Nuclear Matrix Elements for Forbidden Beta Decay of Spheroidal-Shaped Nuclei*

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Theoretical expressions for both the nonrelativistic and relativistic nuclear matrix elements for β transitions of arbitrary forbiddenness are derived by using the Nilsson model. The β-decay operators are expressed in terms of irreducible spherical tensors. One- and two-particle intrinsic wave functions are used for odd- and even-mass nuclei, respectively. For even-mass nuclei, the results are found to be strongly dependent on the coupling of the two particles. The results are expected to be applicable to decays involving both intrinsic ground states and excited intrinsic states. The theoretical expressions are used to calculate matrix-element parameters for β decays in Tm^{170} and Re^{186}. The theoretical values are compared with experimentally determined values of the Tm^{170} and Re^{186} parameters.

I. INTRODUCTION

DURING recent years experimental work (β-γ directional correlation, etc.) has yielded values for the nuclear matrix elements for several forbidden β decays. It is of interest to compare these experimental values with the theoretical values predicted by nuclear models. The calculation of β-decay nuclear matrix elements using the Nilsson model[1] has been previously considered. Bogdan[2,3] has derived theoretical expressions for the matrix elements for first-forbidden β decay by using one-particle intrinsic wave functions. Recently, Berthier and Lipnik[4] have considered the use of one- and two-particle intrinsic wave functions. However, they present results only for the nonrelativistic matrix elements. Also, their results for two particle intrinsic wave functions are restricted to transitions between intrinsic ground states.

In the present work, considerably more general theoretical expressions for the β-decay nuclear matrix elements than were previously available are derived by using the Nilsson model. The present results are applicable to both the nonrelativistic and the relativistic matrix elements for β decays of any degree of forbiddenness. This is accomplished by writing the β-decay operators in terms of irreducible spherical tensors of arbitrary rank as defined by Rose and Osborn.[5] Also, the present results are applicable to β decays involving both intrinsic ground states and excited intrinsic states. One- and two-particle intrinsic wave functions are used for odd- and even-mass nuclei, respectively. The two-particle intrinsic wave functions have the general form suggested by Gallagher.[6] As indicated by Gallagher, it is found that the results are strongly dependent on the coupling of the two particles.

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II. DISCUSSION OF WAVE FUNCTION

The appropriately symmetrized total nuclear wave function for strongly deformed nuclei has the form[1]

$$\Psi=\left(\frac{2I+1}{16\pi^2}\right)^{1/2}[D_{MK}{}^I\chi_\Omega+(-)^{I-j}D_{M-K}{}^I\chi_{-\Omega}]. \quad (1)$$

The vibrational wave function has been omitted since it is assumed in this investigation that the nucleus remains in the same vibrational state. It is also assumed that the nucleus has axial symmetry and hence the condition $K=\Omega$ exists throughout the investigation.

The intrinsic wave function χ_Ω is the appropriately symmetrized product of Nilsson single-particle wave functions χ_{Ω_i} where χ_{Ω_i} is the wave function appropriate for the ith nucleon. In the coupling scheme appropriate for large deformations, the intrinsic motion is characterized by the constants of motion Ω_i, the component of the angular momentum j_i of each nucleon along the symmetry axis. Ω $(=\sum_i \Omega_i)$ is the component of the total intrinsic angular momentum j along the symmetry axis and is a good quantum number. Mottelson and Nilsson[7] have described the intrinsic states of odd-mass nuclei with single-particle intrinsic wave functions. In this case, χ_Ω in Eq. (1) is simply the Nilsson state appropriate for the odd particle.

In the present investigation, the intrinsic states of even-mass nuclei are described with two-particle intrinsic wave functions as suggested by Gallagher.[6] In this case, χ_Ω in Eq. (1) becomes the product of two Nilsson single-particle states:

$$\chi_\Omega=\chi_{\Omega_1}\chi_{\pm\Omega_2}, \quad (2a)$$

$$\chi_{-\Omega}=\chi_{-\Omega_1}\chi_{\mp\Omega_2}. \quad (2b)$$

The sign of Ω_2 is determined by the coupling of the particles. This general form applies to proton-proton, neutron-neutron, and proton-neutron systems.

For an odd-odd nucleus the two-particle intrinsic state is formed from the appropriate Nilsson states for the odd proton and the odd neutron. Therefore, the

[7] B. R. Mottelson and S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Skrifter **1**, No. 8 (1958).

following possibilities exist:

$$\chi_\Omega=\chi_{\Omega(p)}{}^{(1)}\chi_{\Omega(n)}{}^{(2)} \quad \text{and} \quad \Omega=\Omega(p)+\Omega(n), \qquad (3a)$$

$$\chi_\Omega=\chi_{\Omega(p)}{}^{(1)}\chi_{-\Omega(n)}{}^{(2)} \quad \text{and} \quad \Omega=|\Omega(p)-\Omega(n)|. \qquad (3b)$$

The proton and neutron have their angular momentum components coupled parallel and antiparallel for Eqs. (3a) and (3b), respectively. The superscripts denote particles (1) and (2) and are superfluous since the particles are not identical.

For an even-even nucleus, the intrinsic ground state consists of pairs of nucleons coupled to spin zero so that $\Omega=0$. Gallagher and Soloviev[8] have interpreted the excited states of even-even nuclei as two-particle intrinsic excitations for which the last two protons (or neutrons) occupy different Nilsson states. In this situation, for a proton-proton intrinsic state the following possibilities exist:

$$\chi_\Omega=\tfrac{1}{2}[\chi_{\Omega(p)}{}^{(1)}\chi_{\Omega(p')}{}^{(2)}-\chi_{\Omega(p')}{}^{(1)}\chi_{\Omega(p)}{}^{(2)}] \quad \text{and} \quad \Omega=\Omega(p)+\Omega(p'), \qquad (4a)$$

$$\chi_\Omega=\tfrac{1}{2}[\chi_{\Omega(p)}{}^{(1)}\chi_{-\Omega(p')}{}^{(2)}-\chi_{-\Omega(p')}{}^{(1)}\chi_{\Omega(p)}{}^{(2)}] \quad \text{and} \quad \Omega=|\Omega(p)-\Omega(p')|. \qquad (4b)$$

For a neutron-neutron intrinsic state the following changes are made in Eqs. (4a) and (4b): $\Omega(p)\rightarrow\Omega(n)$ and $\Omega(p')\rightarrow\Omega(n')$. For Eqs. (4a) and (4b), the two protons have their angular momentum components coupled parallel and antiparallel, respectively. Since the particles are identical, the wave functions have been symmetrized. For the intrinsic ground state of an even-even nucleus, the two protons will be in the same Nilsson state. Therefore, the condition $\chi_{\Omega(p')}=\chi_{\Omega(p)}$ exists.

III. DISCUSSION OF OPERATORS

In the present work, the β-decay operators are represented in terms of spherical tensors as suggested by Rose and Osborn.[5] The tensors are defined as follows:

$$T_{\lambda Lm}(\mathbf{r},\boldsymbol{\omega})=\sum_{m'}(1\,L-m',m'+m|\lambda m)\times\mathcal{Y}_{L,m'+m}(\mathbf{r})\mathcal{Y}_{1,-m'}(\boldsymbol{\omega}), \qquad (5)$$

where the $\mathcal{Y}$'s are solid spherical harmonics and $(1\,L-m',m'+m|\lambda m)$ is a Clebsch-Gordan coefficient. For β decay of arbitrary forbiddenness the operators fall into the following five types: $\mathcal{Y}_{\lambda m}(\mathbf{r})$, $T_{\lambda Lm}(\mathbf{r},\boldsymbol{\sigma})$, $T_{\lambda Lm}(\mathbf{r},\mathbf{p})$, $\mathcal{Y}_{\lambda m}(\mathbf{r})\boldsymbol{\sigma}\cdot\mathbf{p}$, and $T_{\lambda Lm}(\mathbf{r},\boldsymbol{\sigma}\times\mathbf{p})$. Rose and Osborn give the relation between the reduced matrix elements for the five operator types listed above and the corresponding reduced matrix elements in the customary notation of Konopinski and Uhlenbeck.[9] For any of the five operator types, the transition matrix element between the initial nuclear state $|\Psi_i\rangle$ and the final nuclear state $\langle\Psi_f|$ can be expressed in the form

$$\langle\Psi_f{}^\dagger|T_{\lambda Lm}|\Psi_i\rangle=\langle I_i\lambda M_im|I_fK_f\rangle\langle\Psi_f\|T_{\lambda L}\|\Psi_i\rangle, \qquad (6)$$

where $\langle\Psi_f\|T_{\lambda L}\|\Psi_i\rangle$ is the reduced matrix element of the operator.

IV. EXPRESSION OF MATRIX ELEMENTS IN TERMS OF SINGLE-PARTICLE MATRIX ELEMENTS

From Eq. (6), the reduced matrix elements are proportional to matrix elements of the form

$$\langle\Psi_f{}^\dagger|T_{\lambda Lm}|\Psi_i\rangle.$$

$T_{\lambda Lm}$ represents any of the five types of spherical tensor operators defined by Rose and Osborn[5] and is expressed in terms of a coordinate system fixed in space. The wave functions are given by Eq. (1). Without specifying its exact form, $T_{\lambda Lm}$ may be transformed to a coordinate system fixed in the nucleus and the integration over the variables of the rotational wave functions can be performed. For odd-A nuclei with single-particle intrinsic wave functions, the expression for the reduced matrix elements becomes (see Ref. 1, p. 31)

$$\langle\Psi_f\|T_{\lambda L}\|\Psi_i\rangle=\left(\frac{2I_i+1}{2I_f+1}\right)^{1/2}\times[\langle I_i\lambda K_i,K_f-K_i|I_fK_f\rangle\langle\chi_{\Omega_f}{}^\dagger|T_{\lambda L,K_f-K_i}|\chi_{\Omega_i}\rangle+(-)^{I_f-j_f}\langle I_i\lambda K_i,-K_f-K_i|I_f-K_f\rangle\times\langle\chi_{-\Omega_f}{}^\dagger|T_{\lambda L,-K_f-K_i}|\chi_{\Omega_i}\rangle]. \qquad (7)$$

In this expression the reduced matrix element is expressed in terms of single-particle intrinsic matrix elements of the form $\langle\chi_{\Omega_f}{}^\dagger|T_{\lambda Lm}|\chi_{\Omega_i}\rangle$, where χ_{Ω_i} and χ_{Ω_f} are, respectively, the initial and final Nilsson states of the transforming particle, and $T_{\lambda Lm}$ represents the spherical operators expressed in terms of a coordinate system fixed in the nucleus.

For even-mass nuclei, equations for the β-decay reduced matrix elements that are the analog of Eq. (7) may be developed. In this case, two-particle intrinsic wave functions are used. The present development is for β decays from odd-odd to even-even nuclei. For this situation the initial proton-neutron intrinsic wave function is given by either Eq. (3a) or Eq. (3b). For a final proton-proton state, the intrinsic wave function is given by either Eq. (4a) or Eq. (4b). Therefore, there are four possible combinations of initial and final intrinsic states. These may be divided into two categories based on the angular-momentum couplings of the two nucleons before and after the transition. The transition is between states of the same relative coupling if the two particles are coupled parallel (or antiparallel) in both the initial and final states. The transition is between states of different relative coupling if the two

[8] C. J. Gallagher and V. G. Soloviev, Kgl. Danske Videnskab. Selskab, Mat. Fys. Skrifter **2**, No. 2 (1962).

[9] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **60**, 308 (1941).

particles are coupled parallel in the initial state and antiparallel in the final state or vice versa.

The derivation of the expressions for the reduced matrix elements for even mass nuclei in terms of the Nilsson single-particle matrix elements proceeds in a manner similar to that used for odd-mass nuclei. The total nuclear wave function [Eq. (1)] with χ_Ω representing the appropriate two-particle intrinsic wave function is introduced into Eq. (6). Since for the initial state the superscript (2) is associated with the neutron, for β^- decay, the transforming particle is denoted as particle (2) and the nontransforming particle as particle (1). Hence, the scalar product of two single particle intrinsic wave functions for the nontransforming particle will appear in the calculations. Integrals such as $\langle\chi_{-\Omega(p_f)}{}^{\dagger(1)}|\chi_{\Omega(p_i)}{}^{(1)}\rangle$ are identically zero because of the orthogonality of the single-particle wave functions. Integrals such as $\langle\chi_{\Omega(p_f)}{}^{\dagger(1)}|\chi_{\Omega(p_i)}{}^{(1)}\rangle$ are assumed to be equal to one if $\chi_{\Omega(p_f)}{}^{(1)}=\chi_{\Omega(p_i)}{}^{(1)}$ or zero if $\chi_{\Omega(p_f)}{}^{(1)}\neq\chi_{\Omega(p_i)}{}^{(1)}$.

Equations for the reduced matrix elements in terms of single-particle intrinsic matrix elements for a β^- transition from an odd-odd to an even-even nucleus are given below. The equations are valid for a transition from a proton-neutron intrinsic state to a proton-proton intrinsic state. Both initially and finally the states may be either the intrinsic ground state or an excited intrinsic state. The results depend on the initial and final coupling (parallel or antiparallel) of the two particles. The appropriate coupling is given below for each equation.

Transitions between States of the Same Relative Coupling

$$\Omega_i=\Omega(p_i)+\Omega(n_i) \quad\text{and}\quad \Omega_f=\Omega(p_f)+\Omega(p_f'),$$

$$\langle\Psi_f\|T_{\lambda L}\|\Psi_i\rangle=\left(\frac{1}{2}\frac{2I_i+1}{2I_f+1}\right)^{1/2}\{(I_i\lambda K_i,K_f-K_i|I_fK_f)\langle\chi_{\Omega(p_f')}{}^\dagger|T_{\lambda L,K_f-K_i}|\chi_{\Omega(n_i)}\rangle\langle\chi_{\Omega(p_f)}{}^\dagger|\chi_{\Omega(p_i)}\rangle$$
$$-(I_i\lambda K_i,K_f-K_i|I_fK_f)\langle\chi_{\Omega(p_f)}{}^\dagger|T_{\lambda L,K_f-K_i}|\chi_{\Omega(n_i)}\rangle\langle\chi_{\Omega(p_f')}{}^\dagger|\chi_{\Omega(p_i)}\rangle\}. \tag{8a}$$

$$\Omega_i=\Omega(p_i)-\Omega(n_i) \quad\text{and}\quad \Omega_f=\Omega(p_f)-\Omega(p_f'),$$

$$\langle\Psi_f\|T_{\lambda L}\|\Psi_i\rangle=\left(\frac{1}{2}\frac{2I_i+1}{2I_f+1}\right)^{1/2}\{(I_i\lambda K_i,K_f-K_i|I_fK_f)\langle\chi_{-\Omega(p_f')}{}^\dagger|T_{\lambda L,K_f-K_i}|\chi_{-\Omega(n_i)}\rangle\langle\chi_{\Omega(p_f)}{}^\dagger|\chi_{\Omega(p_i)}\rangle$$
$$+(-)^{I_f-j_{p_f}-j_{p_f'}+1}(I_i\lambda K_i,-K_f-K_i|I_f-K_f)\langle\chi_{-\Omega(p_f)}{}^\dagger|T_{\lambda L,-K_f-K_i}|\chi_{-\Omega(n_i)}\rangle\langle\chi_{\Omega(p_f')}{}^\dagger|\chi_{\Omega(p_i)}\rangle\}. \tag{8b}$$

Transitions between States of Different Relative Coupling

$$\Omega_i=\Omega(p_i)+\Omega(n_i) \quad\text{and}\quad \Omega_f=\Omega(p_f)-\Omega(p_f'),$$

$$\langle\Psi_f\|T_{\lambda L}\|\Psi_i\rangle=\left(\frac{1}{2}\frac{2I_i+1}{2I_f+1}\right)^{1/2}\{(I_i\lambda K_i,K_f-K_i|I_fK_f)\langle\chi_{-\Omega(p_f')}{}^\dagger|T_{\lambda L,K_f-K_i}|\chi_{\Omega(n_i)}\rangle\langle\chi_{\Omega(p_f)}{}^\dagger|\chi_{\Omega(p_i)}\rangle$$
$$+(-)^{I_f-j_{p_f}-j_{p_f'}+1}(I_i\lambda K_i,-K_f-K_i|I_f-K_f)\langle\chi_{-\Omega(p_f)}{}^\dagger|T_{\lambda L,-K_f-K_i}|\chi_{\Omega(n_i)}\rangle\langle\chi_{\Omega(p_f')}{}^\dagger|\chi_{\Omega(p_i)}\rangle\}. \tag{8c}$$

$$\Omega_i=\Omega(p_i)-\Omega(n_i) \quad\text{and}\quad \Omega_f=\Omega(p_f)+\Omega(p_f'),$$

$$\langle\Psi_f\|T_{\lambda L}\|\Psi_i\rangle=\left(\frac{1}{2}\frac{2I_i+1}{2I_f+1}\right)^{1/2}(I_i\lambda K_i,K_f-K_i|I_fK_f)\{\langle\chi_{\Omega(p_f')}{}^\dagger|T_{\lambda L,K_f-K_i}|\chi_{-\Omega(n_i)}\rangle\langle\chi_{\Omega(p_f)}{}^\dagger|\chi_{\Omega(p_i)}\rangle$$
$$-\langle\chi_{\Omega(p_f)}{}^\dagger|T_{\lambda L,K_f-K_i}|\chi_{-\Omega(n_i)}\rangle\langle\chi_{\Omega(p_f')}{}^\dagger|\chi_{\Omega(p_i)}\rangle\}. \tag{8d}$$

If the final state is the intrinsic ground state of the even-even nucleus, the proton pair is in the same Nilsson single particle level and is coupled antiparallel. In this case, the following conditions exist: $\chi_{\Omega(p_f)}=\chi_{\Omega(p_f')}$ and $K_f=\Omega(p_f)-\Omega(p_f')=0$. For this special situation Eqs. (8a) through (8d) reduce to the following two cases:

$$\Omega_i=\Omega(p_i)-\Omega(n_i),$$

$$\langle\Psi_f\|T_{\lambda L}\|\Psi_i\rangle=\left(\frac{2I_i+1}{2I_f+1}\right)^{1/2}(I_i\lambda K_i,-K_i|I_f0)\langle\chi_{-\Omega(p_f)}{}^\dagger|T_{\lambda L,-K_i}|\chi_{-\Omega(n_i)}\rangle\langle\chi_{\Omega(p_f)}{}^\dagger|\chi_{\Omega(p_i)}\rangle; \tag{9a}$$

$$\Omega_i=\Omega(p_i)+\Omega(n_i),$$

$$\langle\Psi_f\|T_{\lambda L}\|\Psi_i\rangle=\left(\frac{2I_i+1}{2I_f+1}\right)^{1/2}(I_i\lambda K_i,-K_i|I_f0)\langle\chi_{-\Omega(p_f)}{}^\dagger|T_{\lambda L,-K_i}|\chi_{\Omega(n_i)}\rangle\langle\chi_{\Omega(p_f)}{}^\dagger|\chi_{\Omega(p_i)}\rangle. \tag{9b}$$

In Eqs. (8a) through (8d) and Eqs. (9a) and (9b), $K_f=\Omega_f$ and $K_i=\Omega_i$ are always treated as positive quantities. Therefore, for the situation in which the particles are initially coupled antiparallel the equations are written specifically for the case $\Omega(p_i)\geq\Omega(n_i)$. In order to retain the same functional dependence on $\Omega(p_i)$ and $\Omega(n_i)$ for the case $\Omega(n_i)\geq\Omega(p_i)$ [$\Omega_i=\Omega(n_i)-\Omega(p_i)$], the modification $K_i\rightarrow -K_i$ must be made in Eqs. (8b), (8d), and (9a).

Equations (8a) through (8d) and Eqs. (9a) and (9b) are made applicable to a β^+ transition from an odd-odd nucleus to an even-even nucleus by systematically replacing the subscript n with the subscript p and vice versa throughout the equations. This change must be made in both the matrix-element equations and the equations that state the Ω couplings for the various cases. When so modified the equations are valid for a β^+ transition between an initial proton-neutron intrinsic state and a final neutron-neutron intrinsic state. The initial and final states may be either the intrinsic ground state or an excited intrinsic state.

The corrections to β-decay matrix elements due to nuclear superfluidity have been determined by Soloviev.[10] In order to include this correction the reduced-matrix elements of Eqs. (7), (8), and (9) must be multiplied by the factor L where $L^2=R_ZR_N$. For both the proton and neutron systems, which are considered independently, R has the following form

$$R=\gamma\prod_s(U_sU_s'+V_sV_s')^2.$$

The primes refer to the final state and the product is over all levels in which there are no quasiparticles. According to Soloviev,[10] if the number of paired particles does not change during the transition, then $\gamma=U_f^2$; whereas if the number of paired particles changes during the transition, then $\gamma=V_f^2$, where f refers to the level in which the particle appears or disappears. U_f^2 and V_f^2 characterize the system with the smaller number of quasiparticles.

For the transitions considered in the present work the following situation exists. For the β^- decay of an odd Z, even N nucleus [Eq. (7)], the number of paired particles remains constant for both protons and neutrons. For the β^- decay of an odd-odd nucleus to the intrinsic ground state of an even-even nucleus [Eqs. (9a) and (9b)], the number of paired neutrons remains constant and the number of paired protons increases by 1. For the β^- decay of an odd-odd nucleus to an excited proton-proton state of an even-even nucleus [Eqs. (8a) through (8d)], the number of paired particles remains constant for both protons and neutrons. Gallagher and Soloviev[8] have determined numerical values for the correction factor for a number of β decays involving even mass nuclei for which the intrinsic states are described as two-particle Nilsson states.

V. NILSSON SINGLE-PARTICLE INTRINSIC MATRIX ELEMENTS

In this section, theoretical expressions are presented for the single-particle intrinsic matrix elements, $\langle\chi_{\Omega_f}{}^\dagger|T_{\lambda Lm}|\chi_{\Omega_i}\rangle$, for each of the five spherical tensor operators defined by Rose and Osborn.[5] For β decay of any order forbiddenness, all of the operators and the corresponding matrix elements may be expressed in terms of these five types. $T_{\lambda Lm}$ represents the spherical operators expressed in terms of a coordinate system fixed in the nuclear frame in which the Nilsson single-particle wave functions are described.

The Nilsson single-particle intrinsic wave function may be expressed as

$$\chi_\Omega=\sum_{l\Lambda\Sigma}a_{l\Lambda}|Nl\Lambda\Sigma\rangle, \tag{10}$$

where

$$|Nl\Lambda\Sigma\rangle\sim\rho^le^{-\rho^2/2}\,{}_1F_1(-n,\,l+\tfrac{3}{2},\,\rho^2)Y_{l\Lambda}f_{s\Sigma} \tag{11}$$

and

$$\Omega=\Lambda+\Sigma. \tag{12}$$

${}_1F_1(-n,\,l+\frac{3}{2},\,\rho^2)$ is the confluent hypergeometric function, $Y_{l\Lambda}$ the spherical harmonic, and $f_{s\Sigma}$ the spin wave function with $s=\frac{1}{2}$. Details of the wave function are given by Nilsson.[1]

The variable ρ appearing in the radial part of the wave function is dimensionless and is related to the variable $\boldsymbol{r}$ appearing in the various operators as follows,

$$\boldsymbol{r}=(\hbar/M\omega_0)^{1/2}\rho. \tag{13}$$

M is the nucleon mass and ω_0 is a parameter defined by Nilsson.[1] The variables $\boldsymbol{r}$ and ρ in the present work are denoted as $\boldsymbol{r}'$ and $\boldsymbol{r}$, respectively, by Nilsson.

The evaluation of $\langle\chi_{\Omega_f}{}^\dagger|T_{\lambda Lm}|\chi_{\Omega_i}\rangle$ proceeds in a straightforward manner from the definitions of χ_Ω and $T_{\lambda Lm}$. The details are omitted here and the final results are given below:

$$\langle\chi_{\Omega_f}{}^\dagger|\mathcal{Y}_{\lambda m}(\mathbf{r})|\chi_{\Omega_i}\rangle=(-1)^\lambda\left(\frac{2\lambda+1}{4\pi}\right)^{1/2}\sum_{if}a_{l_i\Lambda_i}a_{l_f\Lambda_f}\mathfrak{F}_\lambda\langle l_f\lambda00|l_i0\rangle\langle l_i\lambda\Lambda_im|l_f\Lambda_f\rangle\delta_{\Sigma_i\Sigma_f}. \tag{14}$$

The radial integral $\mathfrak{F}_\lambda$ is defined as follows,[11]

$$\mathfrak{F}_\lambda(N_fl_f,N_il_i)=(\hbar/M\omega_0)^{\lambda/2}\langle N_fl_f|\rho^\lambda|N_il_i\rangle. \tag{15}$$

[10] V. G. Soloviev, Dokl. Akad. Nauk SSSR **137**, 1350 (1961) [English transl.: Soviet Phys.—Doklady **6**, 346 (1961)].
[11] The formula for the integral $\langle N_fl_f|\rho^\lambda|N_il_i\rangle$ is given by Eq. (41) of Ref. 1.

$$\langle\chi_{\Omega_f}{}^\dagger|T_{\lambda Lm}(\mathbf{r},\boldsymbol{\sigma})|\chi_{\Omega_i}\rangle$$
$$=(-1)^L\frac{3}{4\pi}\left(\frac{2L+1}{3}\right)^{1/2}\sum_{i,f}a_{l_i\Lambda_i}a_{l_f\Lambda_f}\mathfrak{F}_L\langle l_fL00|l_i0\rangle\{\langle L,1,-m,0|\lambda-m\rangle\langle l_iL\Lambda_im|l_f\Lambda_f\rangle(-1)^{\Sigma_i-1/2}\delta_{\Sigma_i\Sigma_f}$$
$$+(2)^{1/2}\langle L,1,-1-m,1|\lambda-m\rangle\langle l_iL\Lambda_i,m+1|l_f\Lambda_f\rangle\delta_{\Sigma_i,1/2}\delta_{\Sigma_f,-1/2}$$
$$-(2)^{1/2}\langle L,1,1-m,-1|\lambda-m\rangle\langle l_iL\Lambda_i,m-1|l_f\Lambda_f\rangle\delta_{\Sigma_i,-1/2}\delta_{\Sigma_f,1/2}\}\,. \quad (16)$$

$$\langle\chi_{\Omega_f}{}^\dagger|T_{\lambda Lm}(\mathbf{r},\mathbf{p})|\chi_{\Omega_i}\rangle$$
$$=-i\frac{(3)^{1/2}}{4\pi}(2\lambda+1)^{1/2}\sum_{i,f}a_{l_i\Lambda_i}a_{l_f\Lambda_f}(-1)^{l_i+1}\{[(l_i+1)(2l_i+3)]^{1/2}\mathfrak{F}_L{}^-\langle l_i+1,l_f00|L0\rangle W(l_i1l_fL;l_i+1,\lambda)$$
$$-[l_i(2l_i-1)]^{1/2}\mathfrak{F}_L{}^+\langle l_i-1,l_f00|L0\rangle W(l_i1l_fL;l_i-1,\lambda)\}\langle l_i\lambda\Lambda_im|l_f\Lambda_f\rangle\delta_{\Sigma_i\Sigma_f}\,, \quad (17)$$

where the W's are Racah coefficients. The radial integrals $\mathfrak{F}_L{}^+$ and $\mathfrak{F}_L{}^-$ are defined as follows,

$$\mathfrak{F}_L{}^\pm(N_fl_f,N_il_i)=\left(\frac{\hbar}{M\omega_0}\right)^{(L-1)/2}\langle N_fl_f|\rho^LD_\pm|N_il_i\rangle\,, \quad (18)$$

where $D_+=d/d\rho+(l_i+1)/\rho$ and $D_-=d/d\rho-l_i/\rho$. According to Rose,[12] the matrix elements $\langle N_fl_f|\rho^LD_\pm|N_il_i\rangle$ may be expressed as follows[13]:

$$(L+1)\langle N_fl_f|\rho^LD_+|N_il_i\rangle=\tfrac{1}{2}(l_f-l_i+L+1)(l_f+l_i-L)\langle N_fl_f|\rho^{L-1}|N_il_i\rangle+[(W_i-W_f)/\hbar\omega_0]\langle N_fl_f|\rho^{L+1}|N_il_i\rangle \quad (19)$$

and

$$(L+1)\langle N_fl_f|\rho^LD_-|N_il_i\rangle=\tfrac{1}{2}(l_f-l_i-L-1)(l_f+l_i+L+2)\langle N_fl_f|\rho^{L-1}|N_il_i\rangle$$
$$+[(W_i-W_f)/\hbar\omega_0]\langle N_fl_f|\rho^{L+1}|N_il_i\rangle\,. \quad (20)$$

In Eqs. (19) and (20), W_i and W_f are the initial and final energy eigenvalues of the spherically symmetric term in the Nilsson single-particle Hamiltonian. Hence, one obtains the following, $W_i-W_f=(N_i-N_f)\hbar\omega_0$. In order to obtain Eqs. (19) and (20) it is assumed that the neutron and proton mass are equal.

$$\langle\chi_{\Omega_f}{}^\dagger|\mathcal{Y}_{\lambda m}(\mathbf{r})\boldsymbol{\sigma}\cdot\mathbf{p}|\chi_{\Omega_i}\rangle=(-1)^\lambda i\left(\frac{2\lambda+1}{4\pi}\right)^{1/2}\sum_{if}a_{l_i\Lambda_i}a_{l_f\Lambda_f}$$
$$\times\left\{\left(\frac{l_i+1}{2l_i+3}\right)^{1/2}\langle l_f\lambda00|l_i+1,0\rangle\mathfrak{F}_\lambda{}^-B(\lambda,l_i+1,m\Lambda_i\Sigma_i\Lambda_f\Sigma_f)\right.$$
$$\left.-\left(\frac{l_i}{2l_i-1}\right)^{1/2}\langle l_f\lambda00|l_i-1,0\rangle\mathfrak{F}_\lambda{}^+B(\lambda,l_i-1,m\Lambda_i\Sigma_i\Lambda_f\Sigma_f)\right\}\,, \quad (21)$$

where

$$B(\lambda,l_i\pm1,m\Lambda_i\Sigma_i\Lambda_f\Sigma_f)=\langle l_i\pm1,1\Lambda_i0|l_i\Lambda_i\rangle\langle l_i\pm1,\lambda\Lambda_im|l_f\Lambda_f\rangle(-1)^{\Sigma_i-1/2}\delta_{\Sigma_i\Sigma_f}$$
$$+(2)^{1/2}\langle l_i\pm1,1,\Lambda_i+1,-1|l_i\Lambda_i\rangle\langle l_i\pm1,\lambda,\Lambda_i+1,m|l_f\Lambda_f\rangle\delta_{\Sigma_i,1/2}\delta_{\Sigma_f,-1/2}$$
$$-(2)^{1/2}\langle l_i\pm1,1,\Lambda_i-1,1|l_i\Lambda_i\rangle\langle l_i\pm1,\lambda,\Lambda_i-1,m|l_f\Lambda_f\rangle\delta_{\Sigma_i,-1/2}\delta_{\Sigma_f,1/2}\,. \quad (22)$$

$$\langle\chi_{\Omega_f}{}^\dagger|T_{\lambda Lm}(\mathbf{r},\boldsymbol{\sigma}\times\mathbf{p})|\chi_{\Omega_i}\rangle=\frac{3\sqrt{2}}{4\pi}\sum_{if}a_{l_i\Lambda_i}a_{l_f\Lambda_f}(-)^{l_i}\sum_l(2l+1)W(11\lambda L,1l)$$
$$\times\{[(l_i+1)(2l_i+3)]^{1/2}\langle l_i+1,l_f00|L0\rangle W(l_i1l_fL,l_i+1l)\mathfrak{F}_L{}^-$$
$$-[l_i(2l_i-1)]^{1/2}\langle l_i-1,l_f00|L0\rangle W(l_i1l_fL,l_i-1l)\mathfrak{F}_L{}^+\}D(\lambda lm,\Lambda_i\Sigma_i\Lambda_f\Sigma_f)\,, \quad (23)$$

[12] M. E. Rose, *Multipole Fields* (John Wiley & Sons, Inc., New York, 1955), p. 81.
[13] The formula for the integral $\langle N_fl_f|\rho^{L\pm1}|N_il_i\rangle$ is given by Eq. (41) of Ref. 1.

where

$$D(\lambda lm,\Lambda_i\Sigma_i\Lambda_f\Sigma_f)=\langle 1l0m|\lambda m\rangle\langle l_il\Lambda_im|l_f\Lambda_f\rangle(-)^{\Sigma_i-1/2}\delta_{\Sigma_i\Sigma_f} +\sqrt{2}\langle 1l-1,1+m|\lambda m\rangle\langle l_il\Lambda_i1+m|l_f\Lambda_f\rangle\delta_{\Sigma_i,1/2}\delta_{\Sigma_f,-1/2} -\sqrt{2}\langle 1l1,-1+m|\lambda m\rangle\langle l_il\Lambda_i,-1+m|l_f\Lambda_f\rangle\delta_{\Sigma_i,-1/2}\delta_{\Sigma_f,1/2}. \quad (24)$$

VI. NUCLEAR PARAMETERS FOR FIRST FORBIDDEN β DECAY

The theoretical expressions presented in the preceding sections will now be used to determine the nuclear matrix elements for first forbidden β decays in Tm^{170} and Re^{186}. In both isotopes, the transition under consideration takes place from the ground state of an odd-odd nucleus with $J_i=1$ to the first excited state of an even-even nucleus with $J_f=2$. The nuclear matrix elements involved in this situation are

$$\int \mathbf{r},\quad \int i\boldsymbol{\alpha},\quad \int i\boldsymbol{\sigma}\times\mathbf{r},\quad \text{and}\quad \int B_{ij}$$

in the Konopinski and Uhlenbeck notation[9] or $\langle f\|\mathcal{Y}_1(\mathbf{r})\|i\rangle$, $\langle f\|T_{10}(\mathbf{r},\mathbf{p})\|i\rangle$, $\langle f\|T_{11}(\mathbf{r},\boldsymbol{\sigma})\|i\rangle$, and $\langle f\|T_{21}(\mathbf{r},\boldsymbol{\sigma})\|i\rangle$ in the spherical-tensor notation. In the present calculation the following parameters are evaluated:

$$x=-\frac{C_V}{C_A}\frac{\int\mathbf{r}}{\int B_{ij}},\quad u=\frac{\int i\boldsymbol{\sigma}\times\mathbf{r}}{\int B_{ij}},\quad \text{and}\quad \Lambda=\frac{\int i\boldsymbol{\alpha}}{\xi\int\mathbf{r}},$$

where $\xi=\alpha Z/2\rho$ and ρ is the nuclear radius.

For the transitions in Tm^{170} and Re^{186}, two-particle intrinsic wave functions for both the initial and final states should adequately describe the situation. For both transitions, the initial intrinsic state is described as a proton-neutron state in which the odd proton and odd neutron couple together so that $K=\Omega=\Omega(p)\pm\Omega(n)=1$. The final intrinsic state is described as a proton-proton state in which the two protons pair off in the same Nilsson state so that $K=\Omega=0$.

According to Fig. 9 of Ref. 7 the deformation parameters η are approximately equal to 6 and 4 for Tm^{170} and Re^{186}, respectively. Therefore, the intrinsic wave functions are determined at $\eta=6$ for Tm and $\eta=4$ for Re from the Nilsson energy-level diagrams.[7] However, in order to show the variation of the parameters with respect to deformation the calculations are also made for $\eta=2$ and $\eta=4$ for Tm and $\eta=2$ and $\eta=6$ for Re.

In Nilsson's representation the expansion coefficients $a_{l\Lambda}$ for a particular intrinsic state depend on the parameter μ which determines the relative strength of the spin-orbit and l^2 terms in the single-particle Hamiltonian. Nilsson has assigned values of μ for each N shell for both proton and neutron levels in order to reproduce for zero deformation the proper sequence of shell model levels. For both the Tm^{170} and Re^{186} transitions, the transforming nucleon is initially in a neutron level from the shell $N_i=5$ and finally in a proton level from the shell $N_f=4$. For these levels Nilsson[1] has assigned the values $\mu_i=0.45$ and $\mu_f=0.55$ and calculated the corresponding expansion coefficients. These coefficients were used in the present calculations. In addition, Mottelson and Nilsson[7] have given values for the expansion coefficients for the $N=5$ shell with $\mu=0.70$. Berthier and Lipnik[4] have used the $\mu=0.70$ expansion coefficients for the initial intrinsic state in calculating the matrix elements for the Tm^{170} transition. However, this does not appear to be appropriate since Nilsson and Mottelson specify that the calculation with $\mu=0.7$ is valid only for proton levels, and in the Tm^{170} calculation the transforming nucleon is initially in a neutron level. However in order to compare our results with those of Berthier and Lipnik and to show the variation of nuclear parameters with respect to μ, results are presented for both Re^{186} and Tm^{170} for two sets of μ values: $\mu_i=0.45$, $\mu_f=0.55$, and $\mu_i=0.70$, $\mu_f=0.55$.

Gallagher and Soloviev[8] concluded that for the ground state of Tm^{170}, which is the initial state for the transition under consideration, the odd proton is in the Nilsson state $\frac{1}{2}+[411\downarrow]$, and the odd neutron is in the Nilsson state $\frac{1}{2}-[521\downarrow]$. In the final state both protons are characterized by $\frac{1}{2}+[411\downarrow]$. In order to meet the requirement $K=\Omega$ it is necessary that

$$\Omega_i=\Omega(p_i)+\Omega(n_i)=\tfrac{1}{2}+\tfrac{1}{2}=1=K_i$$

and

$$\Omega_f=\Omega(p_f)-\Omega(p_f)=\tfrac{1}{2}-\tfrac{1}{2}=0=K_f.$$

Thus, the proton and neutron are coupled parallel in the initial state and the two protons are coupled antiparallel in the final state. Therefore, the transition takes place between states of different relative coupling and the general expression for the reduced matrix elements is given by Eq. (9b).

For the Re^{186} transition, as suggested by Gallagher and Soloviev,[8] the initial intrinsic states are taken to be $\frac{5}{2}+[402\uparrow]$ and $\frac{3}{2}-[512\downarrow]$ for the odd proton and odd neutron, respectively. In the final state, both protons are in the $\frac{5}{2}+[402\uparrow]$ state. For this situation

$$\Omega_i=\Omega(p_i)-\Omega(n_i)=\tfrac{5}{2}-\tfrac{3}{2}=1=K_i$$

and

$$\Omega_f=\Omega(p_f)-\Omega(p_f)=\tfrac{5}{2}-\tfrac{5}{2}=0=K_f.$$

Thus, the particles are coupled antiparallel in both the initial and final states. The transition takes place between states of the same relative coupling and the reduced matrix elements are given by Eq. (9a).

In Tables I and II the values of the nuclear parameters x, u, and Λ are given for Tm^{170} and Re^{186}. In order to show the dependence of the parameters on the de-

TABLE I. Nuclear parameters for 883-keV β transition in Tm^{170} as a function of η for $\mu_i=0.45$ and $\mu_f=0.55$. For Tm^{170} the accepted value for η is approximately equal to 6.

Parameter	$\eta=2$	$\eta=4$	$\eta=6$
x	0.055	−0.101	−0.032
u	−3.208	4.185	1.523
Λ	0.966	0.966	0.966

formation, values are given for three values of η. The "best" values are for $\eta=4$ and $\eta=6$ for Re^{186} and Tm^{170}, respectively. The variation of the nuclear parameters with respect to μ_i is illustrated in Table III.

The values given in Tables I, II, and III are independent of the superfluid correction factor since it divides out of the matrix element ratios.

The present results for the nuclear parameters for the Tm^{170} and Re^{186} β transitions apparently have no relation to those obtained previously by Bogdan.[2,3] This indicates that the present use of two-particle intrinsic wave functions for even mass nuclei rather than single-particle intrinsic wave functions has a profound effect on the results. It appears that it is not possible to formulate the total wave function for an odd-odd nucleus such as Tm^{170} with a single-particle intrinsic wave function. In order to satisfy the condition $K=\Omega=\sum_i \Omega_i$ for even mass nuclei it is necessary to sum over an even number of particles since K is an integer.

Berthier and Lipnik[4] have computed values of the nonrelativistic parameters only for the Tm^{170} transition by using two particle intrinsic wave functions. However, for the Nilsson parameter μ they assume an initial value of 0.75 and in the present work μ is initially assumed to be 0.45. However, if the value of μ_i is changed to 0.75, the present results for the nonrelativistic parameters (x and u) agree with those of Berthier and Lipnik.

For both transitions under consideration, the Ahrens-Feenberg[14] relation predicts that Λ is approximately 1 and the conserved vector current relation developed by Fujita[15] predicts that Λ is approximately 2.6. The present results indicate that when Λ is evaluated for a particular transition by using the Nilsson model a value is obtained which is independent of the model

TABLE II. Nuclear parameters for 934-keV β transition in Re^{186} as a function of η for $\mu_i=0.45$ and $\mu_f=0.55$. For Re^{186} the accepted value for η is approximately equal to 4.

Parameter	$\eta=2$	$\eta=4$	$\eta=6$
x	−0.043	−0.029	−0.023
u	−0.189	−0.223	−0.247
Λ	0.890	0.890	0.890

[14] T. Ahrens and E. Feenberg, Phys. Rev. **86**, 64 (1952).
[15] J. I. Fujita, Phys. Rev. **126**, 202 (1962).

parameters such as η and μ. In fact, the Nilsson model predicts the following general expression for Λ,

$$\Lambda=(W_i-W_f)\xi^{-1} \quad (25)$$

$$=(N_i-N_f)(67.64/Z). \quad (26)$$

For both the Tm^{170} and Re^{186} transitions, N_i-N_f equals 1 and therefore the following values are obtained for Λ: 0.966 for Tm^{170} and 0.890 for Re^{186}. The factor (W_i-W_f) in Eq. (25) arises from the use of Eqs. (19) and (20) in the evaluation of $\int i\boldsymbol{\alpha}$. Bogdan[2] has used the total β energy W_0 for W_i-W_f instead of the Nilsson-model energy difference $(N_i-N_f)\hbar\omega_0$ in evaluating Λ. In the present work, if W_0 is used for (W_i-W_f), the following values are obtained for Λ: 0.184 for Tm^{170} and 0.181 for Re^{186}.

Dulaney *et al.*[16] have determined what range of nuclear parameter values are in agreement with the available experimental data for the Tm^{170} and Re^{186} transitions. For Tm^{170} there appears to be no correlation

TABLE III. Nuclear parameters for Tm^{170} and Re^{186} as a function of μ_i for $\mu_f=0.55$. For Tm^{170} and Re^{186} the values are for $\eta=6$ and $\eta=4$, respectively.

	μ_i	x	u	Λ
Tm^{170}	0.45	−0.032	1.523	0.966
	0.70[a]	−0.353	2.135	0.966
Re^{186}	0.45	−0.029	−0.223	0.890
	0.70	0.004	−0.293	0.890

[a] The values of x and u for $\mu_i=0.70$ for Tm^{170} are equivalent to those obtained by Berthier and Lipnik. They are equivalent rather than equal since different standard matrix elements are used.

between the present results and those determined experimentally. According to Dulaney *et al.*, there is no unique experimental solution for the Tm^{170} parameters. For $\Lambda\approx1$ as determined in the present work, Dulaney *et al.*[16] find no suitable solution for the nuclear parameters. However, for Λ in the range 2 to 3 as suggested by Fujita the parameters $x\approx-0.20$ and $u\approx0$ provide the only satisfactory solution. This disagrees with the present result: $x=-0.032$, $u=1.523$. For Re^{186} there is apparently good agreement between the present results and those predicted experimentally. Again, there is no unique experimental solution. However, for Λ in the range from 1 to 5 the parameters $x=0$, $u=-0.21$ provide a satisfactory solution. This is in good agreement with the present results $x=-0.03$, $u=-0.22$.

An additional point of interest is that the Re^{186} results are considerably less sensitive than the Tm^{170} results to changes in either the value of η or μ_i.

[16] H. Dulaney, C. H. Braden, and L. D. Wyly, Nucl. Phys. **52**, 79 (1964).