\pm}$ and $B^{\pm}$ have the following crossing relations[4]:

$$A^{\pm}(s,t,u)=\pm A^{\pm}(u,t,s),$$
$$B^{\pm}(s,t,u)=\mp B^{\pm}(u,t,s), \quad (2)$$

and are connected with the amplitudes A^I and B^I with fixed total isospins $I=\frac{1}{2}, \frac{3}{2}$ by

$$A^{+}=\tfrac{1}{3}(A^{1/2}+2A^{3/2}),$$
$$A^{-}=\tfrac{1}{3}(A^{1/2}-A^{3/2}); \quad (3)$$

similar relations hold for the B's.

The theorem will be proved by *reductio ad absurdum*. We take the premise that there are no production processes in the s channel and will show that under this premise, the analyticity domain obtained by Martin[5] may be enlarged repeatedly by use of s-channel elastic unitarity to the whole s-t manifold except for the usual physical cuts, so that we can write down the Mandelstam representations for $A^{\pm}$ and $B^{\pm}$. It is shown then in the next section that consistency between crossing relations and unitarity demands that the invariant amplitudes A^I and B^I must be identically zero.

Martin[5] has combined the t-plane analyticity property of Lehmann[6] and that of Bros, Epstein, and Glaser[7] with the fixed-t dispersion relations,[8]

$$A^{\pm}(s,t,u)=\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{A_s^{\pm}(s',t)}{s'-s}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}du'\frac{A_u^{\pm}(u',t)}{u'-u},$$

$$B^{\pm}(s,t,u)=\frac{g^2}{s-M^2}\mp\frac{g^2}{u-M^2}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{B_s^{\pm}(s',t)}{s'-s}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}du'\frac{B_u^{\pm}(u',t)}{u'-u}, \quad -t_c<t\leq 0 \quad (4)$$

and showed that $A^{\pm}$ and $B^{\pm}$ are analytic in s and t within $\mathfrak{D}$, where

$$\mathfrak{D}=[|t|<R]\otimes[s\text{-cut plane with cut along the real axis from } (M+\mu)^2 \text{ to } \infty \text{ and another parallel cut from } -t+(M-\mu)^2 \text{ to } -\infty-i\,\mathrm{Im}t] \quad (5)$$

and $R\cong 1.83\mu^2$ is the maximum of the right extremity of the small Lehmann ellipse in the t plane.[9] The precise value of R does not concern us here so long as it is a

* Research supported by the U. S. Air Force Office of Scientific Research.
[1] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That* (W. A. Benjamin, Inc., New York, 1964), R. Jost, *General Theory of Quantized Fields* (American Mathematical Society Publication, Providence, Rhode Island, 1963).
[2] S. Aks, J. Math. Phys. **6**, 516 (1965).
[3] F. K. Cheung, thesis, Maryland University Technical Report No. 534, 1965 (unpublished).
[4] J. Bros, H. Epstein, and V. Glaser, Comm. Math. Phys. **1**, 240 (1965).
[5] A. Martin, Nuovo Cimento **42**, 930 (1966); 44, 1219 (1966).
[6] H. Lehmann, Nuovo Cimento **10**, 579 (1958).
[7] J. Bros, H. Epstein, and V. Glaser, Nuovo Cimento **31**, 1265 (1964).
[8] See, for instance, H. J. Bremermann, R. Oehme, and J. G. Taylor, Phys. Rev. **109**, 2178 (1958). For simplicity, subtractions are neglected here as they will not affect our argument; see remarks at the end of this article.
[9] G. Sommer, CERN Report No. 66/159/5-TH731 (unpublished).

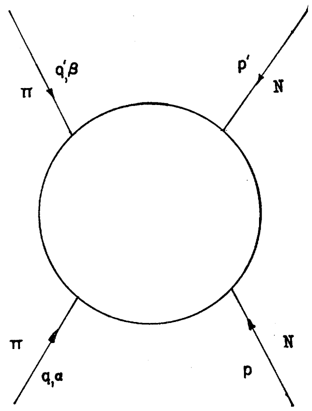

FIG. 1. Kinematics of the pion-nucleon scattering.

finite constant number.[10] From the Martin result and that of Lehmann, it further follows[5,9] that the boundary values of the absorptive parts as one approaches the real axis from above, $A_s^{\pm}$ and $B_s^{\pm}$, are analytic in t within the ellipse,

$$E(0,-4q^2|R)\,, \tag{6}$$

with foci 0 and $-4q^2$ and right extremity at $t=R$ and with q the c.m. momentum of either particle in the s channel, $4q^2=[s-(M+\mu)^2][s-(M-\mu)^2]/s$. $A_s^{\pm}(s,t)$ and $B_s^{\pm}(s,t)$ will then be analytic for all s real, $\infty>s\geqslant s_0=(M+\mu)^2$, within the intersection of these ellipses[11]:

$$\bigcap_{s_0\leq s<\infty} E(0,-4q^2|R)=E(0,-4q_0^2(s_0)|R)\,. \tag{7}$$

The s and u channels' absorptive parts are related to each other by crossing relations similar to Eq. (2),

$$A_s^{\pm}(s,t)=\pm A_u^{\pm}(u,t)\,,$$
$$B_s^{\pm}(s,t)=\mp B_u^{\pm}(u,t)\,. \tag{8}$$

Hence $A_u^{\pm}(u,t)$ and $B_u^{\pm}(u,t)$ will also be analytic within $E(0,-4q_0^2|R)$ for all real u, $u_0=s_0\leq u<\infty$, so that for any given $t\in E(0,-4q_0^2|R)$, dispersion relations as given by Eq. (4) exist and the scattering amplitudes $A^{\pm}$ and $B^{\pm}$ are analytic within the domain,

$$[t\in E(0,-4q_0^2|R)]\otimes[s\text{-cut plane}]. \tag{9}$$

For a given s real, we may fit into $E(0,-4q_0^2|R)$ an ellipse of the form $E(0,-4q^2|r)$ and then use elastic unitarity in the s channel to find the analyticity domain of $A_s^{\pm}$ and $B_s^{\pm}$ to be $E(0,-4q^2|4r(1+r/4q^2))$, except possibly for cuts along the real axis starting from $t=4$, and $t=-s+(M-\mu)^2$. Hereafter, we shall use $\mathcal{E}$ instead of E to denote an ellipse that may have been penetrated by these physical cuts. This ellipse is to be compared with $E(0,-4q^2|R)$, the initial analyticity domain of $A_s^{\pm}$ and $B_s^{\pm}$ for given q^2. An enlargement[12] has occurred if $4r(1+r/4q^2)>R$. This condition will be satisfied for $q_0^2\leq q^2<q_0^2+3R/16$ by taking $r(q^2)=R-4(q^2-q_0^2)$, so that Eq. (7) can now be replaced by the larger domain,

$$\bigcap_{q_0^2\leq q^2\leq q_0^2+3R/16} \mathcal{E}\Big(0,-4q^2|4[R-4(q^2-q_0^2)]\times\Big[1+\frac{R-4(q^2-q_0^2)}{4q^2}\Big]\Big)\supset\mathcal{E}(0,-4(q_0^2+3R/16)|R)\,, \tag{10}$$

and by going through the argument leading to Eq. (9), we see that $A^{\pm}$ and $B^{\pm}$ may be analytically continued at least into

$$[t\in\mathcal{E}(0,-4(q_0^2+3R/16)|R)]\otimes[s\text{-cut plane}]. \tag{11}$$

The enlargement procedure above can be carried on by steps indefinitely, insofar as the s-channel elastic unitarity is applicable for all real $s\geq s_0$, because if we have analytically continued $A^{\pm}$ and $B^{\pm}$ into[13]

$$[t\in\mathcal{E}(0,-4(q_0^2+3(n-1)R/16)|R)]\otimes[s\text{-cut plane}], \tag{12}$$

the same argument as given above shows that we can also enlarge the analyticity domain to

$$[t\in\mathcal{E}(0,-4(q_0^2+3nR/16)|R)]\otimes[s\text{-cut plane}]. \tag{13}$$

The domain $\mathcal{E}(0,-4(q_0^2+3nR/16)|R)$ evidently gets bigger as n becomes larger, but it is always limited as it has a fixed right extremity at $t=R$. To further enlarge the analyticity domain, we may again, for a given $q^2\leq q_0^2+3nR/16$, fit an ellipse $\mathcal{E}(0,-4q^2|r=R)$ into $\mathcal{E}(0,-4(q_0^2+3nR/16)|R)$ and use elastic unitarity to find the analyticity domain of $A_s^{\pm}$ and $B_s^{\pm}$ to be $\mathcal{E}(0,-4q^2|4R(1+R/4q^2))$, which contains the ellipse $\mathcal{E}(0,-4q^2|4R)$. Using these ellipses instead of $E(0,-4q^2|R)$ in Eq. (6), we may now repeat the whole enlargement process over again. It is easy to see that in this way we can analytically continue $A^{\pm}$ and $B^{\pm}$ into the product domain (Fig. 2):

$$[t\in\mathcal{E}(0,-4(q_0^2+3nR/16)|4^mR)]\otimes[s\text{-cut plane}]. \tag{14}$$

As n and m can be made as large as we please by repeated use of the above argument, we reach the con-

[10] In contrast, if we have only the Lehmann ellipse at our disposal, the analyticity domain of A and B cannot be extended indefinitely even if elastic unitarity is assumed for all s.

[11] A somewhat larger domain would be obtained had we taken into account also the Lehmann ellipses. The q_0^2 in Eq. (7) is then larger than zero. However, this is immaterial in our subsequent discussion.

[12] S. Mandelstam, Nuovo Cimento **15**, 658 (1960); A. Martin, Ref. 5. See also the latter's note added in proof. Here we follow their assumption that there is no pathological difficulty in using the elastic unitarity to enlarge the analyticity domain of the absorptive parts even if the T matrix is taken as a distribution.

[13] For simplicity we have neglected possible singularities generated by the ellipse $\mathcal{E}(0,-4q^2|r)$ with the physical cuts, $t>4\mu^2$ and $u>(M+\mu)^2$. When they are taken into account, we should replace the ellipse $\mathcal{E}(0,-4(q_0^2+3nR/16)|4^mR)$ by $\mathcal{E}(0,-4(q_0^2+3nR/16)|R16)|R+4m)$; the result after Eq. (14), however, is independent of these complications.

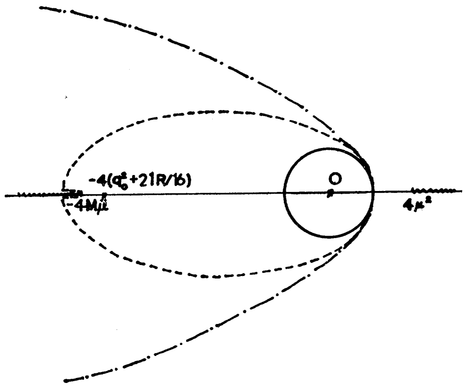

FIG. 2. The diagram shows several stages of enlargements of the analyticity domain in t of A and B; the circle is the original Martin domain, the indented ellipse corresponds to $n=7$, and the parabola $n=\infty$, when both n, $m\to\infty$, we are left only with the physical cuts.

clusion that $A^\pm$ and $B^\pm$ are analytic in the whole $s-t$ manifold except for the usual physical cuts. As a result, we can write down the Mandelstam representations for $A^\pm$ and $B^\pm$,[14]

$$
\begin{aligned}
A^\pm(s,t,u)=&\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}ds'\int_{4\mu^2}^{\infty}dt'\frac{A_{st}^\pm(s',t')}{(s'-s)(t'-t)}\\
&+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}du'\int_{4\mu^2}^{\infty}\frac{dt'A_{tu}^\pm(t',u')}{(t'-t)(u'-u)}\\
&+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}ds'\int_{(M+\mu)^2}^{\infty}du'\frac{A_{su}^\pm(s',u')}{(u'-u)(s'-s)},\\
B^\pm(s,t,u)=&\frac{g^2}{s-M^2}\mp\frac{g^2}{u-M^2}\\
&+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}ds'\int_{4\mu^2}^{\infty}dt'\frac{B_{st}^\pm(s',t')}{(s'-s)(t'-t)}\\
&+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}du'\int_{4\mu^2}^{\infty}dt'\frac{B_{tu}^\pm(t',u')}{(t'-t)(u'-u)}\\
&+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}ds'\int_{(M+\mu)^2}^{\infty}du'\frac{B_{su}^\pm(s',u')}{(u'-u)(s'-s)},
\end{aligned}
\tag{15}
$$

where subtractions are neglected but will be discussed at the end of the article.

III. PROOF THAT PRODUCTION AMPLITUDES ARE NECESSARY

In terms of the double-density functions, the crossing relations Eq. (5) become

$$
\begin{aligned}
A_{su}^\pm(s,u)&=\pm A_{su}^\pm(u,s),\\
B_{su}^\pm(s,u)&=\mp B_{su}^\pm(u,s).
\end{aligned}
\tag{16}
$$

[14] S. Mandelstam, Phys. Rev. 112, 1344 (1958).

These double-density functions are analytic in the real s-u plane except for possible singularities along the Landau curves $u_n^+(s)$.[15] These Landau curves lie entirely in the first quadrant of the s-u plane and do not cross one another, so that starting from the origin, $u_{n+1}^+(s)$ can be reached only by traversing the curve $u_n^+(s)$. $u_n^+(s)$ has asymptote $u=(M+n\mu)^2$ along the s direction, but because of our premise that there are no production processes in the s channel, they are all asymptotic to $s=(M+\mu)^2$. Furthermore, $A_{su}^\pm(s,u)$ and $B_{su}^\pm(s,u)$ vanish before the first Landau curve is reached. Referring to Fig. 3, $A_{su}^\pm=B_{su}^\pm=0$ in the shaded region. Now supposing $P(s_1,u_1)$ is a point in region II, we will get its image point $Q(u_1,s_1)$ with respect to $s=u$ by interchanging the coordinates (s_1,u_1). $Q(u_1,s_1)$ is a point in region I where $A_{su}^\pm(s,u)$ and $B_{su}^\pm(s,u)$ are zero. By the crossing relations Eq. (16), $A_{su}^\pm(s,u)$ and $B_{su}^\pm(s,u)$ also vanish at $P(s_1,u_1)$,

$$
\begin{aligned}
A_{su}^\pm(s_1,u_1)&=\pm A_{su}(u_1,s_1)=0,\\
B_{su}^\pm(s_1,u_1)&=\mp B_{su}(u_1,s_1)=0.
\end{aligned}
\tag{17}
$$

However, region II contains part of every Landau curve $u_n^+(s)$, so that by analytically continuing out from region II, we see that the double density functions must vanish identically everywhere in the real s-u plane. From Eq. (3), the double-density functions A_{su}^I and B_{su}^I with fixed isotopic spins will also vanish,

$$A_{su}^I=B_{su}^I=0. \tag{18}$$

On the other hand, these double-density functions are related to the absorptive parts A_t^I, A_u^I, B_t^I, B_u^I by

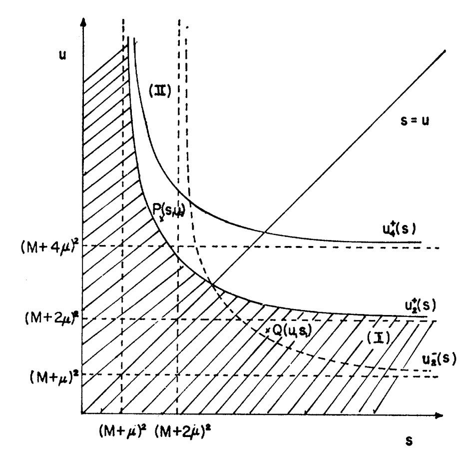

FIG. 3. The Landau curves $u_n^+(s)$ in the real s-u plane. The dashed curve $u_2^-(s)$ is the image of $u_2^+(s)$ reflected with respect to the line $s=u$. Regions I and II refer to those portions of the s-u plane which are bounded between $u_2^-(s)$ and $u_2^+(s)$.

[15] W. Zimmermann, Nuovo Cimento 21, 249 (1961); L. D. Landau, Nucl. Phys. 13, 181 (1959).

the s-channel unitarity integrals[14]

$$A_{su}^{I}(s,z(u))=\frac{q}{4\pi(w+E)}\int_{1+t_0/2q^2}^{z>z_1z_2+(z_1^2-1)^{1/2}(z_2^2-1)^{1/2}}dz_1\int_{1+\{u_0-[(q^2+M^2)^{1/2}-(q^2+\mu^2)^{1/2}]^2\}/2q^2}dz_2\,K(z,z_1,z_2)$$
$$\times\{f_{11}[A_t^{I*}(s,z_1)A_t^{I}(s,z_2)+A_u^{I*}(s,z_1)A_u^{I}(s,z_2)]+f_{12}[A_t^{I*}(s,z_1)B_t^{I}(s,z_2)+A_u^{I*}(s,z_1)B_u^{I}(s,z_2)]$$
$$+f_{21}[B_t^{I*}(s,z_1)A_t^{I}(s,z_2)+B_u^{I*}(s,z_1)A_u^{I}(s,z_2)]+f_{22}[B_t^{I*}(s,z_1)B_t^{I}(s,z_2)+B_u^{I*}(s,z_1)B_u^{I}(s,z_2)]\},\quad(19a)$$

$$B_{su}^{I}(s,z(u))=\frac{q}{4\pi(w+E)}\int_{1+t_0/2q^2}^{z>z_1z_2+(z_1^2-1)^{1/2}(z_2^2-1)^{1/2}}dz_1\int_{1+\{u_0-[(q^2+M^2)^{1/2}-(q^2+\mu^2)^{1/2}]^2\}/2q^2}dz_2\,K(z,z_1,z_2)$$
$$\times\{g_{11}[A_t^{I*}(s,z_1)A_t^{I}(s,z_2)+A_u^{I*}(s,z_1)A_u^{I}(s,z_2)]+g_{12}[A_t^{I*}(s,z_1)B_t^{I*}(s,z_2)+A_u^{I*}(s,z_1)B_u^{I}(s,z_2)]$$
$$+g_{21}[B_t^{I*}(s,z_1)A_t^{I}(s,z_2)+B_u^{I*}(s,z_1)A_u^{I}(s,z_2)]+g_{22}[B_t^{I*}(s,z_1)B_t^{I}(s,z_2)+B_u^{I*}(s,z_1)B_u^{I}(s,z_2)]\},\quad(19b)$$

where E and w are the c.m. energy of the incoming nucleon and pion, respectively, $s=(E+w)^2$, and z is the cosine of the c.m. scattering angle with $t=-2q^2(1+z)$. $K(z,z_1,z_2)$ is the elastic unitarity kernel; $(K(z,z_1,z_2)=(z^2+z_1^2+z_2^2-1-2zz_1z_2)^{-1/2}$. The f's and g's depend linearly on z_1 and z_2 and are given explicitly by

$$f_{11}=M\left(1-\frac{w}{2(E+w)}\frac{1-z_1-z_2+z}{1+z}\right),$$
$$f_{12}=\frac{E^2-Ew-M^2}{2}\frac{1+z_1-z_2-z}{1-z}+\frac{M^2w}{2(w+E)}\frac{1-z_1-z_2+z}{1+z},$$
$$f_{21}=\frac{E^2+Ew-M^2}{2}\frac{1-z_1+z_2-z}{1-z}+\frac{M^2w}{2(w+E)}\frac{1-z_1-z_2+z}{1+z},\quad(20a)$$
$$f_{22}=\frac{Mw}{2(w+E)}((E+w)^2-M^2)\frac{1+z-z_1-z_2}{1+z},$$
$$g_{11}=\frac{E}{w+E}\left(\frac{1+z-z_1-z_2}{1+z}\right),$$
$$g_{12}=\frac{M}{2}\frac{1-z+z_1-z_2}{1-z}-\frac{EM}{2(w+E)}\frac{1+z-z_1-z_2}{1+z},$$
$$g_{21}=\frac{M}{2}\frac{1-z-z_1+z_2}{1-z}-\frac{EM}{2(w+E)}\frac{1+z-z_1-z_2}{1+z},$$
$$g_{22}=(E^2+Ew-M^2)-\frac{E}{2(w+E)}\frac{1+z-z_1-z_2}{1+z}.\quad(20b)$$

We will also need to use later another set of amplitudes defined by

$$C^I=\frac{E+M}{2(E+w)}\left(\frac{A^I+(E+w-M)B^I}{4\pi}\right),$$
$$D^I=\frac{E-M}{2(E+w)}\left(\frac{-A^I+(E+w+M)B^I}{4\pi}\right).\quad(21)$$

The absorptive parts C_t^I, D_t^I and C_u^I, D_u^I are similarly related to A_t^I, B_t^I and A_u^I, B_u^I, respectively. Within the integration range of Eq. (19), $K(z,z_1,z_2)$ is a positive function. By the mean-value theorem,[16] this factor may be taken out of the integral and evaluated at some intermediate point (z_{10},z_{20}) within the integration range. Disregarding $K(z,z_{10},z_{20})$, it is then easy to see that for the left-hand side of Eq. (19) to vanish for all z, the quantities within the brackets of the integrands must be identically zero. In particular, for $z_1=z_2$, we have

$$0=f_{11}(A_t^I(z_2)A_t^I(z_2)+A_u^{I*}(z_2)A_u^I(z_2))+f_{12}(A_t^{I*}(z_2)B_t^I(z_2)+A_u^{I*}(z_2)B_u^I(z_2))+f_{21}(B_t^{I*}(z_2)A_t^I(z_2)+B_u^{I*}(z_2)A_u^I(z_2))+f_{22}(B_t^{I*}(z_2)B_t^I(z_2)+B_u^{I*}(z_2)B_u^I(z_2)),$$
$$0=g_{11}(A_t^{I*}(z_2)A_t^I(z_2)+A_u^{I*}(z_2)A_u^I(z_2))+g_{12}(A_t^{I*}(z_2)B_t^I(z_2)+A_u^{I*}(z_2)B_u^I(z_2))+g_{21}(B_t^{I*}(z_2)A_t^I(z_2)+B_u^{I*}(z_2)A_u^I(z_2))+g_{22}(B_t^{I*}(z_2)B_t^I(z_2)+B_u^{I*}(z_2)B_u^I(z_2)).\quad(22)$$

If we write out the real and imaginary parts of the above equations, we get a set of four homogeneous equations which has only the solution

$$|A_t^I(s,z_2)|^2+|A_u^I(s,z_2)|^2=|B_t^I(z_2)|^2+|B_u^I(z_2)|^2=|A_t^{I*}(s,z_2)B_t^I(s,z_2)+A_u^{I*}(s,z_2)B_u^I(s,z_2)|=0.\quad(23)$$

Since this remains true as z_2 and s are varied, we see that $A_t^I(z_2)$, $B_t^I(z_2)$, $A_u^I(z_2)$, and $B_u^I(z_2)$ are identically zero; and by Eq. (21) so are C_t^I, D_t^I, C_u^I, and D_u^I. To see the further consequences, let us take for instance, $C_u^I=D_u^I=0$ and consider the following partial-wave

[16] R. Courant, *Differential and Integral Calculus* (Nordemann Publishing Company, Inc., New York, 1938).

expansion in the u channel:

$$C^I(u,z(s))=\sum_{l=0}^{\infty} f_{l+}(u)P_{l+1}'(\cos\theta_u) -\sum_{l=2}^{\infty} f_{l-}(u)P_{l-1}'(\cos\theta_u)\,, \quad (24)$$

$$D^I(u,z(s))=\sum_{l=0}^{\infty} (f_{l-}-f_{l+})P_l(\cos\theta_u)\,, \quad (25)$$

where $f=(\eta_{l\pm}e^{2i\delta_{l\pm}}-1)/q_u$, $\eta_{l\pm}$ is the inelasticity factor with $0\leq\eta_{l\pm}\leq 1$, and

$$\mathrm{Im}f_{l\pm}(u)=\frac{1}{2}\int_{-1}^{1} dz[C_u{}^I(z)P_l(z)+D_u{}^I(z)P_{l\pm1}(z)]. \quad (26)$$

From this we see that $\mathrm{Im}f_{l\pm}$ vanishes along with $C_u{}^I$ and $D_u{}^I$,

$$\mathrm{Im}f_{l\pm}(u)=0 \quad \text{for all } l. \quad (27)$$

However, $\mathrm{Im}f_{l\pm}(u)$ are related to the modulus of $f_{l\pm}(u)$ by the partial-wave unitarity condition,

$$\mathrm{Im}f_{l\pm}(u)=q_u|f_{l\pm}(u)|^2+q_u(1-\eta_{l\pm}/4)\,, \quad (28)$$

where $q_u((1-\eta_{l\pm})/4)$ is the inelastic contribution to $\mathrm{Im}f_{l\pm}(u)$ in the u channel and is non-negative, so that

$$f_{l\pm}(u)=0 \quad \text{for all } l. \quad (29)$$

Putting this back into Eqs. (24) and (25), we see that C^I and D^I, and hence A^I and B^I, vanish in the physical region of the u channel. By analytic continuation, they must be identically zero and we reach the final result that there can be no scattering at all if there are no production processes in one channel.

It is worth remarking that the result we have obtained remains true even if there is a finite number of subtractions in the Mandelstam representations for $A^\pm$ and $B^\pm$ in Eq. (15). This is so because the result is based only on the crossing relations Eq. (2), the structure of the Landau curves on the real s-u plane, and the elastic-unitarity integrals Eq. (19), and none of these is changed by the presence of a finite number of subtractions in Eq. (15).

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Quantum Theory of Parametric Amplification. I*

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The amplification of electromagnetic fields is analyzed in a quantum-mechanical context by discussing the behavior of a simple theoretical model of the parametric amplifier. The statistical properties of the amplifier fields are described by means of the time-dependent density operator for the system. In doing this, extensive use is made of the coherent states and the P representation of the density operator, which provide a quantum-mechanical description of the fields closely resembling their classical description. Explicit solutions are obtained for the density operator for either of the two field modes for a variety of initial states of the modes. Initial states considered include combinations of coherent states, chaotic mixtures, and n-quantum states. Particular attention is given the behavior of the amplifier fields in the limit of large amplification. The conditions are established under which the amplification process leads in this limit to the existence of a non-negative P representation for the density operator for a single mode of oscillation.

I. INTRODUCTION

THE fundamental process which has become known as parametric amplification in electronic contexts plays a central role in several physical phenomena of interest. These include the coherent Raman and Brillouin effects and the frequency splitting of light beams in nonlinear media. The most familiar form of the parametric amplifier is designed to amplify an oscillating signal by means of a particular coupling of the mode in which it appears to a second mode of oscillation, the idler mode. The coupling parameter is made to oscillate with time in a way which gives rise to a steady increase of the energy in both the signal and idler modes.

The physical processes we have indicated as depending upon parametric amplification may be described in parallel terms. In the coherent Raman effect, for example, the presence of a monochromatic light wave in a Raman active medium gives rise to parametric coupling between an optical vibrational mode and a mode of the radiation field which represents the scattered (Stokes) wave. In the case of Brillouin scattering a similar form of coupling holds, with the vibrational mode oscillating at an acoustic rather than an optical frequency. The frequency splitting of light beams is an example of parametric amplification in which both of the coupled modes are electromagnetic. An intense light wave in a nonlinear dielectric medium couples pairs of electromagnetic field modes whose frequencies sum

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