TABLE IV. Phase shifts at 27.6 MeV from the three leading models and an energy-independent phase shift analysis (EIPSA).

Model	$^1S_0$	$^3P_0$	$^3P_1$	$^3P_2$
AMIV	48.0	8.2	−4.8	2.7
YRBl(K_0)	49.2	12.1	−5.8	2.6
CR21	46.6	8.7	−5.7	3.2
EIPSA[a]	48.6±0.4	7.6±0.6	−4.1±0.5	2.4±0.2

[a] P. Signell, Phys. Rev. **139**, B315 (1965).

not be extended below 24 MeV was known to the authors and was indicated in their paper. One way they could achieve a good low-energy fit would be to add the effective-range contributions to their representation,[24] as was done in the CR21 representation.

Data which give large χ^2 contributions to any one of the three representations are shown in Table III. Nearly half the high χ^2 contribution to YRBl(K_0) in the second energy range is seen to result from the 25.7-MeV measurements of A_{xx} and A_{yy}. Since these are determined primarily by the ^3P phase parameters, we compare these parameters to the other models in Table IV. It is seen that $^3P_0$ is high and $^1S_0$ is low both compared to the other models and to single-energy phase-shift analyses at that energy. Again, a correlated adjustment of parameters should remove this difficulty.

V. CONCLUSION

Examination of the existing fits to the best proton-proton scattering data reveals discrepancies in the fits which should be taken account of in any application where these discrepancies are potentially important. If this paper encourages more care to be taken in applying these models in specific cases, we will have accomplished our purpose.

ACKNOWLEDGMENTS

The calculations of χ^2 were carried out in the computer laboratory of Michigan State University. We wish to thank our many colleagues in the various accelerator laboratories who have helped us in the compilation and treatment of the data. Receipt of the manuscript of the article by Breit and Haracz on nucleon-nucleon scattering[16] prior to publication is gratefully acknowledged.

PHYSICAL REVIEW VOLUME 159, NUMBER 4 20 JULY 1967

Eigenstates of the $L=0$, Charge- and Spin-Independent Pairing Hamiltonian. I. Seniority-Zero States*

R. W. RICHARDSON
Physics Department, New York University, University Heights, New York, New York
(Received 28 February 1967)

Equations for all the seniority-zero eigenstates of $2N$ nucleons in an arbitrary charge- and spin-independent potential well and interacting through charge- and spin-independent pairing forces are derived. These equations are solved exactly for a large number of states of this system. The interaction in this Hamiltonian is effective in the $L=0$ states of the two-nucleon system, and its strength is independent of the remaining quantum numbers of the two nucleons. We solve our equations exactly for those states whose wave functions are totally symmetric functions of the spin-isospin coordinates of the N $L=0$ pairs of nucleons in the state. The wave functions of these states factor into a spin-isospin-dependent part and a spatially dependent part. The spin-isospin-dependent part of one of these wave functions is an eigenvector of three tridiagonal matrices which insure that the state is a spin, isospin, and supermultiplet eigenstate, respectively. Explicit expressions are given for the eigenvalues of these three matrices in terms of the quantum numbers of the state. The spatially-dependent part of one of these wave functions is given explicitly in terms of N parameters which we call pair energies. These pair energies are shown to satisfy N coupled algebraic equations which depend parametrically upon the supermultiplet quantum numbers of the state. An expression for the occupation probabilities of the levels of the single-particle well is given. Throughout this work, an arbitrary splitting of the single-particle levels is treated exactly.

I. INTRODUCTION

THE simplicity of the pairing Hamiltonian has made it a fruitful model for the study of the approximation techniques used in nuclear-structure calculations.[1] In addition to this, it has also proven to be a useful model for the calculation of specific nuclear properties.[1] In its most elementary form, the pairing model uses jj coupling single-particle states and a pairing interaction that is effective between any two identical nucleons that are coupled to $J=0$. The neutrons and protons therefore act as two independent systems. In this form, the model has been very successful in representing the properties of heavy nuclei. These properties have generally been calculated using

* This research was supported in part by the National Science Foundation.
[1] A. M. Lane, *Nuclear Theory* (W. A. Benjamin, Inc., New York, 1964), Part I and the references cited therein.

the approximations of the BCS theory of superconductivity.[1] However, the model is an exactly solvable many-body system[2] and it has been demonstrated[3] that the agreement between the predictions of the model and experimental observations can be significantly improved by the use of the exact eigenstates of the model Hamiltonian. The calculation of these exact eigenstates is straightforward[4] and it is, in fact, no more difficult than the calculation of approximate ones using any improvement of the BCS theory that gives up the assumption of completely independent quasiparticles.

A deficiency in this simple pairing model that becomes significant when it is applied to medium-weight nuclei is its neglect of any neutron-proton interaction. This deficiency may be removed by using the charge-independent pairing Hamiltonian rather than the charge-dependent one that we have been discussing. The interaction in this Hamiltonian is effective between any two nucleons that are coupled to $J=0$ and therefore to isospin $T=1$. This model has not yet received the extensive treatment that the simple pairing model with identical nucleons has been given. However, it does promise to extend the utility of the pairing model into the domain of intermediate-weight nuclei. Its eigenstates may be calculated, for a small number of nondegenerate single-particle levels, by quasispin methods.[5–7] In the more general case of many single-particle levels, approximation methods have been developed that are generalizations of the methods used for identical nucleons.[8–10] However, these approximations are still in need of some kind of justification. The exact eigenstates of this Hamiltonian have recently been derived by a generalization of the same formalism that led to the exact eigenstates of the pairing model with identical nucleons.[11,12] The calculation of these states would not require any more work than the calculation of the states of a system of identical nucleons.

The charge-independent pairing Hamiltonian is inappropriate for light nuclei due to the absence of any $T=0$ forces between the nucleons in its interaction. This lack may be remedied by the use of the charge- and spin-independent pairing Hamiltonian which is the subject of this paper. In this model, LS-coupling single-particle states and a pairing interaction that is effective between any two nucleons that are coupled to $L=0$ are used. The states of this Hamiltonian have been treated by group-theoretical methods for a single degenerate level[5,13] and by generalized BCS methods for any arbitrary single-particle spectrum.[14] With this paper, we show that many of the states of this Hamiltonian may be calculated exactly using the same methods that have been successfully applied to the simple and charge-independent pairing Hamiltonians.

While it may be argued that these exactly solvable many-body systems are not realistic, our point of view is that they are realistic enough to provide a valid testing ground for the various approximation techniques in use and, in some cases, they may be used as models of real nuclei. Furthermore, much insight into real many-body problems may be gotten from the study of exactly solvable models. For example, the structure of these model wave functions may be generalized somewhat and the resulting state used as an approximate eigenstate of a more realistic Hamiltonian. The details of this generalization will be presented elsewhere.

We begin our discussion of the charge- and spin-independent pairing Hamiltonian in Sec. II, where we set up our notation and discuss some of the general properties of the Hamiltonian and its eigenstates. In Sec. III, we specialize to the seniority-zero states of the Hamiltonian, expand an arbitrary state in a complete set of seniority-zero states and derive a Schrödinger equation for its amplitude. The solutions of this equation are all the seniority-zero states of the system and, in Sec. IV, we consider the special case of all the seniority-zero states of four nucleons. This section is meant to be an introduction to Sec. V, where we consider the spin-isospin symmetric seniority-zero states of $2N$ nucleons. By spin-isospin symmetric we mean that the wave function of the state is totally symmetric with respect to the spin-isospin coordniates of the N $L=0$ pairs in the state. We show that the wave function of one of these states may be written as a product of a spin-isospin part and an orbital part. The spin-isospin part of the wave function is shown to be a function of three indices and to be an eigenvector of three tridiagonal matrices, each matrix operating on one of the three indices. These three matrices insure that our states are spin, isospin, and supermultiplet eigenstates, respectively, and their eigenvalues are given in terms of the appropriate quantum numbers of the state. The orbital part of the wave function is shown to be a symmetrized product of N single-pair functions with each pair function being determined by a single parameter which we call a pair energy. The total energy of the state is shown to be the sum of the N pair energies. These pair energies are determined by a set of N coupled, nonlinear, algebraic equations which depend upon the single-particle spectrum, interaction strength, and supermultiplet quantum numbers of the state under consideration. The supermultiplet quantum numbers[15]

[2] R. W. Richardson and N. Sherman, Nucl. Phys. **52**, 221 (1964).
[3] R. W. Richardson and N. Sherman, Nucl. Phys. **52**, 253 (1964).
[4] R. W. Richardson, Phys. Rev. **141**, 949 (1966).
[5] M. Ichimura, Progr. Theoret. Phys. (Kyoto) **32**, 757 (1964).
[6] B. H. Flowers and S. Szpikowski, Proc. Phys. Soc. (London) **85**, 193 (1964).
[7] K. T. Hecht, Phys. Rev. **139**, B794 (1965).
[8] B. Bredmond and J. G. Valatin, Nucl. Phys. **41**, 640 (1963).
[9] M. Ichimura, Progr. Theoret. Phys. (Kyoto) **31**, 575 (1964).
[10] A. Goswami, Nucl. Phys. **60**, 228 (1964).
[11] R. W. Richardson, Phys. Rev. **144**, 874 (1966).
[12] R. W. Richardson, Phys. Rev. **154**, 1007 (1967).
[13] B. H. Flowers and S. Szpikowski, Proc. Phys. Soc. (London) **84**, 673 (1964).
[14] B. H. Flowers and M. Vujicic, Nucl. Phys. **49**, 586 (1963).
[15] E. P. Wigner, Phys. Rev. **51**, 106 (1937).

of the spin-isospin symmetric seniority-zero states that we consider are ($P\leq N, P'=0, P''=0$) with P even or odd as N is even or odd. The ground state as well as many excited states of the system are included in the supermultiplet with $P=0$ or 1 and the states of a system of identical nucleons are contained in the supermultiplet with $P=N$.

In Sec. VI, we derive an expression for the occupation probabilities of the levels of the potential well. This expression may be evaluated by solving an $N\times N$ system of linear algebraic equations. Finally, in Sec. VII, we discuss some of the properties of our equations and their solutions.

II. THE CHARGE- AND SPIN-INDEPENDENT PAIRING HAMILTONIAN

We consider a system of nucleons in an arbitrary charge- and spin-independent potential well and interacting through pairing forces that are effective in the $L=0$ two-body state and which are independent of the charge and spin of the two nucleons.[5,13,14] The levels of the potential well are labeled by the single-particle quantum numbers (n,l,m,σ,τ), where n is the radial quantum number, l is the orbital angular momentum and m its z component, σ is the spin projection which is $+\frac{1}{2}$ $(-\frac{1}{2})$ for spin up (down), and τ is the isospin projection which is $+\frac{1}{2}$ $(-\frac{1}{2})$ for proton (neutron). For brevity, we will denote the quantum numbers (n,l) by λ. The energy of one of these levels is denoted by ϵ_λ and each level is $4(2l+1)$-fold degenerate. The Hamiltonian for the noninteracting system of nucleons is

$$K=\sum_\lambda \epsilon_\lambda N_\lambda, \tag{2.1}$$

where

$$N_\lambda=\sum_{m,\sigma,\tau} a^\dagger_{\lambda m\sigma\tau} a_{\lambda m\sigma\tau} \tag{2.2}$$

is the occupation number operator for the single-particle level λ. In Eq. (2.2), $a^\dagger$ and a are nucleon creation and annihilation operators satisfying the usual Fermi commutation rules:

$$[a_{\lambda m\sigma\tau}, a^\dagger_{\lambda' m'\sigma'\tau'}]=\delta_{\lambda\lambda'}\delta_{mm'}\delta_{\sigma\sigma'}\delta_{\tau\tau'}. \tag{2.3}$$

Before we write down the pairing interaction that is to be added to (2.1) to complete our Hamiltonian, we must define some pair-annihilation and creation operators and introduce some notation. The pair-annihilation and creation operators destroy or create a pair of nucleons in the single-particle level λ that are coupled to a total orbital angular momentum $L=0$, spin and spin projection SM_S, and isospin and isospin projection TM_T. These operators are given by

$$B_\lambda(SM_S,TM_T)=\sum_{m\sigma\tau\sigma'\tau'}(-)^{l-m}\langle\sigma\sigma'|SM_S\rangle \times\langle\tau\tau'|TM_T\rangle a_{\lambda-m\sigma\tau}a_{\lambda m\sigma'\tau'} \tag{2.4}$$

and

$$B_\lambda^\dagger(SM_S,TM_T)=\sum_{m\sigma\tau\sigma'\tau'}(-)^{l-m}\langle\sigma\sigma'|SM_S\rangle \times\langle\tau\tau'|TM_T\rangle a_{\lambda m\sigma'\tau'}^\dagger a_{\lambda-m\sigma\tau}^\dagger, \tag{2.5}$$

where $\langle mm'|JM\rangle$ is an abbreviation for the Clebsch-Gordan coefficient $\langle\frac{1}{2}m\frac{1}{2}m'|JM\rangle$. In contrast to previous work on this Hamiltonian,[5,13,14] we will use pair operators that are coupled to good spin and isospin. The use of these coupled operators will lead to simple equations that are easily interpreted in Sec. V. However, there are only six spin-isospin states for an $L=0$ pair of nucleons while there are 16 possible values of the spin-isospin quantum numbers of a pair. To remove this superfluity, we introduce a new six-valued label for the spin-isospin state of a pair that retains the advantages of the spin-isospin quantum numbers. We choose to label the spin-isospin states of an $L=0$ pair by giving its spin S and the projection M of its spin *or* its isospin. The relation between the spin-isospin quantum numbers S, M_S, T, M_T and our new label S, M is

$$\begin{aligned} S&=S, \qquad M_S=SM,\\ T&=1-S, \quad M_T=TM. \end{aligned} \tag{2.6}$$

The values of S are 0 and 1 and the values of M are 0 and ±1, independent of the value of S. The utility of this choice will become evident in the succeeding sections of this paper. We will write the pair operators (2.4) and (2.5) as $B_\lambda(SM)$ and $B_\lambda^\dagger(SM)$ always with the relation (2.6) in mind in what follows.

The pairing interaction may now be written in terms of the pair operators as

$$V=-\tfrac{1}{2}g\sum_{S,M}\sum_{\lambda,\lambda'}B_\lambda^\dagger(SM)B_{\lambda'}(SM), \tag{2.7}$$

where the sums on S and M are over their allowable values and the sums on λ and λ' are over a finite set of values of λ which is chosen as part of the definition of V. In what follows, all values of λ will be assumed to belong to this set since particles that occupy levels outside this set do not interact. The pairing Hamiltonian $K+V$ is then given by

$$H=\sum_\lambda \epsilon_\lambda N_\lambda-\tfrac{1}{2}g\sum_{SM}\sum_{\lambda\lambda'}B_\lambda^\dagger(SM)B_{\lambda'}(SM). \tag{2.8}$$

It should be noted that here and throughout this work the single-particle spectrum ϵ_λ is completely arbitrary.

The Hamiltonian (2.8) is both spin- and charge-independent. Its eigenstates may therefore be classified according to the Wigner supermultiplet scheme.[15] That is, the states may be labeled by the irreducible representation of the four-dimensional unitary group $U(4)$ of unitary transformations among the four charge-spin states of a single nucleon to which they belong. This

group is generated by the 16 generators

$$Q_{\sigma\tau,\sigma'\tau'}=\sum_{\lambda m} a_{\lambda m\sigma\tau}{}^{\dagger}a_{\lambda m\sigma'\tau'}. \tag{2.9}$$

The irreducible representations of $U(4)$ may be labeled by the number of particles and the three supermultiplet quantum numbers $PP'P''$. However, for the states that we will treat in this paper, the quantum numbers P' and P'' are zero and the representation is labeled by $2N$, the number of particles, and P. These quantum numbers may be determined from the eigenvalues of the number operator

$$n=\sum_{\lambda} N_{\lambda}=\sum_{\sigma\tau} Q_{\sigma\tau,\sigma\tau}, \tag{2.10}$$

whose eigenvalues are $2N$ and the quadratic Casimir operator

$$C_2=\sum_{\sigma\tau,\sigma'\tau'} Q_{\sigma\tau,\sigma'\tau'}Q_{\sigma'\tau',\sigma\tau}, \tag{2.11}$$

whose eigenvalues are $N^2+P(P+4)+P'(P'+2)+P''^2$, for arbitrary P, P', P''. The construction of eigenstates of n may be easily accomplished and the close relationship between the problems of diagonalizing H and C_2 for the states that we consider is demonstrated in Appendix 2.

On restricting the group $U(4)$ to its (spin) $\times$ (isospin) subgroup, we obtain total spin and isospin quantum numbers for the states. These groups are generated by the operators $\mathbf{S}$ and $\mathbf{T}$, the components of which are

$$\begin{aligned} S_{\pm}&=\sum_{\tau} Q_{\pm\frac{1}{2}\tau,\mp\frac{1}{2}\tau}, \\ S_0&=\sum_{\sigma\tau} \sigma Q_{\sigma\tau,\sigma\tau}, \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} T_{\pm}&=\sum_{\sigma} Q_{\sigma\pm\frac{1}{2},\sigma\mp\frac{1}{2}}, \\ T_0&=\sum_{\sigma\tau} \tau Q_{\sigma\tau,\sigma\tau}. \end{aligned} \tag{2.13}$$

The eigenstates of $\mathbf{S}^2$ and $\mathbf{T}^2$ are treated in Appendix 1.

The Hamiltonian (2.8) also conserves the total number of pairs of particles that are coupled to $L=0$. The explicit demonstration of this peoperty is given in the following section. Since the Hamiltonain also conserves the total number of particles, its eigenstates may be labeled by the number of particles that are not in pairs that are coupled to $L=0$. This number is the seniority quantum v of the state. In this paper, we will only be concerned with the $v=0$ states of H. A complete set of these states may be formed by adding N, $L=0$, pairs of particles to the vacuum state in all possible ways. This observation is the starting point of the next section, where the Shchrödinger equation for the seniority-zero states of the Hamiltonian is derived.

III. SENIORITY-ZERO OF $2N$ NUCLEONS

In this section, we first expand an arbitrary $2N$-particle seniority-zero state in a complete set of states and then derive a Schrödinger equation for the amplitude of the state. Subsequent sections will deal with special cases of this equation.

A complete set of $2N$-particle seniority-zero states is given by the set of states

$$B_{\lambda_1}{}^{\dagger}(S_1M_1)\cdots B_{\lambda_N}{}^{\dagger}(S_NM_N)|0\rangle, \tag{3.1}$$

where $|0\rangle$ is the vacuum state and the $\lambda_iS_iM_i$ range over all their allowed values. We may expand an arbitrary state in the set (3.1) as

$$\begin{aligned} |\psi\rangle=\sum \psi(\lambda_1S_1M_1,\cdots,\lambda_NS_NM_N) \\ \times B_{\lambda_1}{}^{\dagger}(S_1M_1)\cdots B_{\lambda_N}{}^{\dagger}(S_NM_N)|0\rangle. \end{aligned} \tag{3.2}$$

In order to condense some overly long equations, we will use the notation $i=\lambda_iS_iM_i$, $\epsilon_i=\epsilon_{\lambda_i}$, and $B_i{}^{\dagger}=B_{\lambda_i}{}^{\dagger}(S_iM_i)$ when it does not obscure the meaning of the equations. Thus, Eq. (3.2) becomes

$$|\psi\rangle=\sum \psi(1\cdots N)B_1{}^{\dagger}\cdots B_N{}^{\dagger}|0\rangle. \tag{3.2'}$$

Note that the amplitude ψ must be a totally symmetric function of the variables $1\cdots N$.

We determine the amplitude ψ in (3.2) by substituting $|\psi\rangle$ into the Schrödinger equation $H|\psi\rangle=E|\psi\rangle$, for which we need to calculate the quantity $H|\psi\rangle$. After some commutator algebra, this may be written as

$$\begin{aligned} H|\psi\rangle=\sum\psi(1\cdots N)\{\sum_{i=1}^{N}\prod_{k\neq i} B_k{}^{\dagger}[H,B_i{}^{\dagger}] \\ +\tfrac{1}{2}\sum_{ij}{}'\prod_{k\neq ij} B_k{}^{\dagger}[[H,B_i{}^{\dagger}],B_j{}^{\dagger}]\}|0\rangle, \end{aligned} \tag{3.3}$$

where the prime on the sum on i and j will always be taken to mean that the terms with $i=j$ are excluded. There are no triple or higher commutators of H with three or more $B^{\dagger}$'s in (3.3) because these all vanish. To proceed further, we need to evaluate the single and double commutators of H with one and two $B^{\dagger}$'s. Recalling the form of H (2.8), we see that this will involve the single commutators of an N_{λ} and a B with a $B^{\dagger}$ and the double commutator of a B with two $B^{\dagger}$'s. We now turn to the evaluation of these commutators.

The commutator of N_{λ} with $B_i{}^{\dagger}$ follows immediately from the interpretation of N_{λ} as a number operator and is

$$[N_{\lambda},B_i{}^{\dagger}]=2\delta_{\lambda\lambda_i}B^{\dagger}. \tag{3.4}$$

The commutator of B_1 with $B_2{}^{\dagger}$ may be evaluated

using (2.3), (2.4), and (2.5) with the result that

$$[B_1,B_2^\dagger]=2\delta_{\lambda_1\lambda_2}\{(2l_1+1)\delta_{S_1S_2}\delta_{M_1M_2}-2\sum\langle\sigma_1\sigma_1'|S_1M_{S_1}\rangle\langle\tau_1\tau_1'|T_1M_{T_1}\rangle \times\langle\sigma_2\sigma_1'|S_2M_{S_2}\rangle\langle\tau_2\tau_1'|T_2M_{T_2}\rangle a_{\lambda m\sigma_2\tau_2}^\dagger a_{\lambda m\sigma_1\tau_1}\}, \quad (3.5)$$

where the sum is over all the σ's and τ's and the quantum numbers $S_iM_{S_i}T_iM_{T_i}$ are related to S_iM_i by (2.6). The double commutator of B_1 with $B_2^\dagger$ and $B_3^\dagger$ may be evaluated using (3.5), (2.5), and (2.3). We write the result as

$$[[B_1,B_2^\dagger],B_3^\dagger]=-8\delta_{\lambda_1\lambda_2}\delta_{\lambda_1\lambda_3}\sum_{S_4M_4}\mathfrak{M}(S_1M_{S_1},S_2M_{S_2},S_4M_{S_4},S_3M_{S_3}) \times\mathfrak{M}(T_1M_{T_1},T_2M_{T_2},T_4M_{T_4},T_3M_{T_3})B_{\lambda_1}(S_4M_4), \quad (3.6)$$

where the matrix $\mathfrak{M}$ is defined by

$$\mathfrak{M}(a\alpha,b\beta,c\gamma,d\delta)=\sum_{\sigma'\cdots\sigma_4}\langle\sigma_1\sigma_2|a\alpha\rangle\langle\sigma_2\sigma_3|b\beta\rangle\langle\sigma_3\sigma_4|c\gamma\rangle\langle\sigma_4\sigma_1|d\delta\rangle$$

$$=\delta_{\alpha+\gamma,\beta+\delta}\sum_\sigma\langle\sigma,\alpha-\sigma|a\alpha\rangle\langle\alpha-\sigma,\beta-\alpha+\sigma|b\beta\rangle\langle\beta-\alpha+\sigma,\gamma-\beta+\alpha-\sigma|c\gamma\rangle\times\langle\gamma-\beta+\alpha-\sigma,\sigma|d\delta\rangle. \quad (3.7)$$

Using the symmetry properties of the Clebsch-Gordan coefficients and the fact that $\mathfrak{M}$ is the trace of a matrix product it can be shown that $\mathfrak{M}$ is invariant under a cyclic permutation of its arguments and that it satisfies

$$\begin{aligned}\mathfrak{M}(a\alpha,b\beta,c\gamma,d\delta)&=(-)^{a+b+c+d}\mathfrak{M}(c\gamma,b\beta,a\alpha,d\delta)\\&=(-)^{a+b+c+d}\mathfrak{M}(a\alpha,d\delta,c\gamma,b\beta)\\&=(-)^{a+b+c+d}\mathfrak{M}(a-\alpha,d-\delta,c-\gamma,d-\delta).\end{aligned} \quad (3.8)$$

These symmetries plus the values

$$\mathfrak{M}(11,11,11,11)=1,\quad \mathfrak{M}(11,11,c0,d0)=\mathfrak{M}(11,b0,1-1,d0)=\tfrac{1}{2}(-)^{d+1}\quad \mathfrak{M}(a0,b0,c0,d0)=\tfrac{1}{4}[(-)^{a+c}+(-)^{b+d}] \quad (3.9)$$

determine all the nonzero elements of the matrix $\mathfrak{M}$.

Having determined the necessary commutators of the N's and B's we may now return to the evaluation of the single and double commutators of H with one and two $B^\dagger$'s. Using Eq. (2.8) for the Hamiltonian, we have

$$[H,B_i^\dagger]=\sum_\lambda\epsilon_\lambda[N_\lambda,B_i^\dagger]-\tfrac{1}{2}g\sum_{SM}\sum_{\lambda\lambda'}B_\lambda^\dagger(SM)[B_{\lambda'}(SM),B_i^\dagger].$$

Since this is going to be applied to the vacuum state when it is substituted into (3.3), we do so here with the result that the operator part of the commutator (3.5) can be ignored. We then have, using (3.4) and (3.5),

$$[H,B_i^\dagger]|0\rangle=\{2\epsilon_iB_i^\dagger-(2l_i+1)g\sum_\lambda B_\lambda^\dagger(S_iM_i)\}|0\rangle. \quad (3.10)$$

For the double commutator of H with two $B^\dagger$'s, we have

$$[[H,B_i^\dagger],B_j^\dagger]=-\tfrac{1}{2}g\sum_{SM}\sum_{\lambda\lambda'}B_\lambda^\dagger(SM)[[B_{\lambda'}(SM),B_i^\dagger],B_j^\dagger]$$

$$=g\delta_{\lambda_i\lambda_j}\sum_{S_i'M_i'S_j'M_j'}C(S_iM_i,S_jM_j;S_i'M_i',S_j'M_j')\sum_\lambda B_\lambda^\dagger(S_i'M_i')B_{\lambda_j}^\dagger(S_j'M_j'), \quad (3.11)$$

where we have used (3.6) and have defined the matrix C by

$$C(S_iM_i,S_jM_j;S_i'M_j',S_j'M_j')=4\mathfrak{M}(S_i'M_{S_i}',S_iM_{S_i},S_j'M_{S_j}',S_jM_{S_j})\mathfrak{M}(T_i'M_{T_i}',T_iM_{T_i},T_j'M_{T_j}',T_jM_{T_j}). \quad (3.12)$$

From the symmetries (3.8) of the matrix $\mathfrak{M}$, the following symmetries for the matrix C can be derived:

$$C(i,j;i',j')=C(i,j;j',i')=C(j,i;i',j')=C(j,i;j',i')=C(i',j';i,j)=C(-i,-j;-i',-j'), \quad (3.13)$$

where we have written i for S_iM_i and $-i$ for S_i-M_i. Furthermore, it follows from (3.7) that C is diagonal in the spin and isospin projections $M_{S_i}+M_{S_j}$ and $M_{T_i}+M_{T_j}$, which implies that C is diagonal in the quantities M_i+M_j and $S_iM_i+S_jM_j$ in terms of the new variables. All the nonzero values of C may be obtained from those values given in Table I by using the symmetry relations (3.13).

Having evaluated the necessary commutators of H and the $B^\dagger$'s, we may now substitute the results (3.10) and

(3.11) into (3.3) in order to obtain a Schrödinger equation for the amplitude ψ. The resulting equation is

$$(2\epsilon_1+\cdots+2\epsilon_N-E)\psi(1\cdots N)-g\sum_{i=1}^{N}\sum_{\lambda}(2l+1)\psi(1\cdots\lambda S_iM_i\cdots N)$$
$$+\tfrac{1}{2}g\sum_{i,j}{}'\sum_{S_i'M_i',S_j'M_j'}C(S_iM_i,S_jM_j;S_i'M_i',S_j'M_j')\psi(1\cdots\lambda_jS_i'M_i'\cdots\lambda_jS_j'M_j'\cdots N)=0, \quad (3.14)$$

where we have defined the components of ψ that violate the Pauli principle by ignoring the fact that the corresponding state (3.1) vanishes.[2,11]

The solutions of Eq. (3.14) are all seniority-zero states of the Hamiltonian H. The remainder of this paper is devoted to the study of some solutions of (3.14) that can be written down in a simple analytical form. In Sec. IV, we consider the special case of four nucleons ($N=2$) for which we can write down all the solutions of (3.14). Then, in Sec. V, we consider the spin-isospin symmetric solutions of (3.14) for arbitrary N. By spin-isospin symmetric solutions we mean those solutions of (3.14) that are totally symmetric functions of the spin-isospin variables $S_1M_1\cdots S_NM_N$ or equivalently of the orbital variables $\lambda_1\cdots\lambda_N$.

IV. THE SENIORITY-ZERO STATES OF FOUR NUCLEONS

As an introduction to the structure of Eq. (3.14), we will consider the seniority-zero states of four nucleons in some detail. For $N=2$, Eq. (3.14) becomes

$$(2\epsilon_{\lambda_1}+\epsilon_{\lambda_2}-E)\psi(\lambda_1S_1M_1,\lambda_2S_2M_2)-g\sum_{\lambda}(2l+1)[\psi(\lambda S_1M_1,\lambda_2S_2M_2)+\psi(\lambda_1S_1M_1,\lambda S_2M_2)]$$
$$+\tfrac{1}{2}g\sum_{S_1'M_1',S_2'M_2'}C(S_1M_1,S_2M_2;S_1'M_1',S_2'M_2')[\psi(\lambda_1S_1'M_1',\lambda_1S_2'M_2')+\psi(\lambda_2S_1'M_1',\lambda_2S_2'M_2')]=0. \quad (4.1)$$

We lose no generality by studying the $S_0=T_0=0$ solutions of this equation, i.e., those states which satisfy $M_1+M_2=S_1M_1+S_2M_2=0$. There are eight spin-isospin states of two pairs that satisfy this requirement and we now turn to the corresponding eight classes of solutions of Eq. (4.1).

Inspection of Eq. (4.1) shows that we may factor ψ as

$$\psi(\lambda_1S_1M_1,\lambda_2S_2M_2)=\alpha(S_1M_1,S_2M_2)\psi(\lambda_1\lambda_2), \quad (4.2)$$

where α is determined by the eigenvalue problem

$$\sum_{S_1'M_1',S_2'M_2'}C(S_1M_1,S_2M_2;S_1'M_1',S_2'M_2')\alpha(S_1'M_1',S_2'M_2')=\kappa\alpha(S_1M_1,S_1M_2), \quad (4.3)$$

and $\psi(\lambda_1\lambda_2)$ is determined by the eigenvalue problem

$$(2\epsilon_{\lambda_1}+2\epsilon_{\lambda_2}-E)\psi(\lambda_1\lambda_2)-g\sum_{\lambda}(2l+1)[\psi(\lambda\lambda_2)+\psi(\lambda_1\lambda)]+\tfrac{1}{2}\kappa g[\psi(\lambda_1\lambda_1)+\psi(\lambda_2\lambda_2)]=0. \quad (4.4)$$

The matrix elements of C needed for (4.3) may be obtained from Table I. When (4.3) is written out in detail, it becomes clear that its solutions fall into two classes—three antisymmetric solutions and five symmetric solutions. The orbital part of the wave function must of course have the same symmetry that α has in order to maintain the total symmetry of the wave function (4.2). We will first consider the antisymmetric solutions of (4.3) and (4.4) after which we will turn to the more interesting symmetric solutions.

For the antisymmetric solutions of (4.3), we have α given by

$$\begin{aligned}\alpha(11;1-1)&=-\alpha(1-1,11)=a,\\ \alpha(01,0-1)&=-\alpha(0-1,01)=c,\\ \alpha(10,00)&=-\alpha(00,10)=b,\\ \alpha(10,10)&=\alpha(00,00)=0,\end{aligned} \quad (4.5)$$

and (4.3) is satisfied identically with $\kappa=0$. The corresponding orbital parts of the wave functions of these states are the antisymmetric solutions of (4.4) with $\kappa=0$. These solutions are just antisymmetrized products of single-pair functions and the corresponding energy is the sum of the two single-pair energies, i.e.,

$$\psi(\lambda_1\lambda_2)=A(2\epsilon_{\lambda_1}-E_1)^{-1}(2\epsilon_{\lambda_1}-E_2)^{-1} \quad (4.6)$$

and

$$E=E_1+E_2, \quad (4.7)$$

where A is an antisymmetrizer and the pair energies E_1 and E_2 are distinct roots of the equation

$$1/g=\sum_{\lambda}(2l+1)/(2\epsilon_\lambda-E), \quad i=1,2. \quad (4.8)$$

If we further require these states to be spin and isospin eigenstates, then only one of the quantities a, b, and

TABLE I. The elements of the matrix C from which all nonzero elements may be obtained by use of the symmetries of C given in Eq. (3.13).

S_iM_i	S_jM_j	$S_i'M_i'$	$S_j'M_j'$	C
11	11	11	11	2
01	01	01	01	2
11	01	11	01	1
11	10	11	10	1
11	00	11	00	1
10	01	10	01	1
00	01	00	01	1
11	0 −1	11	0 −1	1
10	10	10	10	1
10	00	10	00	1
00	00	00	00	1
11	1 −1	10	10	1
11	1 −1	01	0 −1	1
10	10	00	00	1
01	0 −1	00	00	1
11	1 −1	00	00	−1
10	10	01	0 −1	−1

c of Eq. (4.5) can be different from zero. If we label the corresponding spins and isospins with a subscript a, b, or c, we then have $S_a=1$, $T_a=0$; $S_b=1$, $T_b=1$; and $S_c=0$, $T_c=1$. Using the results of Appendix 2, it can be shown that these states belong to the supermultiplet whose quantum numbers (P,P',P'') are given by (1,1,0).

For the symmetric solutions of (4.3), we use a notation that will be used in the next section for arbitrary N. In this notation, we have α given by

$$\begin{aligned}\alpha(11,1\,-1)&=\alpha(1\,-1,11)=\alpha(220),\\ \alpha(10,00)&=\alpha(00,10)=\alpha(100),\\ \alpha(01,0\,-1)&=\alpha(0\,-1,01)=\alpha(002),\\ \alpha(10,10)&=\alpha(200)\alpha(00,00)\alpha(000).\end{aligned}\tag{4.9}$$

The three variables n, n_S, and n_T in $\alpha(n,n_S,n_T)$ on the right-hand side of (4.9) are defined by

$$\begin{aligned}n=S_1+S_2,\quad n_S&=S_1|M_1|+S_2|M_2|,\\ n_T&=(1-S_1)|M_1|+(1-S_2)|M_2|.\end{aligned}\tag{4.10}$$

Equations for $\alpha(n,n_S,n_T)$ may be obtained from (4.3) using Table I with the results that

$$\begin{aligned}-\kappa\alpha(220)+\alpha(200)-\alpha(000)+2\alpha(002)&=0,\\ 2\alpha(220)+(1-\kappa)\alpha(200)+\alpha(000)-2\alpha(001)&=0,\\ -2\alpha(220)+\alpha(200)+(1-\kappa)\alpha(000)+2\alpha(002)&=0,\\ 2\alpha(220)-\alpha(200)+\alpha(000)-\kappa\alpha(002)&=0,\\ (2-\kappa)\alpha(100)&=0.\end{aligned}\tag{4.11}$$

This system of equations has one eigenvalue $\kappa=-4$ and four eigenvalues $\kappa=2$. If we further require our states to be spin and isospin eigenstates, using the methods of Appendix 1, then we obtain the solutions listed in Table II for Eqs. (4.11). In Table II, we have listed the supermultiplet quantum numbers, the eigenvalue κ of (4.11), the spin and isospin quantum numbers, and the corresponding solution of (4.11) with an arbitrary normalization.

TABLE II. The solution $\alpha(nn_Sn_T)$ of Eq. (4.11) with an arbitrary normalization.

				nn_Sn_T				
$(PP'P'')$	κ	S	T	220	200	000	002	100
(000)	−4	0	0	1	−1	1	−1	0
(200)	2	0	0	1	−1	−1	1	0
(200)	2	1	1	0	0	0	0	1
(200)	2	2	0	1	2	0	0	0
(200)	2	0	2	0	0	2	1	0

The symmetric solutions of Eq. (4.4) have been derived elsewhere.[2,11] We list here only the results, which are

$$\psi(\lambda_1\lambda_2)=S(2\epsilon_{\lambda_1}-E_1)^{-1}(2\epsilon_{\lambda_2}-E_2)^{-1}\tag{4.12}$$

and

$$E=E_1+E_2,\tag{4.13}$$

where S is a symmetrizer and E_1 and E_2 are distinct roots of the equations

$$\begin{aligned}1/g+\kappa/(E_2-E_1)&=\sum_\lambda(2l+1)/(2\epsilon_\lambda-E_1),\\ 1/g+\kappa/(E_1-E_2)&=\sum_\lambda(2l+1)/(2\epsilon_\lambda-E_2).\end{aligned}\tag{4.14}$$

This completes the solution of the four-nucleon system. We will show in the next section how the results for the symmetric states can be generalized to an arbitrary even number of nucleons.

V. SPIN-ISOSPIN SYMMETRIC, SENIORITY-ZERO STATES OF $2N$ NUCLEONS

We now generalize the results derived in Sec. IV for the symmetric states of 4 nucleons to the states of $2N$ nucleons. We will consider the states of $2N$ nucleons that are totally symmetric in the spin-isospin variables S_iM_i. It will be shown that the wave functions of these states can be factored into a product of a spin-isospin–dependent part times an orbital part as was done in Eq. (4.2) for $N=2$. Equations that are generalizations of Eqs. (4.11)–(4.14) will then be derived. Again, as in Sec. IV, no generality is lost by considering only the $S_0=T_0=0$ states of the system.

We seek the solutions of Eq. (3.14) that are totally symmetric functions of the variables λ_i and the variables S_iM_i separately. In order to determine the value of a totally symmetric function of N six-valued variables such as the S_iM_i in the wave function of one of these states, one only needs to know six numbers that specify how many of the variables have each of the six possible values. Therefore, let us introduce the six numbers $N(SM)$ which are functions of $S_iM_i\cdots S_NM_N$ and which are defined as the number of S_iM_i in the set $S_1M_1\cdots S_NM_N$ that satisfy $S_i=S$ and $M_i=M$, i.e.,

$$N(SM)=\sum_{i=1}^{N}\delta_{SS_i}\delta_{MM_i}.$$

Then, from the preceding remark, we know that the dependence of the wave function ψ on the spin-isospin variables $S_iM_i\cdots S_NM_N$ can only be through a dependence on the six numbers $N(SM)$ for the totally symmetric states. Furthermore, since we know the spin and isospin projections and the total number of particles of the state, we have the relations

$$S_0=\sum_{i=1}^{N} S_iM_i=N(11)-N(1\,-1),$$

$$T_0=\sum_{i=1}^{N} (1-S_i)M_i=N(01)-N(0\,-1),$$

and

$$N=\sum_{SM} N(SM), \tag{5.1}$$

which must be satisfied by the $N(SM)$. Therefore, only three of the $N(SM)$ are independent. In order to describe the dependence of a symmetric state on the three independent $N(SM)$, we introduce the variables

$$n=\sum_{i=1}^{N} S_i=N(11)+N(1\,0)+N(1\,-1),$$

$$n_S=\sum_{i=1}^{N} S_i|M_i|=N(11)+N(1\,-1),$$

and

$$n_T=\sum_{i=1}^{N} (1-S_i)|M_i|=N(01)+N(0\,-1). \tag{5.2}$$

These variables are the generalization of the variables introduced in Sec. IV in Eq. (4.10). The interpretation of these variables is as follows: n is the number of spin-one, isospin-zero pairs and therefore N-n is the number of spin-zero, isospin-one pairs in the component of the state; n_S is the number of spin-one, isospin-zero pairs that have $|M_S|=1$ and n_T is the number of spin-zero, isospin-one pairs that have $|M_T|=1$ in the component of the state. The variables n_S and n_T are analogous to the variable m introduced in Ref. 11 while n or N-n plays a role similar to that of the total number of pairs in that paper. The inverse relations to (5.1) and (5.2) are

$$\begin{aligned} &N(00)=N-n-n_T, \quad && N(0\,\pm1)=\tfrac{1}{2}(n_T\pm T_0),\\ &N(10)=n-n_S, && N(1\,\pm1)=\tfrac{1}{2}(n_S\pm S_0). \end{aligned} \tag{5.3}$$

Having defined the variables n, n_S, and n_T that determine the spin-isospin dependence of the symmetric wave functions, we now look for solutions of (3.14) of the form

$$\psi(1\cdots N)=\delta(\sum_i S_iM_i,0)\delta(\sum_i(1-S_i)M_i,0)\alpha(\sum S_i,\sum S_i|M_i|,\sum(1-S_i)|M_i|)\psi(\lambda_1\cdots\lambda_N), \tag{5.4}$$

where the first two factors are Kronecker deltas which insure that $S_0=T_0=0$, α is the spin-isospin wave function depending upon n, n_S, and n_T, and $\psi(\lambda_1\cdots\lambda_N)$ is the orbital part of the wave function. We obtain equations for α and by symmetrizing Eq. (3.14). This is done by multiplying the equation by

$$\delta(\sum_i S_iM_i,0)\delta(\sum_i(1-S_i)M_i,0)\delta(\sum_i S_i,n)\delta(\sum_i S_i|M_i|,n_S)\delta(\sum_i(1-S_i)|M_i|,n_T) \tag{5.5}$$

and summing over the variables $S_1M_1\cdots S_NM_N$. Before doing this, however, we will derive a combinatorial result that is needed in order to carry out this program.

In symmetrizing Eq. (3.14), we will have to evaluate many expressions of the form

$$A_N(S_0,T_0,n,n_s,n_T)=\sum_{S_1M_1\cdots S_NM_N}\delta(\sum S_iM_i,S_0)\delta(\sum(1-S_i)M_i,T_0)\delta(\sum S_i,n)$$
$$\times\delta(\sum S_i|M_i|,n_S)\delta(\sum(1-S_i)|M_i|,n_T). \tag{5.6}$$

This is most easily evaluated by introducing the generating function

$$G_N(uvxyz)=\sum_{S_0,T_0,n_S,n,n_T} A_N(S_0T_0nn_Sn_T)u^{S_0}v^{T_0}x^ny^{n_S}z^{n_T}, \tag{5.7}$$

which can be easily evaluated using (5.6). The result of this evaluation is

$$G_N(uvxyz)=(1+z/v+x+xy/u+vz+uxy)^N. \tag{5.8}$$

Expanding this expression, using the multinomial theorem, and identifying the resulting expansion with (5.7) yields the result

$$A_N(S_0T_0nn_Sn_T)=\frac{N!}{(n-n_S)!(N-n-n_T)!\left[\frac{1}{2}(n_S-S_0)\right]!\left[\frac{1}{2}(n_S+S_0)\right]!\left[\frac{1}{2}(n_T-T_0)\right]!\left[\frac{1}{2}(n_T+T_0)\right]!}. \tag{5.9}$$

This expression will be used many times in the symmetrization of Eq. (3.14).

The symmetrization of (3.14) is carried out by multiplying it by (15.5) and summing on $S_1M_1\cdots S_NM_N$. Using (5.4), we have for the first two terms of (3.14):

$$[(2\epsilon_{\lambda_1}+\cdots+2\epsilon_{\lambda_N}-E)\psi(\lambda_1\cdots\lambda_N)-g\sum_{i=1}^{N}\sum_{\lambda}(2l+1)\psi(\lambda_1\cdots\lambda\cdots\lambda_N)]A_N(00nn_Sn_T)\alpha(nn_Sn_T). \tag{5.10}$$

In order to symmetrize the third term of (3.14) we need to evaluate expressions of the form

$$\sum_{S_1M_1\cdots S_NM_N}\sum_{S_1'M_1',S_2'M_2'}\delta(\sum S_iM_i,0)\delta(\sum(1-S_i)M_i,0)\delta(\sum S_i,n)\delta(\sum S_i|M_i|,n)_S\delta(\sum(1-S_i)|M_i|,n_T)$$
$$\times C(S_1M_1,S_2M_2;S_1'M_1',S_2'M_2')\alpha(n+S_1'+S_2'-S_1-S_2,n_S+S_1'|M_1'|+S_2'|M_2'|-S_1|M_1|,$$
$$-S_2|M_2|n_T+(1-S_1')|M_1'|+(1-S_2')|M_2'|-(1-S_1)|M_1|-(1-S_2)|M_2|), \tag{5.11}$$

where the arguments of α are those appropriate for the spin-isospin variables $S_1'M_1',S_2'M_2',S_3M_3,\cdots,S_NM_N$. The sums on $S_3M_3\cdots S_NM_N$ in (5.11) may be performed using (5.6) with the result that (5.11) becomes

$$\sum_{S_1M_1S_2M_2S_1'M_1'S_2'M_2'}A_{N-2}(-S_1M_1-S_2M_2,-(1-S_1)M_1-(1-S_2)M_2,n-S_1-S_2,n_S-S_1|M_1|$$
$$-S_2|M_2|,n_T-(1-S_1)|M_1|-(1-S_2)|M_2|)C(S_1M_1,S_2M_2;S_1'M_1',S_2'M_2')\alpha(n+S_1'+S_2'-S_1$$
$$-S_2,n_S+S_1'|M_1'|+S_2'|M_2'|-S_1|M_1|-S_2|M_2|,n_T+(1-S_1')|M_1'|+(1-S_2')|M_2'|$$
$$-(1-S_1)|M_1|-(1-S_2)|M_2|). \tag{5.12}$$

The evaluation of (5.12) is rather long but it is completely straightforward using the values of C given in Table I. We will not reproduce the details of this evaluation here since it would take about as long to read it as it would to redo it. Rather, we will record here the result of (5.12), which is

$$[4A_{N-2}(2,0,n-2,n_S-2,n_T)+8A_{N-2}(1,1,n-1,n_S-1,n_T-1)+4A_{N-2}(0,2,n,n_S,n_T-2)+8A_{N-2}(1,0,n-2,n_S-1,n_T)$$
$$+8A_{N-2}(1,0,n-1,n_S-1,n_T)+8A_{N-2}(0,1,n-1,n_S,n_T-1)+8A_{N-2}(0,1,n,n_S,n_T-1)$$
$$+8A_{N-2}(1,-1,n-1,n_S-1,n_T-1)+A_{N-2}(0,0,n-2,n_S,n_T)+4A_{N-2}(0,0,n-1,n_S,n_T)+A_{N-2}(0,0,n,n_S,n_T)]$$
$$\times\alpha(n,n_S,n_T)+A_{N-2}(0,0,n-2,n_S,n_T)[\alpha(n-2,n_S,n_T)+2\alpha(n,n_S+2,n_T)-2\alpha(n-2,n_S,n_T+2)]$$
$$+2A_{N-2}(0,0,n-2,n_S-2,n_T)[\alpha(n,n_S-2,n_T)-\alpha(n-2,n_S-2,n_T)+2\alpha(n-2,n_S-2,n_T+2)]$$
$$+2A_{N-2}(0,0,n,n_S,n_T-2)[\alpha(n,n_S,n_T-2)-\alpha(n+2,n_S,n_T-2)+2\alpha(n+2,n_S+2,n_T-2)]$$
$$+A_{N-2}(0,0,n,n_S,n_T)[\alpha(n+2,n_S,n_T)+2\alpha(n,n_S,n_T+2)-2\alpha(n+2,n_S+2,n_T)]. \tag{5.13}$$

A further simplification of (5.13) can be accomplished by using (5.9) to express all the A_{N-2}'s that appear in (5.13) as numerical factors times $A_N(00nn_Sn_T)$. Substituting these results into (5.13), we obtain

$$\{[N(N-1)+2(n-n_S)(N-n-n_T)+(2N-1-2n_S-2n_T)(n_S+n_T)+2n_Sn_T]\alpha(n,n_S,n_T)+(n-n_S)(n-n_S-1)$$
$$\times[\alpha(n-2,n_S,n_T)+2\alpha(n,n_S+2,n_T)-2\alpha(n-2,n_S,n_T+2)]+\tfrac{1}{2}n_S^2[\alpha(n,n_S-2,n_T)-\alpha(n-2,n_S-2,n_T)$$
$$+2\alpha(n-2,n_S-2,n_T+2)]+\tfrac{1}{2}n_T^2[\alpha(n,n_S,n_T-2)-\alpha(n+2,n_S,n_T-2)+2\alpha(n+2,n_S+2,n_T-2)]$$
$$+(N-n-n_T)(N-n-n_T-1)[\alpha(n+2,n_S,n_T)+2\alpha(n,n_S,n_T+2)-2\alpha(n+2,n_S+2,n_T)]\}$$
$$\times A_N(00nn_Sn_T)/N(N-1). \tag{5.14}$$

Following the development of Sec. IV, we determine α so that the curly brackets in (5.14) equal $N(N-1)\kappa\times\alpha(nn_Sn_T)$. When this is satisfied, the third term of (3.14) becomes

$$\tfrac{1}{2}\kappa g\sum_{ij}{}'\psi(\lambda_1\cdots\lambda_j\cdots\lambda_j\cdots\lambda_N)A_N(00nn_Sn_T)\alpha(nn_Sn_T). \tag{5.15}$$

Combining (5.10) and (5.15), we obtain the equation

$$(2\epsilon_{\lambda_1}+\cdots+2\epsilon_{\lambda_N}-E)\psi(\lambda_1\cdots\lambda_N)-g\sum_{i=1}^{N}\sum_{\lambda}(2l+1)\psi(\lambda_1\cdots\lambda\cdots\lambda_N)+\tfrac{1}{2}\kappa g\sum_{ij}{}'\psi(\lambda_1\cdots\lambda_j\cdots\lambda_j\cdots\lambda_N)=0 \tag{5.16}$$

for the orbital part of the wave function. This equation, together with the equation for α derived from (5.14),

$$[N(N-1)(1-\kappa)+2(n-n_S)(N-n-n_T)+(2N-1-2n_S-2n_T)(n_S+n_T)+2n_Sn_T]\alpha(n,n_S,n_T)$$
$$+(n-n_S)(n-n_S-1)[\alpha(n-2,n_S,n_T)+2\alpha(n,n_S+2,n_T)-2\alpha(n-2,n_S,n_T+2)]+\tfrac{1}{2}n_S^2[\alpha(n,n_S-2,n_T)$$
$$-\alpha(n-2,n_S-2,n_T)+2\alpha(n-2,n_S-2,n_T+2)]+\tfrac{1}{2}n_T^2[\alpha(n,n_S,n_T-2)-\alpha(n+2,n_S,n_T-2)$$
$$+2\alpha(n+2,n_S+2,n_T-2)]+(N-n-n_T)(N-n-n_T-1)[\alpha(n+2,n_S,n_T)$$
$$+2\alpha(n,n_S,n_T+2)-2\alpha(n+2,n_S+2,n_T)]=0, \quad (5.17)$$

determine the wave function (5.4). The remainder of this paper is devoted to the properties of the solutions of Eqs. (5.16) and (5.17).

We first consider the solutions of (5.17). From their definitions (5.2), the ranges of the variables n,n_S, and n_T are given by

$$0\leq n\leq N,\ 0\leq n_S\leq n,\ 0\leq n_T\leq N-n. \quad (5.18)$$

Furthermore, n_S and n_T must be even numbers. If we require the states to be spin and isospin eigenstates, then we have two additional equations to be satisfied by α. These equations determine the n_S and n_T dependence of α and are derived in Appendix I, where it is shown that if the state is an eigenstate of $\mathbf{S}^2$ and $\mathbf{T}^2$ with eigenvalues $S(S+1)$ and $T(T+1)$, respectively, then α must satisfy the equations

$$\tfrac{1}{2}n_S^2\alpha(n,n_S-2,n_T)+[2n+(2n-1)n_S-2n_S^2-S(S+1)]\alpha(n,n_S,n_T)+2(n-n_S)(n-n_S-1)\alpha(n,n_S+2,n_T)=0 \quad (5.19)$$

and

$$\tfrac{1}{2}n_T^2\alpha(n,n_S,n_T-2)+[2N-2n+(2N-2n-1)n_T-2n_T^2-T(T+1)]\alpha(n,n_S,n_T)$$
$$+2(N-n-n_T)(N-n-n_T-1)\alpha(n,n_S,n_T+2)=0. \quad (5.20)$$

Thus, if $\alpha(n,0,0)$ is known, Eqs. (5.19) and (5.20) can be used to determine $\alpha(n,n_S,n_T)$ for all values of n_S and n_T. Conversely, if S and T are known, then (5.17) can be written for $n_S=n_T=0$ and (5.19) and (5.20) can then be used to eliminate $\alpha(n02)$ and $\alpha(n20)$ from it. The result of this is a single equation for $\alpha(n00)$. We will now carry out this reduction. We first write Eqs. (5.17), (5.19), and (5.20) for $n_S=n_T=0$, i.e.,

$$[N(N-1)(1-\kappa)+2n(N-n)]\alpha(n,0,0)+n(n-1)[\alpha(n-2,0,0)+2\alpha(n,2,0)-2\alpha(n-2,0,2)]$$
$$+(N-n)(N-n-1)[\alpha(n+2,0,0)+2\alpha(n,0,2)-2\alpha(n+2,2,0)]=0, \quad (5.21)$$

$$[2n-S(S+1)]\alpha(n,0,0)+2n(n-1)\alpha(n,2,0)=0, \quad (5.22)$$

and

$$[2N-2n-T(T+1)]\alpha(n,0,0)+2(N-n)(N-n-1)\alpha(n,0,2)=0. \quad (5.23)$$

Solving (5.22) and (5.23) for $\alpha(n20)$ and $\alpha(n02)$ in terms of $\alpha(n00)$ and substituting the results back into (5.21) yields the equation

$$\frac{n(n-1)[(N-n+3)(N-n+2)-T(T+1)]}{(N-n+2)(N-n+1)}\alpha(n-2,0,0)+[N(N-3)+S(S+1)+T(T+1)$$
$$\times 2n(N-n)-N(N-1)\kappa]\alpha(n,0,0)+\frac{(N-n)(N-n-1)[(n+3)(n+2)-S(S+1)]}{(n+2)(n+1)}\alpha(n+2,0,0)=0 \quad (5.24)$$

for $\alpha(n00)$. We thus have $\alpha(nn_Sn_T)$ as an eigenvector of three tridiagonal matrices, one in each of its three arguments.

We have explicit expressions for the eigenvalues of (5.19) and (5.20) in terms of the spin and isospin of the state. However, we have not yet derived an expression for the eigenvalue κ of (5.24). We will give a simple derivation of this expression after we consider the orbital part of the wave function. A detailed algebraic derivation of κ in terms of the eigenvalues of the Casimir operator C_2, (2.11), is given in Appendix 2. The result of these derivations is

$$\kappa=\frac{N(N-6)+P(P+4)+P'(P'+2)+P''^2}{N(N-1)}, \quad (5.25)$$

where the P's are the supermultiplet quantum numbers of the state. We now turn to consider the orbital part of the wave function.

The orbital parts of the wave functions of these symmetric states are given by the totally symmetric solutions of Eq. (5.16). These solutions have been derived

elsewhere[2,11] and we list here the results of this derivation. The function ψ is given by

$$\psi(\lambda_1\cdots\lambda_N)=S\prod_{i=1}^{N}(2\epsilon_{\lambda i}-E_i)^{-1} \tag{5.26}$$

and the energy of the state is given by

$$E=\sum_{i=1}^{N}E_i, \tag{5.27}$$

where S is a symmetrizer and the pair energies E_i satisfy the system of coupled algebraic equations

$$\frac{1}{g}+\kappa\sum_{j=1}{}'\frac{1}{(E_j-E_i)}=\sum_{\lambda}\frac{(2l+1)}{(2\epsilon_\lambda-E_i)},\quad i=1\cdots N, \tag{5.28}$$

plus the subsidiary conditions

$$E_i\neq E_j,\quad \text{for all}\quad i\neq j. \tag{5.29}$$

Thus, these states have the structure of the states of a system of independent pairs with all the many-body effects included in the coupling term of (5.28). It should be emphasized that the single-particle spectrum ϵ_λ in (5.28) is completely arbitrary.

If we compare the results of (5.27) and (5.28) for a single degenerate shell with those of Flowers and Szpikowski,[13] then we can derive the expression (5.25) for κ. For a single degenerate shell, whose single-particle energy we set equal to zero, Eq. (5.28) becomes

$$1/g+\kappa\sum_{j=1}^{N}{}'1/(E_j-E_i)=-(2l+1)/E_i,\qquad i=1\cdots N. \tag{5.30}$$

If (5.30) is multiplied by E_i and then summed on i, the expression

$$E/g=-N(2l+1)+\tfrac{1}{2}N(N-1)\kappa \tag{5.31}$$

for the energy is obtained. Comparing this with the expression

$$E/g=-N(2l+1)+\tfrac{1}{2}[N(N-6)+P(P+4)+P'(P'+2)+P''^2] \tag{5.32}$$

obtained from Eq. (26) of Ref. 13, we get the desired result (5.25). This result is rederived in Appendix 1, where the eigenvalue κ is related to the eigenvalues of the Casmir operator C_2 defined in Eq. (2.11).

The values that κ may take on can be given more explicitly by specifying which supermultiplets occur in the set of spin-isospin symmetric states. The values of the supermultiplet quantum numbers P, P', and P'' that are compatible with spin-isospin symmetry are

$$P=N, N-2, \cdots, 0 \text{ or } 1,\ P'=0,\ P''=0. \tag{5.33}$$

The fact that $P''=0$ follows from

$$[Y_0,B_\lambda^\dagger(SM)]=0,$$

where Y_0 is the space-exchange operator

$$Y_0=2\sum_{\sigma\tau}\sigma\tau Q_{\sigma\tau,\sigma\tau}. \tag{5.34}$$

Therefore, all the states have the same space exchange quantum number, $Y_0=0$. The values of P and P' follow from the spin-isospin symmetry. Eq. (5.33) may be checked by counting states and verifying that all the spin-isospin symmetric states are included in the supermultiplets (5.33). In order to count the number of spin-isospin symmetric states, we use the result[11] that the allowable values of S and T, which are obtained from Eqs. (5.19) and (5.20) for fixed n and $N-n$, are

$$S=n, n-2, \cdots, 0 \text{ or } 1$$
$$\text{and} \tag{5.35}$$
$$T=N-n, N-n-2, \cdots, 0 \text{ or } 1,$$

where the alternatives 0 or 1 are taken as n or $N-n$ are even or odd. The total number of states, for fixed n and N, is then

$$\sum_S(2S+1)\sum_T(2T+1), \tag{5.36}$$

where the sums are over the values given in (5.35). The total number of states is then found by summing (5.36) over the possible values of n, i.e., $n=0, 1, \cdots N$. Carrying out this program, we have

$$\sum_{n=0}^{N}\sum_{ST}(2S+1)(2T+1)=\tfrac{1}{4}\sum_{n=0}^{N}(n+1)(n+2)\times(N-n+1)(N-n+2)=\binom{N+5}{5}, \tag{5.37}$$

where the first equality is the result of doing the sums on S and T over the values given in (5.35) and the final expression, given in terms of a binomial coefficient, may be obtained by doing the sum on n explicitly. To check this number with the number of states in the supermultiplets (5.33), we use the result[16]

$$\dim\{(P00)\}=\tfrac{1}{12}(P+3)(P+2)^2(P+1) \tag{5.38}$$

for the number of states in the supermultiplet (P,0,0). Summing (5.38) over the allowable values of P given in (5.33) gives Eq. (5.37) for the total number of states. Thus, all the spin-isospin symmetric states are contained in the supermultiplets (5.33) and a complete set of spin-isospin quantum numbers for these states is the set (PSS_0TT_0).

We conclude this section by summarizing its results. The wave functions of the seniority-zero, spin-isospin

[16] H. A. Jahn, Proc. Roy. Soc. (London) **A201**, 516 (1950).

symmetric states of the Hamiltonian (2.8) are given by

$$\psi(1\cdots N)=\delta(\sum S_iM_i,S_0)\delta(\sum(1-S_i)M_i,T_0)\alpha(\sum S_i,\sum S_i|M_i|,\sum(1-S_i)|M_i|)\psi(\lambda_1\cdots\lambda_N). \quad (5.39)$$

The spin-isospin-dependent part of the wave function α is determined by the three eigenvalue problems

$$\frac{n(n-1)[(N-n+3)(N-n+2)-T(T+1)]}{(N-n+2)(N-n+1)}\alpha(n-2,0,0)+[N(N-3)+S(S+1)+T(T+1)$$
$$+2n(N-n)-N(N-1)\kappa]\alpha(n,0,0)+\frac{(N-n)(N-n-1)[(n+3)(n+2)-S(S+1)]}{(n+2)(n+1)}\alpha(n+2,0,0)=0, \quad (5.40)$$

$$\tfrac{1}{2}n_S{}^2\alpha(n,n_S-2,n_T)+[2n+(2n-1)n_S-2n_S{}^2-S(S+1)]\alpha(n,n_Sn_T)+2(n-n_S)(n-n_S-1)\alpha(n,n_S+2,n_T)=0, \quad (5.41)$$

and

$$\tfrac{1}{2}n_T{}^2\alpha(n,n_S,n_T-2)+[2N-2n+(2N-2n-1)n_T-2n_T{}^2-T(T+1)]\alpha(n,n_S,n_T)$$
$$+2(N-n-n_T)(N-n_S-n_T-1)\times\alpha(n,n_Sn_T+2)=0, \quad (5.42)$$

for $S_0=T_0=0$. The eigenvalues κ of (5.40) are given by

$$\kappa=\frac{N(N-6)+P(P+4)}{N(N-1)}, \quad (5.43)$$

where $P=N,\ N-2,\ \cdots,\ 0$ or 1. The eigenvalues of (5.41) and (5.42) are given by $S=n,\ n-2,\ \cdots,\ 0$ or 1 and $T=N-n,\ N-n-2,\ \cdots\ 0$ or 1. The orbital part of the wave function is given by

$$\psi(\lambda_1\cdots\lambda_N)=S\prod_{i=1}^{N}(2\epsilon_\lambda-E_i)^{-1}, \quad (5.44)$$

where S is a symmetrizer and the pair energies E_i satisfy the system of equations

$$1/g+\kappa\sum_j{}'1/(E_j-E_i)=\sum_\lambda(2l+1)/(2\epsilon_\lambda-E_i),$$
$$i=1\cdots N, \quad (5.45)$$

which depend upon the value of κ. The energy of these states is given by the sum of the E_i:

$$E=\sum_{i=1}^{N}E_i. \quad (5.46)$$

The states are labeled by the spin-isospin quantum numbers PSS_0TT_0 and by the root of Eq. (5.45) to which they belong.

We will return to these equations in Sec. VII, where we will discuss some features of their solutions. In Sec. VI, we give a brief derivation of the occupation probabilities for the single-particle levels λ.

VI. OCCUPATION PROBABILITIES

The occupation probabilities for the levels λ of the potential well, $\bar{N}_\lambda=\langle\psi|N_\lambda|\psi\rangle$, may be derived using a general theorem applicable to Hamiltonians that are linear in a parameter. In this case, the parameter is the single-particle energy ϵ_λ and the theorem states that

$$\bar{N}_\lambda=\frac{\partial E}{\partial\epsilon_\lambda}. \quad (6.1)$$

This may be easily verified by using the stationary property of the expectation value of the Hamiltonian in one of its eigenstates. Using (5.45) and (5.46) for E in (6.1), we get

$$\bar{N}_\lambda=\sum_{i=1}^{N}\frac{\partial E_i}{\partial\epsilon_\lambda}, \quad (6.2)$$

where the partial derivatives $\partial E_i/\partial E_\lambda$ satisfy the system of linear algebraic equations

$$\left[C_i-\kappa\sum_{j=1}^{N}{}'\frac{1}{(E_j-E_i)^2}\right]\frac{\partial E_i}{\partial\epsilon_\lambda}+\kappa\sum_{j=1}^{N}{}'\frac{1}{(E_j-E_i)^2}\frac{\partial E_j}{\partial\epsilon_\lambda}$$
$$\times\frac{2(2l+1)}{(2\epsilon_\lambda-E_i)^2},\quad i=1\cdots N, \quad (6.3)$$

obtained by differentiating Eq. (5.45) with respect to ϵ_λ. In Eq. (6.3), C_i is defined by

$$C_i=\sum_\lambda(2l+1)/(2\epsilon_\lambda-E_i)^2. \quad (6.4)$$

Equation (6.2) may be further simplified by using Cramer's rule to solve (6.3) and substituting the results into (6.2). The result of this substitution is

$$\bar{N}_\lambda=2(2l+1)\sum_{i=1}^{N}a_i/(2\epsilon_\lambda-E_i)^2, \quad (6.5)$$

where the coefficients a_i satisfy the system of equations

$$\left[C_i-\kappa\sum_{j=1}^{N}{}'\frac{1}{(E_j-E_i)^2}\right]a_i+\kappa\sum_{j=1}^{N}{}'\frac{a_j}{(E_j-E_i)^2}=1,$$
$$i=1\cdots N. \quad (6.6)$$

Since (6.5) implies

$$\sum_\lambda \bar{N}_\lambda=2\sum_i a_i c_i$$

and (6.6) implies that

$$\sum_i a_i c_i=N,$$

it may be easily shown that the relation

$$\sum_\lambda \bar{N}_\lambda=2N$$

is satisfied.

Algebraic techniques for calculating the expectation values of other operators, similar to those used in the case of $J=0$ pairing between identical nucleons,[17] can now be developed. However, we will not do this here since it leads to a long calculation that does not shed much light upon the structure of the eigenstates.

VII. CONCLUSION

We have derived equations for the seniority-zero spin-isospin symmetric states of $2N$ nucleons in a potential well with an arbitrary splitting of the single-particle levels. These states have the feature that the spin-isospin dependence of the wave function is completely separated from the orbital dependence. Thus, the calculation of one of these states is in two separate parts. First, the spin-isospin wave function is calculated and then the orbital wave function is calculated. The only coupling between these two calculations is through the eigenvalue κ whose value is given by Eq. (5.43). Thus, if only energies and occupation probabilities are wanted, then the spin-isospin problem can be ignored and Eq. (5.45) solved for the pair energies E_i and Eq. (6.5) evaluated for the occupation probabilities. The existence of real and complex pair energies as roots of Eq. (5.45) may be studied using the methods of Ref. 17. That this is a practical solution has been demonstrated by some extensive numerical calculations for $J=0$ pairing between identical nucleons,[4] where the equations to be solved are the same as Eq. (5.45) with $\kappa=2$. If, in addition to the orbital part of the wave function, the spin-isospin part is needed, then it may be obtained from Eqs. (5.40)–(5.42) with relative ease, since the eigenvalues of all three equations are known.

APPENDIX 1. SPIN AND ISOSPIN EIGENSTATES

In this Appendix, we derive Eqs. (5.19) and (5.20) for the spin and isospin eigenstates. The methods used are the same as those that were used in Ref. 12 to calculate isospin eigenstates.

[17] R. W. Richardson, J. Math. Phys. **6**, 1034 (1965).

Let $|ST\rangle$ denote a seniority-zero state with total spin S, total isospin T, and $S_0=T_0=0$. This state may be expanded in the set of states (3.1) as

$$|ST\rangle=\sum_{S_1M_1\cdots S_NM_N}\alpha(S_1M_1\cdots S_NM_N)$$
$$\times B^\dagger(S_1M_1)\cdots B^\dagger(S_NM_N)|0\rangle, \quad (A1.1)$$

where we have suppressed the orbital quantum numbers λ. In what follows, we will denote S_iM_i by i when this notation leads to no confusion. Two equations may be obtained for α by operating on (A1.1) with $\mathbf{S}^2$ and $\mathbf{T}^2$ and equating the results to $S(S+1)|ST\rangle$ and $T(T+1)|ST\rangle$, respectively. For the spin eigenvalue equation, we have

$$\mathbf{S}^2|ST\rangle=S_+S_-|ST\rangle$$
$$=\sum_{1\cdots N}\alpha(1\cdots N)S_+S_-B^\dagger(1)\cdots B^\dagger(N)|0\rangle$$
$$=\sum_{1\cdots N}\alpha(1\cdots N)\{\sum_{i=1}^{N}(\prod_{k\neq i}B^\dagger(k))[S_+,[S_-,B^\dagger(i)]]$$
$$+\sum_{i,j}{}'(\prod_{k\neq i,j}B^\dagger(k))$$
$$\times[S_-,B^\dagger(i)][S_+,B^\dagger(j)]\}|0\rangle. \quad (A1.2)$$

In order to evaluate (A1.2) we define the matrices $C_{S\pm}$ by

$$[S_\pm,B^\dagger(1)]=\sqrt{2}\sum_{i'}C_{S\pm}(i,i')B^\dagger(i'). \quad (A1.3)$$

These matrices may be evaluated by standard techniques with the results that

$$C_{S+}(SM;S'M')=\delta_{S1}\delta_{S'1}(\delta_{M0}\delta_{M'1}+\delta_{M-1}\delta_{M'0}),$$
$$C_{S-}(SM;S'M')=\delta_{S1}\delta_{S'1}(\delta_{M1}\delta_{M'0}+\delta_{M0}\delta_{M'-1}). \quad (A1.4)$$

If (A1.3) is substituted into (A1.2) and the result equated to $S(S+1)|ST\rangle$, then the equation

$$2\{\sum_{i=1}^{N}\sum_{S_i'M_i'S_i''M_i''}\alpha(1\cdots i'\cdots N)C_{S-}(i',i'')C_{S+}(i'',i)$$
$$+\sum_{i,j=1}^{N}{}'\sum_{S_iM_iS_j'M_j'}\alpha(1\cdots i'\cdots j'\cdots N)C_{S-}(i',i)C_{S+}$$
$$\times(j',j)\}=S(S+1)\alpha(1\cdots N) \quad (A1.5)$$

is obtained for α. The isospin eigenvalue problem yields a similar equation in which the matrices $C_{S\pm}$ have been replaced by $C_{T\pm}$ on the left-hand side and S has been replaced by T on the right-hand side. The matrices $C_{T\pm}$ are defined by Eq. (A1.3) with $S_\pm$ replaced by $T_\pm$ and are given by Eq. (A1.4) with the first two factors, $\delta_{S1}\delta_{S'1}$, replaced by $\delta_{S0}\delta_{S'0}$.

For the spin-isospin symmetric states, Eq. (A1.5) takes a particularly simple form in terms of the variables n, n_S, and n_T defined by (5.2). In this case, substitution of (A1.4) into the terms of (A1.5) yields

$$\sum_{i=1}^{N}\sum_{S_i'M_i'S_j'M_j'}\alpha(1\cdots i'\cdots N)C_{S-}(i',i'')C_{S+}(i'',i)$$
$$=(n-\tfrac{1}{2}n_S)\alpha(nn_Sn_T), \quad \text{(A1.6)}$$

$$\sum_{i,j=1}^{N}{}'\sum_{S_i'M_i'S_j'M_j'}\alpha(1\cdots i'\cdots j'\cdots N)C_{S-}(i',i)C_{S+}(j',j)$$
$$=\tfrac{1}{4}n_S{}^2\alpha(n,n_S-2,n_T)+n_S(n-n_S)\alpha(n,n_S,n_T)$$
$$+(n-n_S)(n-n_S-1)\alpha(n,n_S,n_T+2). \quad \text{(A1.7)}$$

Substitution of (A1.6) and (A1.7) into (A1.5) then gives

$$\tfrac{1}{2}n_S{}^2\alpha(n,n_S-2,n_T)+[2n+(2n-1)n_S-2n_S{}^2-S(S+1)]$$
$$\times\alpha(n,n_S,n_T)+2(n-n_S)(n-n_S-1)$$
$$\times\alpha(n,n_S+2,n_T)=0, \quad \text{(A1.8)}$$

which is the Eq. (5.19) that was to be derived.

The isospin eigenvalue equation may be derived in the same way or it may be obtained from Eq. (A1.8) by interchanging n_S and n_T and n and $N-n$. The resulting isospin eigenvalue equation is

$$\tfrac{1}{2}n_T{}^2\alpha(n,n_S,n_T-2)+[2N-2n+(2N-2n-1)n_T$$
$$-2n_T{}^2-T(T+1)]\alpha(n,n_S,n_T)+2(N-n-n_T)$$
$$\times(N-n-n_T-1)\alpha(n,n_S,n_T+2)=0, \quad \text{(A1.9)}$$

which is Eq. (5.20).

APPENDIX 2. CASIMIR-OPERATOR EIGENSTATES

In this Appendix the value of κ, Eq. (5.25), is derived by relating κ to the eigenvalues of the Casimir operator C_2 which is defined by Eq. (2.11).

We denote a $2N$-particle seniority-zero eigenstate of C_2 by $|NPP'P''\rangle$. The corresponding eigenvalue of C_2 is $N^2+P(P+4)+P'(P'+2)+P''^2$. Expanding this state in the set of states (3.1), we have

$$|NPP'P''\rangle=\sum_{1\cdots N}\alpha(1\cdots N)B^\dagger(1)\cdots B^\dagger(N)|0\rangle, \quad \text{(A2.1)}$$

where again the symbol i stands for the spin-isospin variables S_iM_i and the orbital quantum numbers λ have been suppressed. Operating on this state with C_2 which is defined in Eq. (2.13), we have

$$C_2|NPP'P''\rangle=\sum_{1\cdots N}\sum_{\sigma\tau\sigma'\tau'}\alpha(1\cdots N)\{\sum_{i=1}^{N}(\prod_{k\neq i}B^\dagger(k))$$
$$\times[Q_{\sigma\tau,\sigma'\tau'},[Q_{\sigma'\tau',\sigma\tau},B^\dagger(i)]]+\sum_{i,j=1}^{N}{}'(\prod_{k\neq i,j}B^\dagger(k))$$
$$\times[Q_{\sigma\tau,\sigma'\tau'},B^\dagger(i)][Q_{\sigma'\tau',\sigma\tau},B^\dagger(j)]\}|0\rangle. \quad \text{(A2.2)}$$

In order to evaluate the commutators in (A2.2), we introduce the matrices $q_{\sigma\tau,\sigma'\tau'}(SM,S'M')$ which are defined by

$$[Q_{\sigma\tau,\sigma'\tau'},B^\dagger(SM)]=\sum_{S'M'}q_{\sigma\tau,\sigma'\tau'}(SM,S'M')B^\dagger(S'M'). \quad \text{(A2.3)}$$

These matrices may be evaluated using Eq. (2.5) for $B^\dagger$ and Eq. (2.11) for Q, with the result that

$$q_{\sigma\tau,\sigma'\tau'}(SM,S'M')=2\sum_{\sigma_1\tau_1}\langle\sigma_1\sigma'|SM_S\rangle\langle\tau_1\tau'|TM_T\rangle$$
$$\times\langle\sigma_1\sigma|S'M_S'\rangle\langle\tau_1\tau|T'M_T'\rangle. \quad \text{(A2.4)}$$

Consider the first term in Eq. (A2.2). Using Eq. (A2.3), this term may be written as

$$\sum_{\sigma\tau\sigma'\tau'}[Q_{\sigma\tau,\sigma'\tau'},[Q_{\sigma'\tau',\sigma\tau},B^\dagger(i)]]$$
$$=\sum_{\sigma\tau\sigma'\tau'}\sum_{i',i''}q_{\sigma'\tau',\sigma\tau}(i,i'')q_{\sigma\tau,\sigma'\tau'}(i'',i')B^\dagger(i''). \quad \text{(A2.5)}$$

Using Eq. (A2.4), the matrix product of the two q's in (A2.5) may be identified as a product of two matrices $\mathfrak{M}$ of Eq. (3.7) which in turn may be identified with the matrix C of Eq. (3.12). Thus, (A2.5) may be written as

$$\sum_{\sigma\tau\sigma'\tau'}[Q_{\sigma\tau,\sigma'\tau'},[Q_{\sigma'\tau',\sigma\tau},B^\dagger(i)]]$$
$$=\sum_{i',j}C(i,j;i',j)B^\dagger(i') \quad \text{(A2.6)}$$
$$=6B^\dagger(i),$$

where, in the last equality, we have used

$$\sum_{S_jM_j}C(S_iM_i,S_jM_j;S_i'M_i',S_jM_j)=6\delta_{S_iS_i'}\delta_{M_iM_i'}, \quad \text{(A2.7)}$$

which may be derived from the definition of C. The second term of Eq. (A2.2) may be written as

$$\sum_{\sigma\tau,\sigma'\tau'}[Q_{\sigma\tau,\sigma'\tau'},B^\dagger(i)][Q_{\sigma'\tau',\sigma\tau},B^\dagger(j)]$$
$$=\sum_{i'j'}\sum_{\sigma\tau,\sigma'\tau'}q_{\sigma\tau,\sigma'\tau'}(i,i')q_{\sigma'\tau',\sigma\tau}(j,j')B^\dagger(i')B^\dagger(j')$$
$$=\sum_{i'j'}C(i,j;i'j')B^\dagger(i')B^\dagger(j'), \quad \text{(A2.8)}$$

where the second equality follows from the expressions for q in Eq. (A2.4) and for C in Eqs. (3.7) and (3.12).

Substituting Eqs. (A2.6) and (A2.8) into Eq. (A2.2) and equating the result to $[N^2+P(P+4)+P'(P'+2)+P''^2]|NPP'P''\rangle$ yields the equation

$$\sum_{i,j=1}^{N}{}'\sum_{S_i'M_i'S_j'M_j'}C(i,j;i'.j')\alpha(1\cdots i'\cdots j'\cdots N)$$
$$=[N(N-6)+P(P+4)+P(P'+2)+P''^2]$$
$$\times\alpha(1\cdots N) \quad \text{(A2.9)}$$

for α. However, in Sec. V, the eigenvalues of the matrix C were denoted by $N(N-1)\kappa$. Comparison with (A2.9) then yields the value

$$\kappa=\frac{N(N-6)+P(P+4)+P'(P'+2)+P''^2}{N(N-1)} \quad \text{(A2.10)}$$

for κ, which is the value (5.25) that was to be derived.