THE CALCULATION OF DETECTING AND AMPLIFYING PROPERTIES OF AN ELECTRON TUBE FROM ITS STATIC CHARACTERISTICS.

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Synopsis.

Calculation of the Detecting Efficiency of an Electron Tube.—The detecting efficiency of an electron tube is calculated. In the calculation it is assumed that experimental knowledge of the static characteristics of the tube is available. The constants of the circuits used with the tube are also taken as known. From these quantities the average change in the plate current for a given amplitude impressed grid voltage is derived. The impressed grid voltage is taken to be of the form $A \cos \omega t$ where A is constant. The case when A varies slowly is also discussed.

Calculation of the Input Impedance of an Electron Tube.—The input impedance of an electron tube is calculated for the case of both positive and negative grid voltages, no assumption being made as to the mathematical form of the tube characteristics.

Generalization of the Concept of Internal Resistance and Amplification Factor.—The concept of the complex internal resistance is introduced in treating the amplification due to a single tube. This quantity is defined by

$$\bar{\xi}_\omega = \frac{1}{\dfrac{1}{r_p} + \dfrac{1}{\left(\dfrac{1}{jC_2\omega}\right)}},$$

where r_p is the internal resistance, C_2 the grid plate capacity, $j = \sqrt{-1}$, and $\omega/2\pi$ = frequency. Similarly the amplification factor is generalized. (See (4.3)).

Condition for the Vanishing of Incoming Signal.—The condition for the vanishing of the incoming signal has been worked out. For the case of zero grid current this condition is just satisfied by tubes obeying Van der Bijl's relation.

It is also seen that the condition is not satisfied by the values of plate circuit constants predicted from the simple theory neglecting internal capacities.

The first rigorous calculation of the detecting action of an electron tube is to my knowledge due to Carson.[1] Carson treats the special case of an electron tube used with sufficiently high negative grid voltage to make the grid current zero. He does not take into account the possibility of having a different value for the resistance of the plate circuit at high and low frequencies, as is seen from his formula. He also confines himself to the consideration of an electron tube for which Van der Bijl's relation holds for the plate current in terms of the plate and grid voltages.

The following treatment concerns itself with an electron tube having

[1] See Carson, Proceedings of the Institute of Radio Engineers, Vol. 7, April, 1919.

three electrodes: (1) A cathode consisting usually of a wire heated to incandescence by an electric current. (2) An anode which is usually made of a flat or cylindrical metallic plate and (3) a wire control or grid which is placed between the cathode and the anode.

In what follows I shall refer to the cathode as the *filament*, to the anode as the *plate*, and to the intermediary electrode as the *grid* of the tube.

The filament of the tube has two terminals which are used to heat the filament. It is clear that when the filament is heated by the passage of an electric current through it the potential of its two terminals is different. For this reason when the filament is used as a cathode we cannot speak of the potential of the cathode as a whole. However, if the filament current is known and if the potential of one of its terminals is known the potential of all the remaining points of the filament is determined uniquely. We thus choose arbitrarily as our standard the potential of one of the terminals of the filament and we shall refer to it as the *potential* of the *filament*. The grid and plate may in all practical cases be regarded as equipotential surfaces. The terms "grid potential" and "plate potential" are thus clear. The difference between the potential of the grid and that of the filament we shall call Eg; similarly the difference between the potential of the plate and that of the filament will be denoted by E_p.

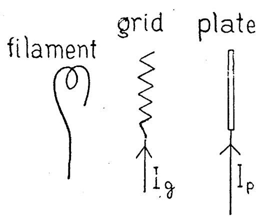

Fig. 1.

Experiments with direct currents show that two quantities E_g, E_p determine completely the currents which flow from the grid and from the plate inside the tube. If these currents, reckoned positive in the directions shown on Fig. 1, be called I_g, I_p respectively we can write,

$$I_g = f(E_g, E_p),$$
$$I_p = \varphi(E_g, E_p),$$

where f and φ are functions of the two arguments E_g and E_p. The form of these functions can be determined mathematically with fair precision for some tubes. The general action of the tube, however, can be predicted without having an accurate knowledge of the mathematical law connecting I_g, I_p with E_g, E_p.

We shall assume in what follows that these functions f and φ have within the range considered a first, second and third derivative, all of these being finite and single valued, and the function itself with its first two derivatives with respect to each argument being in addition continuous.

We shall also assume that even for unsteady currents the equations (1) give values of I_g and I_p which represent the current carried by conduction by the ions inside the tube provided E_g, E_p denote the instantaneous values of the grid and plate potentials. (The filament potential is taken as *zero*.)

Before proceeding with the mathematical development of the problem I must point out the limitations put on this development by the physical assumptions made. There is no doubt that all of the functions considered are finite and single valued. The differentiability of the functions can be doubted in some cases especially in that of the grid current. Thus it is known that there is an essential distinction between the behavior of the grid current for positive and negative grid potentials. The grid current for negative potentials grid is zero to all intents and purposes. Thus at $E_g = 0$ one would be unable to define the derivative: $\partial I_g/\partial E_g$.

Fortunately this difficulty is only formal in character because a sufficiently close examination of the static characteristics shows that there is some grid current even for negative grid potentials. So that $\partial I_g/\partial E_g$ always exists. Thus we will be safe to apply our theory provided we shall recognize by means of our instruments currents and potentials so small that $\partial I_g/\partial E_g$ is continuous.

Another possibility of a break down in the theory is offered by our second assumption. In fact it is conceivable that the space inside the electron tube may have a resonant frequency. This frequency may be fixed by the dimensions or shape of the space between the electrodes. If such is the case I_g, I_p are no longer single-valued functions of E_g, E_p when unsteady currents are considered. In fact one would expect to have here a dependence of I_g, I_p either on $\partial E_g/\partial t$, dE_p/dt or on the previous history of E_g, E_p.

Such resonant effects have been described by R. Whiddington[1]. They come into play, however, at frequencies very much higher than those ordinarily employed in wireless telegraphy. Also these effects are practically absent in high vacuum tubes. Thus here again we have a wide range within which our theory applies. In many cases electron tubes are used as detectors in direction finder circuits. If such is the case they are connected across the terminals of the tuning condenser the connections being as in Fig. 2. In this type of connection the electromotive force is induced in the coil. The reader will see without difficulty that this circuit is a special case of a slightly more general one drawn in

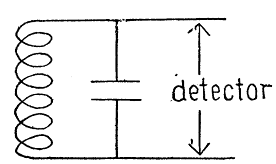

Fig. 2.

[1] Radio Review, 1, 53, 1919.

Fig. 3. Here e represents the place where the electromotive force is induced; Zg' takes the place of the direction finder coil; Zg'' replaces the tuning condenser. For generality Zg was introduced into the circuit so as to include the possibility of a blocking condenser and grid leak. The capacities c_1, c_2 represent the effective capacities between the electrodes. There is, of course, a third capacity between the filament and plate. But this can be incorporated as part of the circuit Z_p. The diagram of Fig. 3 includes most of the circuits which are used with electron tubes. The only essential type of connection left out is that in which there is inductive coupling between the plate and grid circuits.

Notation.

We shall distinguish between the behavior of the various circuits Zg, Zg', Zg'', Z_p at different frequencies. Thus we shall not assume that a definite *resistance* can be assigned to any one of these, because experiments show that such an assumption is illegitimate. Since it is a difficult matter to give *in general* the relation between the properties of an electric circuit at various frequencies we shall content ourselves by taking the impedance [meaning thereby $R + jX(j = \sqrt{-1})$, R denoting the resistance, X the reactance] of a circuit as a function of the frequency which is known either experimentally or by computation. These complex values of the impedance at frequency $\omega/2\pi$ we shall write $Z_{g\omega}$, $Z_{g'\omega}$, $Z_{g''\omega}$, $Z_{p\omega}$, etc. The real part of these expressions (*i.e.*, the resistance) will be written $R_{g\omega}$, $R_{g'\omega}$, $R_{g''\omega}$, $R_{p\omega}$ and the imaginary part (*i.e.*, the reactance) will similarly be denoted by $X_{g\omega}$, $X_{g'\omega}$, $X_{g''\omega}$, $X_{p\omega}$.

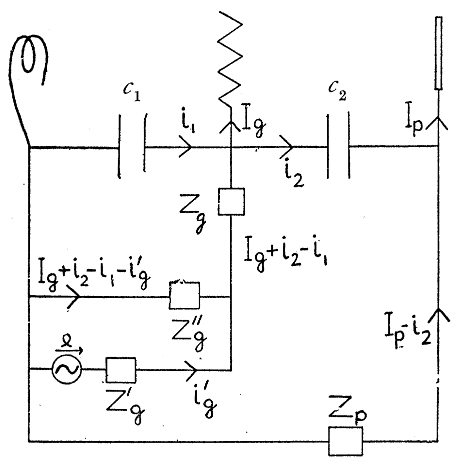

Fig. 3.

We will not be concerned in this problem with transient phenomena. If therefore the impressed electromotive force is periodic or if it consists of a sum of several periodic electromotive forces all the currents are periodic or are sums of periodic functions the periods of the electromotive forces and the currents being equal.

It is convenient for the mathematical treatment to represent the currents and potentials as Fourier series.

We shall first treat the case when the electromotive force impressed (e) is of the form $A \cos \omega t$. Then all the currents and potentials must be Fourier series in ωt. Thus any one of these quantities is of the form

$$
\begin{aligned}
b_0 + b_1 \cos \omega t + b_2 \cos 2\omega t + \cdots + a_1 \sin \omega t + a_2 \sin 2\omega t + \cdots \\
= b_0 + \Sigma(b_m \cos m\omega t + a_m \sin m\omega t) \\
= b_0 + \Sigma B_m e^{jm\omega t},
\end{aligned}
$$

where

$$B_m = \sqrt{a_m^2 + b_m^2} e^{-j \tan^{-1} a_m/b_m}$$

and where the real part only of the expression on the right is to be taken. Using this complex notation we can write

$$i_1 = \Sigma i_{1,\, m\omega} e^{jm\omega t}$$

and similarly for all other currents of Fig. 3.

We shall now express the quantities

$$E_g,\ E_p \text{ in terms of } I_g,\ I_p \text{ and } e.$$

We have

$$
\begin{aligned}
\bar{\imath}_{g'\omega} \bar{Z}_{g'\omega} - \bar{e}_\omega &= \bar{Z}_{g''\omega}(\bar{I}_{g\omega} + \bar{\imath}_{2\omega} - \bar{\imath}_{1\omega} - \bar{\imath}_{g'\omega}) \\
&= \frac{\bar{\imath}_{1\omega}}{jC_{1\omega}} - \bar{Z}_{g\omega}(\bar{I}_{g\omega} + \bar{\imath}_{2\omega} - \bar{\imath}_{1\omega}) \\
&= \bar{Z}_{p\omega}(\bar{I}_{p\omega} - \bar{I}_{2\omega}) - \frac{\bar{\imath}_{2\omega}}{jC_{2\omega}} - \bar{Z}_{g\omega}(\bar{I}_{g\omega} + \bar{\imath}_{2\omega} - \bar{\imath}_{1\omega}),
\end{aligned}
$$

where the currents are as on the figure the positive directions being those of the arrows. (The currents are written to satisfy the first of Kirchhoff's laws.) Solving these equations we find

$$
\begin{aligned}
\bar{\imath}_{1\omega} &= \frac{-\bar{e}_\omega \bar{Z}_{g''\omega} + \bar{I}_{g\omega}\bar{S}_{2\omega}}{\dfrac{\bar{Z}_{g'\omega} + \bar{Z}_{g''\omega}}{jc_1\omega} + \bar{S}_{2\omega}} \left[1 - \frac{\bar{S}_{2\omega}}{\bar{S}_{2\omega} + \left(\bar{Z}_{p\omega} + \dfrac{1}{jc_2\omega}\right)(\bar{Z}_{g'\omega} + \bar{Z}_{g''\omega} + \bar{S}_2 jc_1\omega)} \right] \\
&\quad + \bar{I}_{p\omega} \frac{\bar{Z}_{p\omega}\bar{S}_{2\omega}}{\left(\dfrac{\bar{Z}_{g'\omega} + \bar{Z}_{g''\omega}}{jc_1\omega} + \bar{S}_{2\omega}\right)\left(\dfrac{1}{jc_2\omega} + \bar{Z}_{p\omega} + \dfrac{\bar{S}_{2\omega}}{\bar{Z}_{g'\omega} + \bar{Z}_{g''\omega} + \bar{S}_{2\omega} jc_1\omega}\right)} \\
\bar{\imath}_{2\omega} &= \frac{\dfrac{\bar{e}_\omega \bar{Z}_{g''\omega} - \bar{I}_{g\omega}\bar{S}_{2\omega}}{\bar{Z}_{g'\omega} + \bar{Z}_{g''\omega} + jc_1\omega \bar{S}_{2\omega}} + \bar{Z}_{p\omega}\bar{I}_{p\omega}}{\dfrac{1}{jc_2\omega} + \bar{Z}_{p\omega} + \dfrac{\bar{S}_{2\omega}}{\bar{Z}_{g'\omega} + \bar{Z}_{g''\omega} + \bar{S}_{2\omega} jc_1\omega}},
\end{aligned} \tag{1}
$$

where $\bar{S}_{2\omega} = \bar{Z}_{g'\omega}\bar{Z}_{g''\omega} + \bar{Z}_{g\omega}\bar{Z}_{g'\omega} + \bar{Z}_{g\omega}\bar{Z}_{g''\omega}$. The complex expression for $\bar{E}_{g\omega}$ is obtained as $\bar{\imath}_1/jc_1\omega$ while that for $\bar{E}_{p\omega}$ is $\bar{\imath}_{1\omega}/jc_1\omega + \bar{\imath}_{2\omega}/jc_2\omega$.

Performing the calculation we find

$$(1.1)\qquad \begin{aligned}\bar{E}_{g\omega} &= \bar{G}_{1\omega}\bar{e}_\omega + \bar{G}_{g\omega}\bar{I}_{g\omega} + \bar{G}_{p\omega}\bar{I}_{p\omega},\\ \bar{E}_{p\omega} &= \bar{P}_{1\omega}\bar{e}_\omega + \bar{P}_{g\omega}\bar{I}_{g\omega} + \bar{P}_{p\omega}\bar{I}_{p\omega},\end{aligned}$$

where

$$\bar{G}_{1\omega} = \frac{\bar{Z}_{g''\omega}\left(\bar{Z}_{p\omega} + \frac{1}{jc_2\omega}\right)}{\bar{A}_\omega}, \qquad \bar{G}_{g\omega} = -\frac{\bar{S}_{2\omega}\left(\bar{Z}_{p\omega} + \frac{1}{jc_2\omega}\right)}{\bar{A}_\omega},$$

$$\bar{G}_{p\omega} = -\frac{\bar{S}_{2\omega}\bar{Z}_{p\omega}}{\bar{A}_\omega},$$

$$(1.2)\qquad \bar{P}_{1\omega} = \frac{\bar{Z}_{g''\omega}\bar{Z}_{p\omega}}{\bar{A}_\omega}, \qquad \bar{P}_{g\omega} = -\frac{\bar{S}_{2\omega}\bar{Z}_{p\omega}}{\bar{A}_\omega},$$

$$\bar{P}_{p\omega} = \frac{-\bar{Z}_{p\omega}\left[\bar{S}_{2\omega}\frac{c_1 + c_2}{c_2} + \frac{\bar{Z}_{g'\omega} + \bar{Z}_{g''\omega}}{jc_2\omega}\right]}{\bar{A}_\omega},$$

where

$$\bar{A}_\omega = \bar{S}_{2\omega}\left(1 + \frac{c_1}{c_2} + j\,\bar{Z}_{p\omega}c_1\omega\right) + (\bar{Z}g_\omega' + \bar{Z}g_\omega'')\left(\bar{Z}_{p\omega} + \frac{1}{jc_2\omega}\right).$$

We called the impressed electromotive force $A\cos\omega t$. To each value of A there corresponds a definite plate current. It is clear that if $A = 0$ the plate current is steady. Similarly the grid current is steady. Let these values of grid and plate current be I_{g_0}, I_{p_0} respectively. For a value of A distinct from zero

$$\begin{aligned}I_g &= I_{g_0} + \Delta I_g,\\ I_p &= I_{p_0} + \Delta I_p,\end{aligned}$$

where ΔI_g, ΔI_p are Fourier Series in ωt. We shall thus write

$$\Delta I_p = b_0 + b_1\cos\omega t + b_2\cos 2\omega t + \cdots + a_1\sin\omega t + a_2\sin 2\omega t + \cdots$$

$$\Delta I_g = \beta_0 + \beta_1\cos\omega t + \beta_2\cos 2\omega t + \cdots + a_1\sin\omega t + a_2\sin 2\omega t + \cdots.$$

Let us now write

$$\bar{G}_{1\omega} = G_{1\omega}e^{j\tan^{-1}\frac{G_{1\omega}''}{G_{1\omega}'}},$$

$$\bar{G}_{g\omega} = G_{g\omega}e^{j\tan^{-1}\frac{G_{g\omega}''}{G_{g\omega}'}},$$

then denoting by ΔE_g, ΔE_p the changes in the grid and plate voltage from the value corresponding to $A = 0$ we have:

$$\Delta E_g = G_{g_0}\beta_0 + G_{p_0}b_0 + G_{1\omega}A\cos\left(\omega t + \tan^{-1}\frac{G_{1\omega}''}{G_{1\omega}'}\right)$$

$$+G_{g\omega}\left[\beta_1 \cos\left(\omega t+\tan^{-1}\frac{G_{g\omega}''}{G_{g\omega}'}\right)+\alpha_1 \sin\left(\omega t+\tan^{-1}\frac{G_{g\omega}''}{G_{g\omega}'}\right)\right]$$

$$+G_{p\omega}\left[b_1 \cos\left(\omega t+\tan^{-1}\frac{G_{p\omega}''}{G_{p\omega}'}\right)+a_1 \sin\left(\omega t+\tan^{-1}\frac{G_{p\omega}''}{G_{p\omega}'}\right)\right]$$

$$+G_{g2\omega}\left[\beta_2 \cos\left(2\omega t+\tan^{-1}\frac{G_{g2\omega}''}{G_{g2\omega}'}\right)+\alpha_2 \sin\left(2\omega t+\tan^{-1}\frac{G_{g2\omega}''}{G_{g2\omega}'}\right)\right]$$

$$+G_{p2\omega}\left[b_2 \cos\left(2\omega t+\tan^{-1}\frac{G_{p2\omega}''}{G_{p2\omega}'}\right)+a_2 \sin\left(2\omega t+\tan^{-1}\frac{G_{p2\omega}''}{G_{p2\omega}'}\right)\right]$$

$$+\cdots,$$

(1.3)

$$\Delta E_p=P_{g_0}\beta_0+P_{p_0}b_0+P_{1\omega}A \cos\left(\omega t+\tan^{-1}\frac{P_{1\omega}''}{P_{1\omega}'}\right)$$

$$+P_{g\omega}\left[\beta_1 \cos\left(\omega t+\tan^{-1}\frac{P_{g\omega}''}{P_{g\omega}'}\right)+\alpha_1 \sin\left(\omega t+\tan^{-1}\frac{P_{g\omega}''}{P_{g\omega}'}\right)\right]$$

$$+P_{p\omega}\left[b_1 \cos\left(\omega t+\tan^{-1}\frac{P_{p\omega}''}{P_{p\omega}'}\right)+a_1 \sin\left(\omega t+\tan^{-1}\frac{P_{p\omega}''}{P_{p\omega}'}\right)\right]$$

$$+P_{g2\omega}\left[\beta_2 \cos\left(2\omega t+\tan^{-1}\frac{P_{g2\omega}''}{P_{g2\omega}'}\right)+\alpha_2 \sin\left(2\omega t+\tan^{-1}\frac{P_{g2\omega}''}{P_{g2\omega}'}\right)\right]$$

$$+P_{p2\omega}\left[b_2 \cos\left(2\omega t+\tan^{-1}\frac{P_{p2\omega}''}{P_{p2\omega}'}\right)+a_2 \sin\left(2\omega t+\tan^{-1}\frac{P_{p2\omega}''}{P_{p2\omega}'}\right)\right]$$

$$+\cdots.$$

But our assumption as to the nature of the relationship between I_p, I_g with E_p, E_g shows that for sufficiently small values of ΔE_p, ΔE_g

$$\Delta I_p = \frac{\partial I_p}{\partial E_p}\Delta E_p + \frac{\partial I_p}{\partial E_g}\Delta E_g + \frac{1}{2}\frac{\partial^2 I_p}{\partial E_p^2}(\Delta E_p)^2 + \frac{\partial^2 I_p}{\partial E_p \partial E_g}\Delta E_p \Delta E_g + \frac{1}{2}\frac{\partial^2 I_p}{\partial E_g^2}(\Delta E_g)^2,$$

$$\Delta I_g = \frac{\partial I_g}{\partial E_p}\Delta E_p + \frac{\partial I_g}{\partial E_p}\Delta E_g + \frac{1}{2}\frac{\partial^2 I_g}{\partial E_p^2}(\Delta E_p)^2 + \frac{\partial^2 I_g}{\partial E_p \partial E_g}\Delta E_p \Delta E_g + \frac{1}{2}\frac{\partial^2 I_g}{\partial E_g^2}(\Delta E_g)^2,$$

where the first derivatives are to be evaluated at $I_p = I_{p_0}$; $I_g = I_g$ while the second derivatives are to be evaluated for some suitably chosen values of E_p, E_g in the intervals

$$(E_{p_0}, E_{p_0} + \Delta E_p)(E_{g_0}, E_{g_0} + \Delta E_g).$$

If ΔE_p, ΔE_g are sufficiently small the values of the second derivatives are sensibly constant and can be taken simply as the values at (E_{p_0}, E_{g_0}).

We can, therefore, regard all the derivatives as constant coefficients.

If the mathematical expressions for I_p, I_g are known the derivatives will be obtained by ordinary differentiation. If no mathematical expression is available the derivatives are still easily obtainable from the graphs of the functions.

First Approximation. Internal Resistance. Amplification Constant. Input Impedance.

As a first approximation we can disregard the squares of ΔE_p, ΔE_g. Then

$$\Delta I_p = \frac{\partial I_p}{\partial E_p}\Delta E_p + \frac{\partial I_p}{\partial E_g}\Delta E_g, \qquad (2)$$

$$\Delta I_g = \frac{\partial I_g}{\partial E_p}\Delta E_p + \frac{\partial I_g}{\partial E_g}\Delta E_g.$$

Let us analyze ΔI_p, ΔI_g, ΔE_p, ΔE_g into a Fourier series, say $\Sigma\overline{\Delta I}_{p\omega}$, $\Sigma\overline{\Delta I}_{g\omega}$, $\Sigma\overline{\Delta E}_{p\omega}$, $\Sigma\overline{\Delta E}_{g\omega}$ where $\overline{\Delta I}_{p\omega}$ is of the form $Ae^{j\omega t}$ $(j = \sqrt{-1})$ and where it is understood that only the real parts of the equations are to be considered. Since the equations (2) are linear we can write

$$\overline{\Delta I}_{p\omega} = \frac{\partial I_p}{\partial E_p}\overline{\Delta E}_{p\omega} + \frac{\partial I_p}{\partial E_g}\overline{\Delta E}_{g\omega},$$

$$\overline{\Delta I}_{g\omega} = \frac{\partial I_g}{\partial E_g}\overline{\Delta E}_{p\omega} + \frac{\partial I_g}{\partial E_g}\overline{\Delta E}_{g\omega}.$$

Combining these with (1.1) we have

$$\bar{I}_{g\omega}\left(\frac{\partial I_p}{\partial E_p}\bar{P}_{g\omega} + \frac{\partial I_p}{\partial E_g}\bar{G}_{g\omega}\right) + \bar{I}_{p\omega}\left(\frac{\partial I_p}{\partial E_p}\bar{P}_{p\omega} + \frac{\partial I_p}{\partial E_g}\bar{G}_{g\omega} - 1\right) = -\bar{e}_\omega\left(\frac{\partial I_p}{\partial E_p}\bar{P}_{1\omega} + \frac{\partial I_p}{\partial E_g}\bar{G}_{1\omega}\right),$$

$$\bar{I}_{g\omega}\left(\frac{\partial I_g}{\partial E_p}\bar{P}_{g\omega} + \frac{\partial I_g}{\partial E_g}\bar{G}_{g\omega} - 1\right) + \bar{I}_{p\omega}\left(\frac{\partial I_g}{\partial E_p}\bar{P}_{p\omega} + \frac{\partial I_g}{\partial E_g}\bar{G}_{p\omega}\right) = -\bar{e}_\omega\left(\frac{\partial I_g}{\partial E_p}\bar{P}_{1\omega} + \frac{\partial I_g}{\partial E_g}\bar{G}_{1\omega}\right).$$

Hence

$$\bar{I}_{p\omega} = e_\omega \frac{\left(\frac{\partial I_g}{\partial E_p}\bar{P}_{g\omega} + \frac{\partial I_g}{\partial E_g}\bar{G}_{g\omega} - 1\right)\left(\frac{\partial I_p}{\partial E_p}\bar{P}_{1\omega} + \frac{\partial I_p}{\partial E_g}\bar{G}_{1\omega}\right) - \left(\frac{\partial I_p}{\partial E_p}\bar{P}_{g\omega} + \frac{\partial I_p}{\partial E_g}\bar{G}_{g\omega}\right)\left(\frac{\partial I_g}{\partial E_p}\bar{P}_{1\omega} + \frac{\partial I_g}{\partial E_g}\bar{G}_{1\omega}\right)}{\Delta}, \qquad (2.1)$$

$$\bar{I}_{g\omega} = e_\omega \frac{\left(\frac{\partial I_p}{\partial E_p}\bar{P}_{p\omega} + \frac{\partial I_p}{\partial E_g}\bar{G}_{p\omega} - 1\right)\left(\frac{\partial I_g}{\partial E_p}\bar{P}_{1\omega} + \frac{\partial I_g}{\partial E_g}\bar{G}_{1\omega}\right) - \left(\frac{\partial I_g}{\partial E_p}\bar{P}_{p\omega} + \frac{\partial I_g}{\partial E_g}\bar{G}_{p\omega}\right)\left(\frac{\partial I_p}{\partial E_p}\bar{P}_{1\omega} + \frac{\partial I_p}{\partial E_g}\bar{G}_{1\omega}\right)}{\Delta},$$

where

$$\Delta = \left(\frac{\partial I_p}{\partial E_p}\bar{P}_{g\omega} + \frac{\partial I_p}{\partial E_g}\bar{G}_{g\omega}\right)\left(\frac{\partial I_g}{\partial E_p}\bar{P}_{p\omega} + \frac{\partial I_g}{\partial E_g}\bar{G}_{p\omega}\right)$$

$$- \left(\frac{\partial I_p}{\partial E_p}\bar{P}_{p\omega} + \frac{\partial I_p}{\partial E_g}\bar{G}_{p\omega} - 1\right)\left(\frac{\partial I_g}{\partial E_p}\bar{P}_{g\omega} + \frac{\partial I_g}{\partial E_g}\bar{G}_{g\omega} - 1\right)$$

$$= -1 + \frac{\partial I_g}{\partial E_p}\bar{P}_{g\omega} + \frac{\partial I_g}{\partial E_g}\bar{G}_{g\omega} + \frac{\partial I_p}{\partial E_p}\bar{P}_{p\omega} + \frac{\partial I_p}{\partial E_g}\bar{G}_{p\omega}$$

$$+ \begin{vmatrix} P_{g\omega} & G_{g\omega} \\ P_{p\omega} & G_{p\omega} \end{vmatrix} \cdot \begin{vmatrix} \dfrac{\partial I_p}{\partial E_p} & \dfrac{\partial I_p}{\partial E_g} \\ \dfrac{\partial E_p}{\partial E_p} & \dfrac{\partial E_g}{\partial E_g} \end{vmatrix}.$$

Before proceeding with the second approximation of the problem we shall interpret the equations given. Let us consider the special case when $Z_{g\omega} = Z_{g'\omega} = 0$. Then (see 1.2) $\bar{S}_{2\omega} = 0$,

$$A_\omega = \bar{Z}_{g''\omega}\left(\bar{Z}_{p\omega} + \frac{1}{jc_2\omega}\right)$$

$$\bar{G}_{1\omega} = 1, \qquad \bar{G}_{g\omega} = 0, \qquad \bar{G}_{p\omega} = 0,$$

$$\bar{P}_{1\omega} = \frac{\bar{Z}_{p\omega}}{\bar{Z}_{p\omega} + \dfrac{1}{jc_2\omega}}, \qquad \bar{P}_{g\omega} = 0, \qquad \bar{P}_{p\omega} = -\frac{\dfrac{\bar{Z}_{p\omega}}{jc_2\omega}}{\bar{Z}_{p\omega} + \dfrac{1}{jc_2\omega}}. \tag{2.2}$$

From (2.1) and (2.2) we derive:

$$\bar{I}_{g\omega} = \bar{e}_\omega \frac{\dfrac{\partial I_g}{\partial E_g} + \dfrac{\partial I_g}{\partial E_p}\dfrac{jc_2\omega\bar{Z}_{p\omega}}{1 + jc_2\omega\bar{Z}_{p\omega}} + \dfrac{\partial(I_p, I_g)}{\partial(E_p, E_g)}\dfrac{\bar{Z}_{p\omega}}{1 + jc_2\omega\bar{Z}_{p\omega}}}{1 + \dfrac{\bar{Z}_{p\omega}}{1 + jc_2\omega\bar{Z}_{p\omega}}\dfrac{\partial \bar{I}_p}{\partial E_p}},$$

$$\bar{I}_{p\omega} = \bar{e}_\omega \frac{\dfrac{\partial I_p}{\partial E_p}\dfrac{jc_2\omega\bar{Z}_{p\omega}}{1 + jc_2\omega\bar{Z}_{p\omega}} + \dfrac{\partial I_p}{\partial E_g}}{1 + \dfrac{Z_{p\omega}}{1 + jc_2\omega\bar{Z}_{p\omega}}\dfrac{\partial I_p}{\partial E_p}}. \tag{2.3}$$

Also from (1′) and (2.2)

$$\bar{\imath}_{2\omega} = \frac{jc_2\omega(\bar{e}_\omega + \bar{Z}_{p\omega}\bar{I}_{p\omega})}{1 + jc_2\omega\bar{Z}_{p\omega}}. \tag{2.4}$$

Hence

$$\bar{I}_{g\omega} + \bar{\imath}_{2\omega} = \bar{e}_\omega\left[\frac{jc_2\omega}{1 + jc_2\omega\bar{Z}_{p\omega}} + \frac{\dfrac{\bar{Z}_{p\omega}^2}{\left(\bar{Z}_{p\omega} + \dfrac{1}{jc_2\omega}\right)^2}\dfrac{\partial I_p}{\partial E_p} + \dfrac{\bar{Z}_{p\omega}}{\bar{Z}_{p\omega} + \dfrac{1}{jc_2\omega}}\dfrac{\partial I_p}{\partial E_g}}{1 + \dfrac{\bar{Z}_{p\omega}}{1 + jc_2\omega\bar{Z}_{p\omega}}\dfrac{\partial I_p}{\partial E_p}}\right.$$

$$+\frac{\dfrac{\partial I_g}{\partial E_g}+\dfrac{\partial I_g}{\partial E_p}\dfrac{\bar{Z}_{p\omega}}{\bar{Z}_{p\omega}+\dfrac{1}{jc_2\omega}}+\dfrac{\partial(I_p,\ I_g)}{\partial(E_p,E_g)}\dfrac{\bar{Z}_{p\omega}/jc_2\omega}{\bar{Z}_{p\omega}+\dfrac{1}{jc_2\omega}}}{1+\dfrac{\bar{Z}_{p\omega}}{1+jc_2\omega\bar{Z}_{p\omega}}\dfrac{\partial I_p}{\partial E_p}}\Bigg]$$

$$=\bar{e}_\omega\frac{1+\bar{Z}_{p\omega}\left(\dfrac{\partial I_p}{\partial E_p}+\dfrac{\partial I_p}{\partial E_g}+\dfrac{\partial I_g}{\partial E_g}+\dfrac{\partial I_g}{\partial E_p}\right)+\dfrac{1}{jc_2\omega}\dfrac{\partial I_g}{\partial E_g}+\dfrac{\bar{Z}_{p\omega}}{jc_2\omega}\dfrac{\partial(I_p,\ I_g)}{\partial(E_p,\ E_g)}}{\bar{Z}_{p\omega}+\dfrac{1}{jc_2\omega}+\dfrac{\bar{Z}_{p\omega}}{jc_2\omega}\dfrac{\partial I_p}{\partial E_p}}.$$

The ratio $\bar{e}_\omega/(\bar{I}_{g\omega}+\bar{\imath}_{2\omega})$ may be called the *intrinsic* input impedance of the tube because (neglecting the current flowing through the capacity C_1) $\bar{I}_{g\omega}+\bar{\imath}_{2\omega}$ is the total current of frequency $\omega/2\pi$ which flows from the outside circuit into the tube. I call the quotient $\bar{e}_\omega/(\bar{I}_{g\omega}+\bar{\imath}_{2\omega})$ the *intrinsic* input impedance of the tube in distinction to the *true* input impedance which is $\bar{e}_\omega/(\bar{I}_{g\omega}+\bar{\imath}_{2\omega}-\bar{\imath}_{1\omega})$. There is really no advantage in knowing the true input impedance of the tube because the tube is always put in parallel with some other circuit, and the capcaity C_1 can, therefore, be considered as part of that other circuit. This procedure as well as the fact that we incorporate the capacity from filament to plate in $\bar{Z}_p$ simplifies our expressions.

Let $\bar{Z}_{i\omega}$ denote the *intrinsic input impedance.* Then

$$(3)\quad Z_{1\omega}=\frac{\bar{Z}_{p\omega}+\dfrac{1}{jc_2\omega}+\dfrac{\bar{Z}_{p\omega}}{jc_2\omega}\dfrac{\partial I_p}{\partial E_p}}{1+\bar{Z}_{p\omega}\left(\dfrac{\partial I_p}{\partial E_p}+\dfrac{\partial I_p}{\partial E_g}+\dfrac{\partial I_g}{\partial E_p}+\dfrac{\partial I_g}{\partial E_g}\right)+\dfrac{1}{jc_2\omega}\dfrac{\partial I_g}{\partial E_g}+\dfrac{\partial(I_p,I_g)}{\partial(E_p,E_g)}\dfrac{Z_{p\omega}}{jc_2\omega}}.$$

An inspection of this equation shows that both the numerator and denominator are linear in $\bar{Z}_{p\omega}$. This fact admits of an interesting interpretation. For it is clear that the relation between $\bar{R}_{p\omega}$ and $\bar{X}_{p\omega}$ $(\bar{Z}_{p\omega}=\bar{R}_{p\omega}+j\bar{X}_{p\omega})$ which will necessitate the real or the imaginary part of $\bar{Z}_{i\omega}$ to be constant is such as to make the pts. $(R_{p\omega}, X_{p\omega})$ referred to cartesian axes lie on a circle. Thus the lines of constant input resistance in the $R_{p\omega}$, $X_{p\omega}$ plane, or constant input reactance are circles. Moreover all of these circles must pass through one point, viz., the point corresponding to $\bar{Z}_{i\omega}=\infty$. Again since in the $\bar{Z}_{i\omega}$ plane the lines of input resistance are orthogonal to the lines of input reactances the same must be true in the $\bar{Z}_{p\omega}$ plane. This gives a general picture of the variation of $\bar{Z}_{i\omega}$ with $\bar{Z}_{p\omega}$. I have treated this matter in more detail for the special case of negative grid voltage.[1] It is seen, therefore, that so far as the general properties of the relationship between the plate circuit and the input impedance are concerned there is no difference between the case of positive and negative grid voltage.

A more thorough investigation of the expression (3) will show, however, an essential distinction between the two cases. This is given by the value of $\bar{Z}_{i\omega}$ for $\omega = 0$. In fact this value is

$$\bar{Z}_{i_0} = \frac{1 + \bar{Z}_{p_0} \dfrac{\partial I_p}{\partial E_p}}{\dfrac{\partial I_g}{\partial E_g} + \dfrac{\partial(I_p, I_g)}{\partial(E_p, E_g)} \bar{Z}_{p_0}}. \tag{3.1}$$

If the grid voltage is sufficiently negative I_g is constant and therefore $\bar{Z}_{i_0} = \infty$. In other words, there is no appreciable current absorbed from the external circuit when the frequency of the alternating current becomes sufficiently low and when $E_g < 0$.

On the contrary if $E_g > 0$ then no matter how low the frequency used may be a sufficient amount of current is absorbed from the external circuit by the tube to oblige us to replace the tube for purposes of computation by the fictitious resistance $\bar{Z}_{i_0}$ given in (3.1).

In most tubes $\partial I_p/\partial E_p$, $\partial I_g/\partial E_g$, $\partial I_p/\partial E_g$ are positive while $\partial I_g/\partial E_p$ is negative. Thus the Jacobian $\partial(I_p, I_g)/\partial(E_p, E_g)$ is positive and $\bar{Z}_{i_0}$ is positive.

The subject of input impedance has been worked out in detail for the case of negative grid voltage by Dr. J. M. Miller.[2] My formula (3) agrees with Dr. Miller's result if I_g = const. and if in Dr. Miller's formula C_1 and C_3 are put equal to zero. In fact both of these become

$$\bar{Z}_{i\omega} = \frac{\dfrac{\bar{Z}_{p\omega}}{jc_2\omega} + r_p\left(\bar{Z}_{p\omega} + \dfrac{1}{jc_2\omega}\right)}{(K+1)\bar{Z}_{p\omega} + r_p},$$

provided we set

$$\frac{\partial I_p}{\partial E_p} = \frac{1}{r_p}, \qquad \frac{\partial I_p}{\partial E_g} = K\frac{\partial I_p}{\partial E_p}.$$

The reasons why the internal resistance and the amplification constant K are to be identified with the expressions given will be discussed later.

In order to compute the input impedance in the general case it is convenient to call

[1] See Bureau of Standards Radio Laboratory Report 5–V.

[2] See Bureau of Standards Scientific Paper No. 351.

$$A_1 = \left(\frac{\partial I_p}{\partial E_p} + \frac{\partial I_p}{\partial E_g} + \frac{\partial I_g}{\partial E_p} + \frac{\partial I_g}{\partial E_g}\right) \Big/ \frac{\partial I_p}{\partial E_p};$$

$$B = \frac{\partial(I_p,\ I_g)}{\partial(E_p,\ E_g)} \Big/ \frac{\partial I_p}{\partial E_p};$$

$$C = \frac{\partial I_g}{\partial E_g} \Big/ \frac{\partial I_p}{\partial E_p}; \qquad r_p = \frac{1}{\dfrac{\partial I_p}{\partial E_p}}.$$

Then

$$\bar{Z}_{i\omega} = \frac{r_p\left(\bar{Z}_{p\omega} + \dfrac{1}{jC_2\omega}\right) + \dfrac{\bar{Z}_{p\omega}}{jC_2\omega}}{r_p + A_1\bar{Z}_{p\omega} + \dfrac{C}{jC_2\omega} + B\dfrac{\bar{Z}_{p\omega}}{jC_2\omega}}.$$

Writing

$$\bar{Z}_{i\omega} = R_{i\omega} + jX_{i\omega}, \qquad \bar{Z}_{p\omega} = R_{p\omega} + jX_{p\omega},$$

$$R_{i\omega} = \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2}; \qquad X_{i\omega} = \frac{-\alpha\delta + \beta\gamma}{\gamma^2 + \delta^2}, \qquad \text{where}$$

$$(3.2) \qquad \alpha = R_{p\omega}r_p + \frac{X_{p\omega}}{C_2\omega}, \qquad \beta = r_pX_{p\omega} - \frac{r_p + R_{p\omega}}{C_2\omega},$$

$$\gamma = r_p + A_1R_{p\omega} + B\frac{X_{p\omega}}{C_2\omega}, \qquad \delta = A_1X_{p\omega} - \frac{C}{C_2\omega} - B\frac{R_{p\omega}}{C_2\omega}.$$

It is thus seen that a knowledge of the four derivatives $\partial I_p/\partial E_p$, $\partial I_p/\partial E_g$, $\partial I_g/\partial E_g$, $\partial E_g/\partial E_p$ of C_2 and of $R_{p\omega}$, $X_{p\omega}$ is sufficient to enable one to calculate the input resistance and input reactance. We can thus consider the problem of input impedance as solved for both positive and negative grid voltages.

Computation of Current in Plate Circuit.

In many cases it is necessary to know the quantity $I_{p\omega} - I_{2\omega}$ because this, to a first approximation is the current in Z_p. Such is, for example the case of a transformer-coupled amplifier. Now from (2.3) and (2.4,

$$\bar{I}_{p\omega} - \bar{i}_{2\omega} = \frac{-jC_2\omega\bar{e}_\omega}{1 + jC_2\omega\bar{Z}_{p\omega}} + \frac{\bar{I}_{p\omega}}{1 + jC_2\omega\bar{Z}_{p\omega}}$$

$$(4) \qquad = \frac{\bar{e}_\omega}{1 + jC_2\omega\bar{Z}_{p\omega}}\left[-jC_2\omega + \frac{\dfrac{\partial I_p}{\partial E_g} + \dfrac{\partial I_p}{\partial E_p}\dfrac{jC_2\omega}{1 + jC_2\omega\bar{Z}_{p\omega}}}{1 + \dfrac{\bar{Z}_{p\omega}}{1 + jC_2\omega\bar{Z}_{p\omega}}\dfrac{\partial I_p}{\partial E_p}}\right]$$

$$= \frac{\bar{e}_\omega\left[-jC_2\omega + \dfrac{\partial I_p}{\partial E_g}\right]}{1 + jC_2\omega\bar{Z}_{p\omega} + \bar{Z}_{p\omega}\dfrac{\partial I_p}{\partial E_p}}.$$

This simple relation connects the current through Z_p with the voltage from the filament to the grid of the tube. It is seen that $\partial I_g/\partial E_p$, $\partial I_g/\partial E_g$ do not come into this expression as could be expected. If $c_2\omega$ is sufficiently small

$$\bar{I}_{p\omega} - \bar{\imath}_{2\omega} = \bar{e}_\omega \frac{\dfrac{\partial I_p}{\partial E_g}}{1 + \bar{Z}_{p\omega}\dfrac{\partial I_p}{\partial E_p}} = \frac{K\bar{e}_\omega}{r_p + \bar{Z}_{p\omega}}, \tag{4.1}$$

where

$$K = \frac{\partial I_p}{\partial E_g}\bigg/\frac{\partial I_p}{\partial E_p} \qquad r_p = \frac{1}{\dfrac{\partial I_p}{\partial E_p}}.$$

A simple interpretation can be given to this last expression. In fact it shows that if C_2 is negligible the current through Z_p can be imagined as due to an electromotive force $K\bar{e}_\omega$ acting in series with $\bar{Z}_{p\omega}$ and a pure resistance, r_p inside the tube. The positive direction of this electromotive force is visibly in the direction from filament to plate outside the tube. For this reason we can call K the amplification factor and r_p the internal resistance of the tube. We note the fact that Van der Bijl's relation is not used in making these definitions.

If C_2 is not negligible we can still write (4) as:

$$\bar{I}_{p\omega} - \bar{\imath}_{2\omega} = \frac{\bar{e}_\omega \dfrac{\dfrac{\partial I_p}{\partial E_g} - jC_2\omega}{jC_2\omega + \dfrac{\partial I_p}{\partial E_p}}}{\bar{Z}_{p\omega} + \dfrac{1}{\dfrac{\partial I_p}{\partial E_p} + jC_2\omega}}$$

and writing

$$\bar{\zeta}_\omega = \frac{1}{\dfrac{\partial I_p}{\partial E_p} + jC_2\omega},$$

$$\bar{k}_\omega = \frac{\dfrac{\partial I_p}{\partial E_g} - jC_2\omega}{jC_2\omega + \dfrac{\partial I_p}{\partial E_p}}, \tag{4.2}$$

$$\bar{I}_{p\omega} - \bar{\imath}_{2\omega} = \frac{\bar{k}_\omega\bar{e}_\omega}{\bar{Z}_{p\omega} + \bar{\zeta}_\omega}. \tag{4.3}$$

Thus $\bar{\zeta}_\omega$ may be spoken of as the complex internal resistance of the tube and $\bar{k}_\omega$ as its complex amplification factor at frequency $\omega/2\pi$.

It is of interest to note the connection between $\bar{\zeta}_\omega$ and r_p. For obviously

$$\bar{\zeta}_\omega = \frac{1}{\dfrac{1}{r_p} + 1 \Big/ \dfrac{1}{jC_2\omega}}.$$

This shows that *the complex internal resistance can be obtained by imagining the capacity C_2 connected in parallel with the internal resistance r_p.*

Second Approximation.

Having calculated our quantities to the first order we can proceed to the calculation of b_0 and β_0. From equations (1.3) we find that the constant terms of $(\Delta E_g)^2$, $(\Delta E_g(\Delta E_p))$, $(\Delta E_p)^2$ are neglecting quantities of higher order than the second. In $(\Delta E_g)^2$:

$$\begin{aligned}
&\frac{G_{1\omega}^2 A^2}{2} + \frac{G_{g\omega}^2}{2}(\beta_1^2 + \alpha_1^2) + \frac{G_{p\omega}^2}{2}(b_1^2 + a_1^2) \\
&\qquad + A\beta_1(G_{g\omega}'G_{1\omega}' + G_{g\omega}''G_{1\omega}'') + A\alpha_1(G_{g\omega}''G_{1\omega}' - G_{g\omega}'G_{1\omega}'') \\
&\qquad + Ab_1(G_{p\omega}'G_{1\omega}' + G_{p\omega}''G_{1\omega}'') + Aa_1(G_{p\omega}''G_{1\omega}' - G_{p\omega}'G_{1\omega}'') \\
&\qquad + (-\alpha_1 a_1 + \beta_1 b_1)(G_{g\omega}'G_{p\omega}' + G_{g\omega}''G_{p\omega}'') \\
(5) &\qquad + (\beta_1 a_1 - \alpha_1 b_1)(G_{p\omega}''G_{g\omega}' - G_{p\omega}'G_{g\omega}'') = 2R_{gg}.
\end{aligned}$$

In $2\Delta E_g \Delta E_p$:

$$\begin{aligned}
&\quad A^2(P_{1\omega}'G_{1\omega}' + P_{1\omega}''G_{1\omega}'') + A\beta_1[G_{1\omega}'P_{g\omega}' + G_{1\omega}''P_{g\omega}'' \\
&\qquad + P_{1\omega}'G_{g\omega}' + P_{1\omega}''G_{g\omega}''] + A\alpha_1[P_{g\omega}''G_{1\omega}' - P_{g\omega}'G_{1\omega}'' \\
&\qquad + P_{1\omega}'G_{g\omega}'' - G_{g\omega}'P_{1\omega}''] + Ab_1(G_{1\omega}'P_{p\omega}' + G_{1\omega}''P_{p\omega}'' \\
&\qquad + P_{1\omega}''G_{p\omega}'' + P_{1\omega}'G_{p\omega}') + Aa_1(P_{p\omega}''G_{1\omega}' - P_{p\omega}'G_{1\omega}'' \\
&\qquad + G_{p\omega}''P_{1\omega}' - G_{p\omega}'P_{1\omega}'') + (P_{g\omega}'G_{g\omega}' + P_{g\omega}''G_{g\omega}'')(\beta_1^2 - \alpha_1^2) \\
&\qquad + (P_{p\omega}'G_{p\omega}' + P_{p\omega}''G_{p\omega}'')(b_1^2 - a_1^2) \\
&\qquad + (\alpha_1 b_1 - a_1\beta_1)(G_{g\omega}''P_{p\omega}' - P_{p\omega}''G_{g\omega}' + P_{g\omega}''G_{p\omega}' - P_{g\omega}'G_{p\omega}'') \\
&\qquad + (\beta_1 b_1 - \alpha_1 a_1)(G_{g\omega}'P_{p\omega}' + G_{g\omega}''P_{p\omega}'' + P_{g\omega}'G_{p\omega}' \\
(5.1) &\quad + P_{g\omega}''G_{p\omega}'') = 2R_{gp}.
\end{aligned}$$

$$\begin{aligned}
&\text{In } (\Delta E_p)^2 = \frac{P_{1\omega}^2 A^2}{2} + \frac{P_{g\omega}^2}{2}(\beta_1^2 + \alpha_1^2) + \frac{P_{p\omega}^2}{2}(b_1^2 + a_1^2) \\
&\qquad + A\beta_1(P_{g\omega}'P_{1\omega}' + P_{g\omega}''P_{1\omega}'') + A\alpha_1(P_{g\omega}''P_{1\omega}' - P_{g\omega}'P_{1\omega}'') \\
&\qquad + Ab_1(P_{p\omega}'P_{1\omega}' + P_{p\omega}''P_{1\omega}'') + Aa_1(P_{p\omega}''P_{1\omega}' - P_{p\omega}'P_{1\omega}'') \\
&\qquad + (\beta_1 b_1 - \alpha_1 a_1)(P_{g\omega}'P_{p\omega}' + P_{g\omega}''P_{p\omega}'') \\
(5.2) &\quad + (\beta_1 a_1 - \alpha_1 b_1)(P_{p\omega}''P_{g\omega}' - P_{p\omega}'P_{g\omega}'') = 2R_{pp}.
\end{aligned}$$

These expressions are calculable if we content ourselves with using values of a_1, b_1, α_1, β_1 as obtained by our first approximation.

In most practical cases the expressions for R_{gg}, R_{gp}, R_{pp} simplify considerably, but for the sake of generality I preferred giving their complete expression.

Identifying now constant terms on both sides of the equations

$$\Delta I_p = \frac{\partial I_p}{\partial E_p}\Delta E_p + \frac{\partial I_p}{\partial E_g}\Delta E_g + \frac{1}{2}\frac{\partial^2 I_p}{\partial E_g^2}(\Delta E_g)^2 + \frac{\partial^2 I_p}{\partial E_p \partial E_g}\Delta E_p \Delta E_g + \frac{\partial^2 I_p}{\partial E_p^2}(\Delta E_p)^2,$$

$$\Delta I_g = \frac{\partial I_g}{\partial E_p}\Delta E_p + \frac{\partial I_g}{\partial E_g}\Delta E_g + \frac{1}{2}\frac{\partial^2 I_g}{\partial E_g^2}(\Delta E_g)^2 + \frac{\partial^2 I_g}{\partial E_p \partial E_g}\Delta E_p \cdot \Delta E_g + \frac{\partial^2 I_g}{\partial E_p^2}(\Delta E_p)^2,$$

we find

$$b_0\left(1 - P_{p0}\frac{\partial I_p}{\partial E_p} - G_{p0}\frac{\partial I_p}{\partial E_g}\right) - \beta_0\left(P_{g0}\frac{\partial I_p}{\partial E_p} + G_{g0}\frac{\partial I_p}{\partial E_g}\right) = C_p,$$

$$-b_0\left(P_{p0}\frac{\partial I_g}{\partial E_p} + G_{p0}\frac{\partial I_g}{\partial E_g}\right) + \beta_0\left(1 - P_{g0}\frac{\partial I_g}{\partial E_p} - G_{g0}\frac{\partial I_g}{\partial E_g}\right) = C_g,$$

where

$$C_p = \frac{\partial^2 I_p}{\partial E_p^2}R_{pp} + \frac{\partial^2 I_p}{\partial E_p \partial E_g}R_{pg} + \frac{\partial^2 I_p}{\partial E_g^2}R_{gg},$$

(5.3)

$$C_g = \frac{\partial^2 I_g}{\partial E_p^2}R_{pp} + \frac{\partial^2 I_g}{\partial E_p \partial E_g}R_{pg} + \frac{\partial^2 I_g}{\partial E_g^2}R_{gg}.$$

Solving these two simultaneous equations in b, β, we obtain

$$b_0 = \frac{C_p\left(1 - P_{g0}\frac{\partial I_g}{\partial E_p} - G_{g0}\frac{\partial I_g}{\partial E_g}\right) + C_g\left(P_{g0}\frac{\partial I_p}{\partial E_p} + G_{g0}\frac{\partial I_p}{\partial E_g}\right)}{\left(1 - P_{p0}\frac{\partial I_p}{\partial E_p} - G_{p0}\frac{\partial I_p}{\partial E_g}\right)\left(1 - P_{g0}\frac{\partial I_g}{\partial E_p} - G_{g0}\frac{\partial I_g}{\partial E_g}\right) - \left(P_{g0}\frac{\partial I_p}{\partial E_p} + G_{g0}\frac{\partial I_p}{\partial E_g}\right)\left(P_{p0}\frac{\partial I_p}{\partial E_p} + G_{p0}\frac{\partial I_g}{\partial E_p}\right)},$$

(5.4)

$$\beta_0 = \frac{C_g\left(1 - P_{p0}\frac{\partial I_p}{\partial E_p} - G_{p0}\frac{\partial I_p}{\partial E_g}\right) + C_p\left(P_{p0}\frac{\partial I_g}{\partial E_p} + G_{p0}\frac{\partial I_g}{\partial E_g}\right)}{\left(1 - P_{p0}\frac{\partial I_p}{\partial E_p} - G_{p0}\frac{\partial I_p}{\partial E_g}\right)\left(1 - P_{g0}\frac{\partial I_g}{\partial E_p} - G_{g0}\frac{\partial I_g}{\partial E_g}\right) - \left(P_{g0}\frac{\partial I_p}{\partial E_p} + G_{g0}\frac{\partial I_p}{\partial E_g}\right)\left(P_{p0}\frac{\partial I_p}{\partial E_p} + G_{p0}\frac{\partial I_g}{\partial E_p}\right)}.$$

The first of these expressions enables us to calculate the average change in plate current due to the electromotive force, $A \cos \omega t$, impressed in series with Zg'. In a similar manner the second equation gives the average change in the grid current. The amplitude A enters in these equations only through C_p, C_g which are homogeneous linear functions of R_{gg}, R_{gp}, R_{pp}. Since α_1, β_1, a_1, b_1, are as a first approximation proportional to A it is readily seen from (5), (5.1), (5.2) that R_{gg}, R_{gp}, R_{pp} are proportional to A^2. Hence our expressions for b_0, β_0 are proportional to A^2. In general, of course, there is no reason why this law should be strictly true. But it is true that the smaller A the more nearly b_0, β_0 are proportional to A^2.

One of the fundamental assumptions in our discussion was that the electromotive force $A \cos \omega t$ has been impressed on the circuit so long that the effect of the initial conditions has been obliterated. The time necessary for this is generally very small. Let us change A slowly enough to justify us in considering all of the currents at any instant as having values identical with the values which they would have if A had had the value which it has at that instant during an infinite time before the instant under consideration. If such is the case the value of b_0 at any instant is proportional to the value which A has at that instant. Thus, if A should vary as $1 + K \cos pt$ where p is sufficiently small to secure conditions discussed above, the average change in the plate current will vary as $1 + (K^2/2) + 2K \cos pt + (K^2/2) \cos 2pt$.

Special Cases.

The expressions (5.4) apply to the most general case under consideration and enable us to calculate b_0, β_0 when the E.M.F. impressed in series with Z_g' is known. We have shown, however, how the action of the tube on the currents in the circuits Z_g, Z_g'', Z_g' can be computed by endowing the tube with an *input impedance*. Since our expressions for R_{pp}, R_{pg}, R_{gg} involve only the squares of first order terms in A we can for convenience of computation divide the problem into two parts. The first will concern itself with a calculation of the voltage from F to G account being taken only of first powers in A. The second will deal with the calculation of b_0 assuming voltage as known. Using formula (3) we have no difficulty in solving the first phase of the problem.

In order to deduce a solution of the second phase from our general expressions we imagine a case when $\bar{Z}_{g'\omega} = \bar{Z}_{g\omega} = 0$. Then equations (1.2) simplify as in (2.2) and $\bar{I}_{g\omega}$, $\bar{I}_{p\omega}$ are given as in (2.3). Also from (5), (5.1), (5.2) we have

$$
(5.5)\quad
\begin{aligned}
2R_{gg} &= \frac{A^2}{2},\\
2R_{gp} &= A^2P_{1\omega}' + A(b_1P_{p\omega}' + a_1P_{p\omega}''),\\
2R_{pp} &= \frac{P_{1\omega}^2}{2}A^2 + \frac{P_{p\omega}^2}{2}(a_1^2+b_1^2) + Ab_1(P_{p\omega}'P_{1\omega}' + P_{p\omega}''P_{1\omega}'')\\
&\qquad + Aa_1(P_{p\omega}''P_{1\omega}' - P_{p\omega}'P_{1\omega}'').
\end{aligned}
$$

But it follows from expressions (2.2) that

$$P_{1\omega}' = + C_2\omega P_{p\omega}'' \text{ and } P_{1\omega}'' = - C_2\omega P_{p\omega}'.$$

Hence

$$P_{p\omega}'P_{1\omega}' + P_{p\omega}''P_{1\omega}'' = 0$$

and

$$P_{p\omega}''P_{1\omega}' - P_{p\omega}'P_{1\omega}'' = C_2\omega P_{p\omega}^2.$$

Thus

$$2R_{pp} = \frac{P_{1\omega}^2}{2}A^2 + \frac{P_{p\omega}^2}{2}(a_1^2+b_1^2) + Aa_1c_2\omega P_{p\omega}^2. \tag{5.5}$$

We note that α_1, β_1 do not occur in these expressions. Now (2.3) can be written

$$
\begin{aligned}
\bar{I}_{p\omega} &= \bar{e}_\omega \frac{-\dfrac{\partial I_p}{\partial E_p} jC_2\omega\bar{P}_{p\omega} + \dfrac{\partial I_p}{\partial E_g}}{1 - \bar{P}_{p\omega}\dfrac{\partial I_p}{\partial E_p}} = \bar{e}_\omega \frac{K - jC_2\omega\bar{P}_{p\omega}}{r_p - \bar{P}_{p\omega}}\\
&= \bar{e}_\omega \frac{(K + C_2\omega P_{p\omega}'') - jC_2\omega P_{p\omega}'}{r_p - P_{p\omega}' - jP_{p\omega}''}\\
&= \bar{e}_\omega \left\{ \frac{(K + C_2\omega P_{p\omega}'')(r_p - P_{p\omega}') + C_2\omega P_{p\omega}'P_{p\omega}''}{(r_p - P_{p\omega}')^2 + P_{p\omega}''^2} \right.\\
&\qquad \left. + j\frac{P_{p\omega}''(K + C_2\omega P_{p\omega}'') - C_2\omega(r_p - P_{p\omega}')P_{p\omega}'}{(r_p - P_{p\omega}')^2 + P_{p\omega}''^2} \right\}\\
&= \bar{e}_\omega \frac{[K(r_p - P_{p\omega}') + r_pC_2\omega P_{p\omega}''] + j[KP_{p\omega}'' + C_2\omega P_{p\omega}^2 - r_pC_2\omega P_{p\omega}']}{(r_p - P_{p\omega}')^2 + P_{p\omega}''^2}.
\end{aligned}
$$

In our case

$$\bar{e}_\omega = A\,(\cos\omega t + j\sin\omega t).$$

Therefore

$$b_1 = A\frac{K(r_p - P_{p\omega}') + r_pC_2\omega P_{p\omega}''}{(r_p - P_{p\omega}')^2 + P_{p\omega}''^2},$$

$$a_1 = A\frac{r_pC_2\omega P_{p\omega}' - C_2\omega P_{p\omega}^2 - KP_{p\omega}''}{(r_p - P_{p\omega}')^2 + P_{p\omega}''^2}.$$

Also

$$a_1^2 + b_1^2 = A^2 \frac{(K + C_2\omega P_{p\omega}'')^2 + C_2^2\omega^2 P_{p\omega}'^2}{(r_p - P_{p\omega}')^2 + P_{p\omega}''^2},$$

$$A(b_1 P_{p\omega}' + a_1 P_{p\omega}'')$$

$$= A^2 \frac{-KP_{p\omega}^2 + Kr_p P_{p\omega}' + 2r_p C_2\omega P_{p\omega}' P_{p\omega}'' - C_2\omega P_{p\omega}'' P_{p\omega}^2}{(r_p - P_{p\omega}')^2 + P_{p\omega}''^2}$$

$$= A^2 \left[- C_2\omega P_{p\omega}'' + \frac{r_p^2 C_2\omega P_{p\omega}'' - KP_{p\omega}^2 + Kr_p P_{p\omega}'}{(r_p - P_{p\omega}')^2 + P_{p\omega}''^2} \right].$$

We thus derive from (5.5) and (5.5)′ that

$$R_{gg} = \frac{A^2}{4}, \qquad R_{gp} = \frac{A^2}{2} \frac{r_p^2 C_2\omega P_{p\omega}'' - KP_{p\omega}^2 + Kr_p P_{p\omega}'}{(r_p - P_{p\omega}')^2 + P_{p\omega}''^2},$$

because $P_{1\omega}' = C_2\omega P_{p\omega}''$,

$$R_{pp} = \frac{A^2}{4} C_2^2\omega^2 P_{p\omega}^2 + \frac{A^2}{4} P_{p\omega}^2 \frac{(K + C_2\omega P_{p\omega}'')^2 + C_2^2\omega^2 P_{p\omega}^2}{(r_p - P_{p\omega}')^2 + P_{p\omega}''^2}$$

$$(5.6) \qquad + \frac{A^2}{2} C_2\omega P_{p\omega}^2 \frac{r_p C_2\omega P_{p\omega}' - C_2\omega P_{p\omega}^2 - KP_{p\omega}''}{(r_p - P_{p\omega}')^2 + P_{p\omega}''^2}$$

$$= \frac{A^2}{4} P_{p\omega}^2 \frac{K^2 + r_p^2 C_2^2\omega^2}{(r_p - P_{p\omega}')^2 + P_{p\omega}''^2}.$$

Now from (5.4) and (5.3) it follows that

$$b_0 = \alpha_{pp} R_{pp} + \alpha_{pg} R_{pg} + \alpha_{gg} R_{gg},$$

where

$$\Delta' \alpha_{pp} = \frac{\partial^2 I_p}{\partial E_p^2}\lambda + \frac{\partial^2 I_g}{\partial E_p^2}\mu, \qquad \text{where} \qquad \lambda = 1 - P_{g_0}\frac{\partial I_g}{\partial E_p} - G_{g_0}\frac{\partial I_g}{\partial E_g},$$

$$\Delta' \alpha_{pg} = \frac{\partial^2 I_p}{\partial E_p \partial E_g}\lambda + \frac{\partial^2 I_g}{\partial E_p \partial E_g}\mu, \qquad \mu = P_{g_0}\frac{\partial I_p}{\partial E_p} + G_{g_0}\frac{\partial I_p}{\partial E_g},$$

$$\Delta' \alpha_{gg} = \frac{\partial^2 I_p}{\partial E_g^2}\lambda + \frac{\partial^2 I_g}{\partial E_g^2}\mu,$$

and

$$\Delta' = \left(1 - P_{p_0}\frac{\partial I_p}{\partial E_p} - G_{p_0}\frac{\partial I_p}{\partial E_g}\right)\left(1 - P_{g_0}\frac{\partial I_g}{\partial E_p} - G_{g_0}\frac{\partial I_g}{\partial E_g}\right)$$

$$- \left(P_{g_0}\frac{\partial I_p}{\partial E_p} + G_{g_0}\frac{\partial I_p}{\partial E_g}\right)\left(P_{p_0}\frac{\partial I_p}{\partial E_p} + G_{p_0}\frac{\partial I_g}{\partial E_p}\right).$$

Using the values of R_{pp}, R_{pg}, R_{gg} obtained in (5.6) we have

$$b_0 = \alpha_{gg}\frac{A^2}{4} + \alpha_{gp}\frac{A^2}{2}\frac{r_p^2 C_2\omega P_{p\omega}'' - KP_{p\omega}^2 + Kr_p P_{p\omega}'}{(r_p - P_{p\omega}')^2 + P_{p\omega}''^2}$$

$$(5.7) \qquad + \alpha_{pp}\frac{A^2}{4} P_{p\omega}^2 \frac{K^2 + r_p^2 \cdot C_2^2\omega^2}{(r_p - P_{p\omega}')^2 + P_{p\omega}''^2}.$$

This expression shows the nature of the dependence of the rectified component of the plate current on the constants of the plate circuit. In fact the quantity

$$\bar{P}_{p\omega} = -\frac{\bar{Z}_{p\omega}}{1 + jc_2\omega\bar{Z}_{p\omega}} = -\frac{1}{\dfrac{1}{\bar{Z}_{p\omega}} + \dfrac{1}{(jc_2\omega)^{-1}}}$$

can be seen to be the negative of the impedance which would be obtained by connecting the capacity C_2 across the plate circuit impedance $\bar{Z}_{p\omega}$.

If $P_{p\omega}$ is very large the predominating terms in (5.7) are

$$\alpha_{gg}\frac{A^2}{4} - \frac{K\alpha_{gp}}{2}A^2 + \alpha_{pp}\frac{A^2}{4}(K^2 + r_p^2C_2^2\omega^2).$$

It is seen, therefore, that in general there is no reason why b_0 should become zero when $P_{p\omega} = \infty$. It is also seen that on account of the term $r_p^2C_2^2\omega^2$ multiplying into α_{pp} even if it were possible to make $b_0 = 0$ by $P_{p\omega} = \infty$ for one value of ω this in general would not be true for a different ω. The condition which is necessary and sufficient to make $b_0 = 0$ when $P_{p\omega} = \infty$ for the case when $r_p^2C_2^2\omega^2$ is negligible in comparison to K^2 is $\alpha_{gg} - 2K\alpha_{gp} + \alpha_{pp}K^2 = 0$.

Case of Negative Grid Voltage.

It is of interest to see how our equations reduce in the case of negative grid voltage when $I_g = \text{const.} = 0$. In this case

$$\Delta' = 1 + \frac{Z_{p_0}}{r_p}, \qquad \alpha_{pp} = \frac{\dfrac{\partial^2 I_p}{\partial E_p^2}}{1 + \dfrac{Z_{p_0}}{r_p}}, \qquad \alpha_{pg} = \frac{\dfrac{\partial^2 I_p}{\partial E_p \partial E_g}}{1 + \dfrac{Z_{p_0}}{r_p}}, \qquad \alpha_{gg} = \frac{\dfrac{\partial^2 I_p}{\partial E_g^2}}{1 + \dfrac{Z_{p_0}}{r_p}}.$$

Hence

$$\left(1 + \frac{Z_{p_0}}{r_p}\right)b_0 = \frac{A^2}{4}\left[\frac{\partial^2 I_p}{\partial E_g^2} + 2\frac{\partial^2 I_p}{\partial E_p \partial E_g}\frac{r_p^2C_2\omega P_{p\omega}'' - KP_{p\omega}^2 + Kr_pP_{p\omega}'}{(r_p - P_{p\omega}')^2 + P_{p\omega}''^2} + \frac{\partial^2 I_p}{\partial E_p^2}\frac{P_{p\omega}^2(K^2 + r_p^2C_2^2\omega^2)}{(r_p - P_{p\omega}')^2 + P_{p\omega}''^2}\right].$$

It is convenient to write $\bar{Z}_\omega = -\bar{P}_{p\omega} = R_\omega + jX_\omega$; R_ω, X_ω being both real. Then

$$\left(1 + \frac{Z_{p_0}}{r_p}\right)b_0 = \frac{A^2}{4}\left[\frac{\partial^2 I_p}{\partial E_g^2} - 2\frac{\partial^2 I_p}{\partial E_p \partial E_g}\frac{r_p^2C_2\omega X_\omega + KZ_\omega^2 + Kr_pR_\omega}{(r_p + R_\omega Z^2 + X_\omega^2} + \frac{\partial^2 I_p}{\partial E_p^2}\frac{Z_\omega^2(K^2 + r_1^2C_2^2\omega^2)}{(r_p + R_\omega)^2 + x_\omega^2}\right].$$

Similarly of course (5.7) could be written

$$(5.7)' \quad b_0 = \frac{A^2}{4}\left[\alpha_{gg} - 2\alpha_{gp}\frac{r_p^2C_2\omega X_\omega + KZ_\omega^2 + Kr_pR_\omega}{(r_p+R_\omega)^2+X_\omega^2} + \alpha_{pp}\frac{Z_\omega^2(K^2+r_p^2C_2^2\omega^2)}{(r_p+R_\omega)^2+X^2_\omega}\right].$$

It is of interest to investigate the conditions for which b_0 can be made equal to zero or else to reverse sign by a proper choice of R_ω and X_ω. Clearing fractions

$$b_0 = \frac{A^2}{4}\frac{(R_\omega^2+X_\omega^2)(\alpha_{gg} - 2K\alpha_{gp} + \alpha_{pp}(K^2+r_p^2C_2^2\omega^2)) + 2r_pR_\omega(\alpha_{gg} - K\alpha_{gp}) - 2\alpha_{gp}r_p^2C_2\omega X_\omega + r_p^2\alpha_{gg}}{(r_p+R_\omega)^2+X_\omega^2}.$$

If $\alpha_{gg} - 2K\alpha_{gp} + \alpha_{pp}(K^2 + r_p^2C_2^2\omega^2) \neq 0$ the denominator cannot become infinite without making the numerator infinite and of the same order. Hence excluding the case mentioned the condition is that the equation

$$(R_\omega^2+X_\omega^2)(\alpha_{gg} - 2K\alpha_{gp} + \alpha_{pp}(K^2+r_p^2C_2^2\omega^2)) + 2r_p(\alpha_{gg} - K\alpha_{gp})R_\omega - 2\alpha_{gp}r_p^2C_2\omega X_\omega + r_p^2\alpha_{gg} = 0 \quad (6.0)$$

be satisfied by a possible pair of values of R_ω and X_ω. The equation written is the equation of a circle in Cartesian coördinates. This circle is real if

$$-r_p^2\alpha_{gg}(\alpha_{gg} - 2K\alpha_{gp} + \alpha_{pp}(K_2 + r_p^2C_2^2\omega^2)) + r_p^2(\alpha_{gg} - K\alpha_{gp})^2 + \alpha_{gp}^2r_p^4C_2^2\omega^2 > 0 \text{ or } \alpha_{gp}^2 > \alpha_{gg}\alpha_{pp}. \quad (6.1)$$

This therefore is the condition which is necessary in order that it be possible to reverse the sign of b_0 by merely changing the plate circuit. If the grid voltage is negative this condition becomes

$$\left(\frac{\partial^2 I_p}{\partial E_p \partial E_g}\right)^2 > \frac{\partial^2 I_p}{\partial E_g^2}\frac{\partial^2 I_p}{\partial E_p^2}.$$

It is remarkable that for tubes obeying Van der Bijl's relation

$$\left(\frac{\partial^2 I_p}{\partial E_p \partial E_g}\right)^2 = \frac{\partial^2 I_p}{\partial E_g^2}\cdot\frac{\partial^2 I_p}{\partial E_p^2}.$$

This corresponds to the case when circle (6.0) collapses to a point. Thus b_0 can always be reduced to zero but cannot be made to reverse sign. As a matter of fact the circle (6.0) reduces in this case to

$$R_\omega^2 + \left(X_\omega - \frac{K}{C_2\omega}\right)^2 = 0.$$

Thus $b_0 = 0$ when $R_\omega = 0$,

$$X_\omega = \frac{K}{C_2\omega}.$$

This explains why for very high values of R_ω, b_0 has been observed to be diminished but not quite to zero.

Proceeding now with the special case of tubes obeying Van der Bijl's relation we have

$$b_0 = \frac{A^2}{4}\frac{\dfrac{\partial^2 I_p}{\partial E_p{}^2}\left[R_\omega{}^2 + \left(X_\omega - \dfrac{K}{C_2\omega}\right)^2\right]}{\left(1 + \dfrac{Z_{p_0}}{r_p}\right)[(r_p + R_\omega)^2 + X_\omega{}^2]}.$$

In particular if C_2 is very small

$$R_\omega = R_{p\omega}, \qquad X_\omega = X_{p\omega}$$

and

$$b_0 = \frac{A^2}{4}\frac{\dfrac{\partial^2 I_p}{\partial E_g{}^2}}{\left(1 + \dfrac{Z_{p_0}}{r_p}\right)[(r_p + R_{p\omega})^2 + X_{p\omega}{}^2]}.$$

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